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A Class of Implicit Fractional ψ-Hilfer Langevin Equation with Time Delay and Impulse in the Weighted Space

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Abstract
In this paper, the Ulam-Hyers-Rassias stability is discussed and the existence and uniqueness of solutions for a class of implicit fractional ψ-Hilfer Langevin equation with impulse and time delay are investigated. A novel form of generalized Gronwall inequality is introduced. Picard operator theory is employed in authour's analysis. An example will be given to support the validity of our findings.

Keywords: Existence and uniqueness, Generalized Gronwall inequality, ψ-Hilfer Langevin equation, Picard operator theory, Ulam-Hyers-Rassias stability
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1. Introduction

The concept of derivatives of arbitrary order, which is essential to fractional calculus and provides a useful tool for characterizing the inherent properties of numerous materials and processes, has maintained its appeal to a large number of scientists in recent years. In [1], Capelas de Oliveira and Sousa presented a generalization concerning these derivatives, in which they combined many formulations, including the traditional Caputo and Riemann-Liouville operators, and proposed a new fractional differential operator, known as the fractional ψ-Hilfer operator.

Parallel to fractional derivation, another theory is growing: fractional differential equations. This theory has numerous applications, especially in the domains of signal processing, biology, physics, engineering, and finance. (Refer to [2, 3]).

One of the best examples of these fractional differential equations is the Langevin equation, which was initially proposed by Paul Langevin in 1908. Its goal is to give descriptions of specific phenomena that physicians, engineers, economists, and other experts may use. The Langevin equation first described the random movement of particles suspended in a liquid, which is commonly referred to as Brownian motion. In addition to being widely applied in all fields, Brownian motion and stochastic differential equations are also commonly used tools in all scientific fields. (see [4]-[13]).

Furthermore, the best modeling method was found to be fractional differential equations with impulse plus delay included.

Since fractional differential equations with impulse are used to simulate evolutionary situations involving fast changes at a finite or infinite number of instants, they represent a fascinating area of study. Similar to this, fractional differential equations with time delay represent real dynamics. They are used in a wide range of fields, including physics, chemistry, biology, road traffic, and medicine. Their goal is to simulate by taking into account the past. Giving someone a drug, for instance, doesn't result in an instant reaction; instead, you must wait a few minutes to see whether the substance has actually had an impact. (Refer to [14, 15]).

The majority of the time, it can be difficult to solve fractional differential equations exactly, and even when it can be done, it can be time-consuming and difficult to compute. Asking whether they are getting close to the exact solutions or if the error we made was not that big makes it simpler to give an appropriate explanation of the approximate solutions.

The concepts of Ulam-Hyers stability, Lyapunov stability, exponential stability, and finite-time stability are employed to evaluate the behavior of solutions to differential equations or dynamical systems under perturbations. Every kind of stability has uses and benefits of its own. Even though Lyapunov, exponential, and finite-time stabilities are important in their own contexts, Ulam-Hyers stability provides a special benefit by emphasizing the robustness of approximations. This makes it particularly desirable for real-world applications where we need to be sure that small deviations won't result in large inaccuracies because exact conditions are rarely realized. It is a useful tool in the stability analysis toolkit due to its adaptability and wide range of applications. (see [16]-[18]).

Abdo et al. [19] have studied the Ulam-Hyers-Mittag-Leffler stability, uniqueness, and existence of a fractional delay differential equation

$$\begin{cases} {}^{H}\mathfrak{D}_{0+,\iota}^{p_1,p_2;\psi} w(\iota) = f(\iota, w_\iota), \quad 0 \le \iota \le b, \\ \mathfrak{I}_{0+,\iota}^{1-\gamma;\psi} w\left(0^+\right) = w_0 \in \mathbb{R}, \\ w(\iota) = \varphi(\iota), \quad -\infty < \iota \le 0. \end{cases}$$

Recently, Lima KB et al. [20] investigated the Ulam-Hyers stability, uniqueness, and existence of the following fractional delay impulsive differential equation:

$$\begin{cases} {}^{H}\mathfrak{D}_{0+,\iota}^{p_1,p_2;\psi} w(\iota) = f(\iota, w_\iota), \quad \iota \in (0,b] \backslash \{\iota_1, \iota_2, \ldots, \iota_m\}, \\ \Delta w(\iota_i) = I_i\left(w\left(\iota_i^-\right)\right) = w\left(\iota_i^+\right) - w\left(\iota_i^-\right), \quad i = 1,2,\ldots,m, \\ \mathfrak{I}_{0+,\iota}^{1-\gamma} w(0) = w_0, \\ w(\iota) = \varphi(\iota), \quad \iota \in [-r, 0]. \end{cases}$$

Motivated by the latter work, we present in this work a fairly exhaustive study of a novel class of implicit ψ-Hilfer fractional Langevin equation with delays and impulses given by the form:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi} + \lambda\right) w(\iota) = f\left(\iota, w(\iota), w(\sigma(\iota)), {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi} + \lambda\right) w(\iota)\right), \quad \iota \in (0,b] \backslash \{\iota_1, \iota_2, \ldots, \iota_m\}, \\ \Delta w(\iota_i) = I_i\left(w\left(\iota_i^-\right)\right) = w\left(\iota_i^+\right) - w\left(\iota_i^-\right), \quad i = 1,2,\ldots,m, \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi} w(0) = w_0, \quad \gamma = q + (p_1+p_2)(1-q), \\ w(\iota) = \varphi(\iota), \quad \iota \in [-r, 0], \quad 0 \le r < \infty. \end{cases} \tag{1.1}$$

Where $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}$ and $\mathfrak{D}_{0^+,\iota}^{\vartheta;\psi}$ represent ψ-fractional integrals in order $1-\gamma$ and ψ-Hilfer fractional derivative in order $\vartheta \in \{p_1, p_2\}$ and type q respectively. $0 < \vartheta < 1$, $0 \le q \le 1$. Also, $f : [0,b] \times \Omega \to \mathbb{R}$ a given function, $I_i : \mathbb{R} \to \mathbb{R}$ and $\varphi : [-r, 0] \to \mathbb{R}$ continuous functions, $w\left(\iota_i^+\right) = \lim_{\tau \to 0^+} w(\iota_i + \tau), w\left(\iota_i^-\right) = \lim_{\tau \to 0^-} w(\iota_i - \tau), \iota_i$ satisfies $0 = \iota_0 < \iota_1 < \iota_2 < \cdots < \iota_m < \iota_{m+1} = b < \infty$ and $\sigma : [0,b] \to [-r, b]$ is a delay function that is continuous and ensures $\sigma(\iota) \le \iota, \iota \in [0,b]$.

Let $J = [0,b]$, and let $C(J,\mathbb{R})$ and $C^n(J,\mathbb{R})$ be the Banach spaces of continuous functions, n-times continuously differentiable functions on J, respectively. Moreover, for any $f \in C(J,\mathbb{R})$, we have $\|f\|_C = \sup\{|f(\iota)| : \iota \in J\}$. On the other hand, we consider the weighted space in [20], defined by

$$C_{1-\gamma;\psi}(J,\mathbb{R}) = \left\{w : J \to \mathbb{R} : (\psi(\iota) - \psi(0))^{1-\gamma} w(\iota) \in C(J,\mathbb{R})\right\}, \quad 0 < \gamma \le 1.$$

Define the Banach space

$$PC_{1-\gamma;\psi}(J,\mathbb{R}) = \left\{ \begin{array}{l} w : J \to \mathbb{R}; \quad w \in C_{1-\gamma;\psi}([\iota_i, \iota_{i+1}], \mathbb{R}), i = 0, \ldots, m, \\ \text{and there exist } w\left(\iota_i^+\right), w\left(\iota_i^-\right), \text{with } w(\iota_i) = w\left(\iota_i^-\right), i = 1,2,\ldots,m \end{array} \right\}, \quad 0 < \gamma \le 1,$$

using the norm

$$\|w\|_{PC_{1-\gamma;\psi}} = \sup_{\iota\in J}\left|(\psi(\iota)-\psi(0))^{1-\gamma}w(\iota)\right|.$$

We then specify the space.

$$\Omega_{\gamma;\psi} = \left\{w:[-r,b]\to\mathbb{R} : w\in C([-r,0],\mathbb{R})\cap PC_{1-\gamma;\psi}(J,\mathbb{R})\right\},$$

using the norm $\|w\|_{\Omega_{\gamma;\psi}} = \max\left\{\|w\|_C, \|w\|_{PC_{1-\gamma;\psi}}\right\}$. One can verify that $\left(\Omega_{\gamma;\psi}, \|\cdot\|_{\Omega_{\gamma;\psi}}\right)$ is a Banach space (see [19, 20]).

2. Preliminaries

Definition 2.1. *([1]) For $p>0$, and $\psi\in C^1(J,\mathbb{R})$, the fractional ψ-Riemann-Liouville operator with order p for an integrable function w can be written as*

$$\mathfrak{I}^{p;\psi}_{0^+,\iota}w(\iota) = \frac{1}{\Gamma(p)}\int_0^\iota \psi'(s)(\psi(\iota)-\psi(s))^{p-1}w(s)ds, \tag{2.1}$$

in which $\psi'(\iota)>0$, $\forall\iota\in J$.

Definition 2.2. *([1]) For $0<p<1$, $w\in C(J,\mathbb{R})$, $\psi\in C^1(J,\mathbb{R})$ with $\psi'(\iota)>0$, $\forall\iota\in J$, the fractional ψ-Hilfer derivative operator with order p and type $0\le q\le 1$ of w is represented as*

$$^{H}\mathfrak{D}^{p,q;\psi}_{0^+,\iota}w(\iota) = \mathfrak{I}^{q(1-p);\psi}_{0^+,\iota}\left(\frac{1}{\psi'(\iota)}\frac{d}{d\iota}\right)\mathfrak{I}^{(1-q)(1-p);\psi}_{0^+,\iota}w(\iota). \tag{2.2}$$

Lemma 2.3. *([1]) Let $0<p<1$, $0\le q\le 1$, $w\in C^1(J,\mathbb{R})$, then*

$$\mathfrak{I}^{p;\psi}_{0^+,\iota}\,{}^{H}\mathfrak{D}^{p,q;\psi}_{0^+,\iota}w(\iota) = w(\iota) - \frac{(\psi(\iota)-\psi(0))^{\rho-1}}{\Gamma(\rho)}\mathfrak{I}^{1-\rho;\psi}_{0^+,\iota}w(0), \tag{2.3}$$

where $\rho = p+q(1-p)$.

Lemma 2.4. *([1, 21]) Let $p,q>0, \delta>p$ and $w\in C(J,\mathbb{R})$. Following that $\forall\iota\in J$ there are*
(1) $\mathfrak{I}^{p;\psi}_{0^+,\iota}\mathfrak{I}^{q;\psi}_{0^+,\iota}w(\iota) = \mathfrak{I}^{p+q;\psi}_{0^+,\iota}w(\iota)$,
(2) ${}^{H}\mathfrak{D}^{p,q;\psi}_{0^+,\iota}\mathfrak{I}^{p;\psi}_{0^+,\iota}w(\iota) = w(\iota)$,
(3) $\mathfrak{I}^{p;\psi}_{0^+,\iota}(\psi(\iota)-\psi(0))^{q-1} = \frac{\Gamma(p)}{\Gamma(p+q)}(\psi(\iota)-\psi(0))^{p+q-1}$,
(4) $\mathfrak{I}^{q;\psi}_{0^+,\iota}(1) = \frac{(\psi(\iota)-\psi(0))^{q}}{\Gamma(q+1)}$,
(5) ${}^{H}\mathfrak{D}^{p,q;\psi}_{0^+,\iota}(\psi(\iota)-\psi(0))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta-p)}(\psi(\iota)-\psi(0))^{\delta-p-1}$,
(6) ${}^{H}\mathfrak{D}^{p,q;\psi}_{0^+,\iota}(\psi(\iota)-\psi(0))^{\delta-1} = 0, \quad 0<\delta<1$.

Lemma 2.5. *([1]) Let $0\le\gamma<1$ and $f\in C_{1-\gamma;\psi}[0,b]$. Then*

$$\mathfrak{I}^{p;\psi}_{0^+,\iota}f(0) = \lim_{\iota\to 0+}\mathfrak{I}^{p;\psi}_{0^+,\iota}f(\iota) = 0, \quad 0\le 1-\gamma<p.$$

To show the Ulam-Hyers-Rassias stability for problem (1.1), we generalise the definitions for ψ-Hilfer given by Rizwan et al in [22].

Take $\varepsilon>0, \theta>0$, $\phi\in\Omega_{\gamma;\psi}$, and considering

$$\begin{cases} \left|{}^{H}\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^{H}\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota) - f\left(\iota, w(\iota), w(\sigma(\iota)), {}^{H}\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^{H}\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)\right)\right| \le \varepsilon\phi(\iota), \quad \iota\in J,\\ \left|\Delta w(\iota_k) - I_k\left(w\left(\iota_k^-\right)\right)\right| \le \varepsilon\theta, \quad k=1,\ldots,m. \end{cases} \tag{2.4}$$

Definition 2.6. *([20]) Problem (1.1) is Ulam-Hyers-Rassias stable in terms of $(\phi(\iota),\theta)$, when a real number $c_{F,m,\phi}>0$ exists in which, for all $\varepsilon>0$ and all $v\in\Omega_{\gamma;\psi}$ solution of (2.4), there is a solution $w\in\Omega_{\gamma;\psi}$ to the problem (1.1) with*

$$\begin{cases} |v(\iota)-w(\iota)| = 0, \quad \iota\in[-r,0],\\ \left|(\psi(\iota)-\psi(0))^{1-\gamma}(v(\iota)-w(\iota))\right| \le c_{F,m,\phi}\,\varepsilon(\phi(\iota)+\theta), \quad \iota\in J. \end{cases}$$

Remark 2.7. *([20]) A continious function $v \in \Omega_{\gamma;\psi}$ is a solution of (2.4) only if $g \in \Omega_{\gamma;\psi}$ a function and g_k , $k = 1,2,\ldots,m$ a sequence (both depends on v) exist in which*
(1) $|g(\iota)| \leq \varepsilon\phi(\iota)$, $\iota \in J$, $|g_k| \leq \varepsilon\theta$, $\quad k = 1,2,\ldots,m$,
(2) ${}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota) = f\left(\iota, w(\iota), w(\sigma(\iota)), {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right) + g(\iota), \quad \iota \in J,$
(3) $\Delta w(\iota_k) = I_k\left(w\left(\iota_k^-\right)\right) + g_k, \quad k = 1,2,\ldots,m.$

Definition 2.8. *([23]) Consider the metric space $(\mathscr{E}, d)$. If there is a $w^* \in \mathscr{E}$ in which*
1. $\mathscr{F}_{\mathscr{T}} = \{w^*\}$, in which $\mathscr{F}_{\mathscr{T}} = \{w \in \mathscr{E} : \mathscr{T}(w) = w\}$.
2. $\{\mathscr{T}^n(w_0)\}_{n\in\mathbb{N}}$ converges to w^* for each $w_0 \in \mathscr{E}$.
Then the operator $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ is a Picard operator.

Lemma 2.9. *([24]) Take $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ an increasing Picard operator with $\mathscr{F}_{\mathscr{T}} = \left\{w^*_{\mathscr{T}}\right\}$, and take $(\mathscr{E}, d, \leq)$ an ordered metric space. Then, for each $w \in \mathscr{E}, w \leq \mathscr{T}(w)$ shows $w \leq w^*_{\mathscr{T}}$.*

Lemma 2.10. *([25]) Take w, v be two functions on J that are integrable. Consider $\psi \in C^1(J,\mathbb{R})$ is an increasing function in which $\psi'(\iota) \neq 0, \forall \iota \in J$. Suppose that*
(i) Both w and v are positive.
(ii) For any J, $(g_i)_{i=1,..n}$ are bounded and monotonic increasing functions.
(iii) $p_i > 0 (i = 1,2,\ldots,n)$. If

$$w(\iota) \leq v(\iota) + \sum_{i=1}^{n} g_i(\iota)\int_a^{\iota} \psi'(s)(\psi(\iota)-\psi(s))^{p_i-1}w(s)\mathrm{d}s,$$

then

$$w(\iota) \leq v(\iota) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\ldots k'=1}^{n} \frac{\prod_{i=1}^{k}(g_{i'}(\iota)\Gamma(p_{i'}))}{\Gamma\left(\sum_{i=1}^{k} p_{i'}\right)}\int_a^{\iota}\left[\psi'(s)(\psi(\iota)-\psi(s))^{\sum_{i=1}^{k} p_{i'}-1}\right]v(s)\mathrm{d}s\right).$$

Assume further that $v(\iota)$ is a nondecreasing function on J. Next, the inequality given by [25, Corollary 2.1], for $n = 2$, gives us

$$w(\iota) \leq v(\iota)\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right) + E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right],$$

where $\psi_g^{p}(\iota,0) := g(\iota)\Gamma(p)(\psi(\iota)-\psi(0))^p$, and E_p is the Mittag-Leffler function defined in [2] by

$$E_p(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(np+1)}, z \in \mathbb{C}, \mathrm{Re}(p) > 0.$$

Lemma 2.11. *For $n = 2$. Let $w \in \Omega_{\gamma;\psi}$ satisfying the following inequality*

$$w(\iota) \leq v(\iota) + \sum_{l=1}^{2} g_l(\iota)\int_0^{\iota}\psi'(\tau)(\psi(\iota)-\psi(\tau))^{p_l-1}w(s)ds + \sum_{0<\iota_k<\iota}\beta_k w(\iota_k), \quad \iota \geq 0, \tag{2.5}$$

where $\beta_k > 0, k = 1,\ldots,m$ is a nonnegative constant and $v \in \Omega_{\gamma;\psi}$ is nonnegative too. Following that

$$w(\iota) \leq v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^k\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right], \quad \iota \in (\iota_k, \iota_{k+1}], \tag{2.6}$$

where $\beta = \max\{\beta_k : k = 1,2,\ldots,m\}$.

Proof. For $n = 2$, and by lemma 2.10, we derive

$$w(\iota) \leq v(\iota)\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right], \quad \iota \in [0,\iota_1], \tag{2.7}$$

$$w(\iota) \leq \left[v(\iota) + \sum_{j=0}^{k}\beta_j w(\iota_j)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right], \quad \iota \in [\iota_k, \iota_{k+1}]. \tag{2.8}$$

By induction, for $k=0$, inequality (2.6) holds by (2.7). Suppose that for $k=j<m$, (2.6) holds. After that, using (2.8) and the nondecreasing nature of v and E_p, we obtain for $\iota \in \left(\iota_{j+1}, \iota_{j+2}\right]$,

$$
\begin{aligned}
w(\iota) &\leq \left[v(\iota)+\sum_{i=0}^{j+1} \beta_i w(\iota_i)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right] \\
&\leq \left[v(\iota)+\sum_{i=1}^{j+1} \beta_i v(\iota_i)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota_i,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota_i,0)\right)\right]\right)^{i-1}\left[E_{p_1}\left(\psi_g^{p_1}(\iota_i,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota_i,0)\right)\right]\right] \\
&\quad \times\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right] \\
&\leq \left[v(\iota)+\beta\sum_{i=1}^{j+1} v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^{i-1}\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right] \\
&\quad \times\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right],
\end{aligned}
$$

then

$$
\begin{aligned}
w(\iota) &\leq \left[v(\iota)+\beta v(\iota)\frac{1-\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^{j+1}}{1-\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)}\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right] \\
&\quad \times\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right] \\
&= \left[v(\iota)+v(\iota)\left(\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^{j+1}-1\right)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right] \\
&\leq \left[v(\iota)+v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^{j+1}-v(\iota)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right] \\
&= v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^{j+1}\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right].
\end{aligned}
$$

This finishes the proof. □

3. Formula of Solutions

Lemma 3.1. *Let $0<p_1,p_2<1, \quad 0\leq q\leq 1$, and $h: J\to\mathbb{R}$ be continuous. A function $w\in\Omega_{\gamma;\psi}$ given by*

$$
w(\iota)=\left[\frac{w_0-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h^a+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(a)}{(\psi(a)-\psi(0))^{\gamma-1}}\right](\psi(\iota)-\psi(0))^{\gamma-1}+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota) \tag{3.1}
$$

is the unique solution for the problem that follows

$$
\begin{cases}
{}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=h(\iota), & \iota\in J,\\
w(a)=w_0, \qquad a>0,
\end{cases} \tag{3.2}
$$

in which $\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h^a=\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h(\iota)\Big|_{\iota=a}$.

Proof. Taking the fractional ψ-integral operator of order p_1+p_2 on each side of (3.2). Then utilizing Lemma 2.3, we arrive at

$$
w(\iota)-e_1(\psi(\iota)-\psi(0))^{\gamma-1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}\left(w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathfrak{I}_{0^+,\iota}^{1-\gamma_1;\psi}w(0)\right)=\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h(\iota), \tag{3.3}
$$

where $\gamma_1=q+p_1(1-q)$, and e_1 is an arbitrary constant.
Since $1-\gamma<1-\gamma_1$, lemma 2.5 implies that $\mathfrak{I}_{0^+,\iota}^{1-\gamma_1;\psi}w(0)=0$.
Hence (3.3) reduces to

$$
w(\iota)=e_1(\psi(\iota)-\psi(0))^{\gamma-1}+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota). \tag{3.4}
$$

In (3.4), the boundary condition $w(a)=w_0$ leads to $e_1=\frac{w_0-\mathfrak{I}^{p_1+p_2;\psi}_{0^+,\iota}h^a+\lambda\mathfrak{I}^{p_2;\psi}_{0^+,\iota}w(a)}{(\psi(a)-\psi(0))^{\gamma-1}}$. We substitute e_1 in (3.4), we obtain (3.1).

On the other hand, suppose w can be the unique solution satisfying (3.1), taking the fractional ψ-Hilfer derivative ${}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}$ on either side of (3.1), and using lemma 2.4, we can obtain

$$ {}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}w(\iota)=\mathfrak{I}^{p_1;\psi}_{0^+,\iota}h(\iota)-\lambda w(\iota), $$

then taking fractional ${}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}$ again, it follows

$$ {}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=h(\iota). $$

Hence, the proof is complete.

□

We obtain the following result from 3.1, which is useful in what follows.

Lemma 3.2. *A function $w\in\Omega_{\gamma;\psi}$ has a solution of (1.1) if and only if $w\in\Omega_{\gamma;\psi}$ is a solution of the given fractional integral equation*

$$ w(\iota)=\begin{cases}\varphi(\iota), & \iota\in[-r,0],\\ \left[\frac{w_0}{\Gamma(\gamma)}+\sum_{i=1}^m\frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}^\gamma_\psi(\iota_i,0)}\right]\mathfrak{R}^\gamma_\psi(\iota,0)+\mathfrak{I}^{p_1+p_2;\psi}_{0^+,\iota}F_w(\iota)-\lambda\mathfrak{I}^{p_2;\psi}_{0^+,\iota}w(\iota), & \iota\in J,\end{cases} \tag{3.5} $$

in which $F_w(\iota)=f(\iota,w(\iota),w(\sigma(\iota)),F_w(\iota))$, and $\mathfrak{R}^\gamma_\psi(\iota,0)=(\psi(\iota)-\psi(0))^{\gamma-1}$.

Proof. Assume that w satisfies (1.1), then w satisfies

$$ {}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=f\left(\iota,w(\iota),w(\sigma(\iota)),{}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)\right). $$

Take ${}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=F_w(\iota)$. It follows that $F_w(\iota)=f(\iota,w(\iota),w(\sigma(\iota)),F_w(\iota))$. Then, we have

$$ {}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=F_w(\iota). $$

If $\iota\in[-r,0]$, clearly that $w(\iota)=\varphi(\iota)$. For $\iota\in[0,\iota_1]$, ${}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=F_w(\iota)$ can be writen as

$$ {}^H\mathfrak{D}^{p_1+p_2,q;\psi}_{0^+,\iota}w(\iota)+\lambda\,{}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}w(\iota)=F_w(\iota). \tag{3.6} $$

Taking the fractional ψ-integral operator of order p_1+p_2 on each side of (3.6). Then utilizing Lemma 2.3, we arrive at

$$ w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}^{1-\gamma;\psi}_{0^+,\iota}w(0)+\lambda\mathfrak{I}^{p_2;\psi}_{0^+,\iota}\left(\mathfrak{I}^{p_1;\psi}_{0^+,\iota}{}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}w(\iota)\right)=\mathfrak{I}^{p_1+p_2;\psi}_{0^+,\iota}F_w(\iota). \tag{3.7} $$

Utilizing again Lemma 2.3, we can get

$$ w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}^{1-\gamma;\psi}_{0^+,\iota}w(0)+\lambda\mathfrak{I}^{p_2;\psi}_{0^+,\iota}\left(w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathfrak{I}^{1-\gamma_1;\psi}_{0^+,\iota}w(0)\right)=\mathfrak{I}^{p_1+p_2;\psi}_{0^+,\iota}F_w(\iota), \tag{3.8} $$

where $\gamma_1=q+p_1(1-q)$.
Since $1-\gamma<1-\gamma_1$, lemma 2.5 implies that $\mathfrak{I}^{1-\gamma_1;\psi}_{0^+,\iota}w(0)=0$.
Hence by $\mathfrak{I}^{1-\gamma;\psi}_{0^+,\iota}w(0)=w_0$ and $\mathfrak{R}^\gamma_\psi(\iota,0)=(\psi(\iota)-\psi(0))^{\gamma-1}$, equation (3.8) reduces to

$$ w(\iota)=\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}^\gamma_\psi(\iota,0)+\mathfrak{I}^{p_1+p_2;\psi}_{0^+,\iota}F_w(\iota)-\lambda\mathfrak{I}^{p_2;\psi}_{0^+,\iota}w(\iota). \tag{3.9} $$

If $\iota\in[\iota_1,\iota_2]$ then ${}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=F_w(\iota)$ with $w\left(\iota_1^+\right)=w\left(\iota_1^-\right)+I_1\left(w\left(\iota_1^-\right)\right)$ By lemma 3.1, one obtain

$$w(\iota)=\left[\frac{w\left(\iota_1^+\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w\left(\iota_1^-\right)+I_1\left(w\left(\iota_1^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota_1,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)+I_1\left(w\left(\iota_1^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)$$
$$-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota_1,0)+I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w_0}{\Gamma(\gamma)}+\frac{I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota).$$

If $\iota\in[\iota_2,\iota_3]$ then again by lemma 3.1

$$w(\iota)=\left[\frac{w\left(\iota_2^+\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w\left(\iota_2^-\right)+I_2\left(w\left(\iota_2^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{\left[\frac{w_0}{\Gamma(\gamma)}+\frac{I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota_2,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)+I_2\left(w\left(\iota_2^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)$$
$$+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w_0}{\Gamma(\gamma)}+\frac{I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}+\frac{I_2\left(w\left(\iota_2^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota).$$

Repeating the same fashion in this way for $\iota\in[\iota_k,\iota_{k+1}]$, we get
$w(\iota)=\left[\frac{w_0}{\Gamma(\gamma)}+\sum_{i=1}^k\frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota),\quad k=1,2,\ldots,m.$

In contrast, Suppose w can be the unique solution satisfying (3.5). If $\iota\in[-r,0]$, clearly that $w(\iota)=\varphi(\iota)$. If $\iota\in[0,\iota_1]$, taking the fractional ψ-Hilfer derivative ${}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}$ on either side of

$$w(\iota)=\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota), \tag{3.10}$$

using lemma 2.4, we can obtain

$${}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}w(\iota)=\mathfrak{I}_{0^+,\iota}^{p_1;\psi}F_w(\iota)-\lambda w(\iota).$$

Then taking fractional ${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}$ again, it follows

$${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota).$$

Now we show that the initial condition $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=w_0$ also holds. We apply $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}$ on both sides of (3.10), then lemma 2.4 implies that

$$\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(\iota)=w_0+\mathfrak{I}_{0^+,\iota}^{1-\gamma+p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{1-\gamma+p_2;\psi}w(\iota).$$

Since $1-\gamma<1-\gamma+p_1+p_2$ and $1-\gamma<1-\gamma+p_2$, lemma 2.5 implies that

$$\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=w_0.$$

If $\iota\in[\iota_k,\iota_{k+1}],\quad k=1,2,\ldots,m$. Using again lemma 2.4, we obtain

$${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota) \text{ and } w\left(\iota_k^+\right)-w\left(\iota_k^-\right)=I_k\left(w\left(\iota_k^-\right)\right).$$

Hence, the proof is complete.

□

4. Existence, Uniqueness and Stability

We present the following hypothesis in order to show the existence, uniqueness, and stability of the solution.

H1: $f:J\times\mathbb{R}^3\to\mathbb{R}$ is continuous and $\mathscr{L}>0$, $0<\mathscr{L}_f<1$ are constants satisfy :

$$|f(\iota,w_1,w_2,w_3)-f(\iota,v_1,v_2,v_3)|\leq\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}\{|w_1-v_1|+|w_2-v_2|\}+\mathscr{L}_f|w_3-v_3|,$$

$\iota\in J$ and $w_1,v_1,w_2,v_2,w_3,v_3\in\mathbb{R}$.

H2: $I_i:\mathbb{R}\to\mathbb{R},(i=1,\ldots,m)$ satisfy :

$$\left|I_i\left(w\left(\iota_i^-\right)\right)-I_i\left(v\left(\iota_i^-\right)\right)\right|\leq\mathscr{L}_{I_i}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right|,$$

where $w,v\in\Omega_{\gamma;\psi}$ and $\mathscr{L}_{I_i}>0$.

H3: The inequality

$$\mathscr{K}:=m\mathscr{L}_I+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}+\frac{|\lambda|\Gamma(\gamma)(\psi(b)-\psi(0))^{p_2}}{\Gamma(p_2+\gamma)}<1,$$

where $\mathscr{L}_I=\max\{\mathscr{L}_{I_i}:i=1,2,\ldots,m\}$.

H4: A non-decreasing function ϕ, bounded in J, and a constant $\lambda_\phi>0$ exist in which, for each $\iota\in J$,

$$\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}\phi(\iota)\leq\lambda_\phi\phi(\iota).$$

Theorem 4.1. *Suppose that H1–H4 are true. Then*
(i). There is a unique solution to problem (1.1) in the space $\Omega_{\gamma;\psi}$.
(ii). Problem (1.1) is Ulam-Hyers-Rassias stable.

Proof. Part 1: In this part we will prove the existence as well as the uniqueness of solutions to problem (1.1). Considering Lemma 3.2, we set the operator $\mathscr{A}:\Omega_{\gamma;\psi}\longrightarrow\Omega_{\gamma;\psi}$

$$(\mathscr{A}w)(\iota)==\begin{cases}\varphi(\iota), & \iota\in[-r,0],\\ \left[\frac{w_0}{\Gamma(\gamma)}+\sum_{i=1}^m\frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota), & \iota\in J,\end{cases} \tag{4.1}$$

where $F_w(\iota)=f(\iota,w(\iota),w(\sigma(\iota)),F_w(\iota))$.

As we can see, the solution to problem (1.1) will be the fixed point of $\mathscr{A}$.

We demonstrate that on $\Omega_{\gamma;\psi}$, $\mathscr{A}$ is a contraction map. Let, $w,v\in\Omega_{\gamma;\psi}$. Then for $\iota\in[-r,0]$, we have:

$$\|\mathscr{A}w-\mathscr{A}v\|_C=0. \tag{4.2}$$

Further, for any $\iota\in J$, we have

$$\begin{aligned}|(\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota)|\leq&\mathfrak{R}_\psi^\gamma(\iota,0)\left[\sum_{i=1}^m\frac{\left|I_i\left(w\left(\iota_i^-\right)\right)-I_i\left(v\left(\iota_i^-\right)\right)\right|}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]+\frac{1}{\Gamma(p_1+p_2)}\int_0^\iota\mathscr{N}_\psi^{p_1+p_2}(\iota,s)|F_w(s)-F_v(s)|ds\\&+\frac{|\lambda|}{\Gamma(p_2)}\int_0^\iota\mathscr{N}_\psi^{p_2}(\iota,s)|w(s)-v(s)|ds,\end{aligned}$$

where $\mathscr{N}_\psi^p(\iota,s)=\psi'(s)(\psi(\iota)-\psi(s))^p, \quad p=p_2,p_1+p_2.$

Using (H1), (H2), and

$$|F_w(\iota)-F_v(\iota)|\leq \mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}\{|w(\iota)-v(\iota)|+|w(\sigma(\iota))-v(\sigma(\iota))|\}+\mathscr{L}_f\,|F_w(\iota)-F_v(\iota)|.$$

It follows that

$$|F_w(\iota)-F_v(\iota)|\leq \frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{1-\mathscr{L}_f}\{|w(\iota)-v(\iota)|+|w(\sigma(\iota))-v(\sigma(\iota))|\}.$$

Therefore, we have

$$\begin{aligned}
&|(\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota)|\\
&\leq \Re_\psi^\gamma(\iota,0)\left[\sum_{i=1}^m \mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right|\right]\\
&+\frac{\mathscr{L}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma}\times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds\\
&+\frac{|\lambda|}{\Gamma(p_2)}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)\,|w(s)-v(s)|\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
|(\psi(\iota)-\psi(0))^{1-\gamma}((\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota))|\leq&\sum_{i=1}^m \mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right|\\
&+\frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma}\\
&\times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds\\
&+\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\\
&\times(\psi(s)-\psi(0))^{1-\gamma}\,|w(s)-v(s)|\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
&|(\psi(\iota)-\psi(0))^{1-\gamma}((\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota))|\\
&\leq m\mathscr{L}_I\|w-v\|_{PC_{1-\gamma;\psi}}\\
&+\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\|w-v\|_{PC_{1-\gamma;\psi}}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)ds\\
&+\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\|w-v\|_{PC_{1-\gamma;\psi}}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds\\
&\leq m\mathscr{L}_I\|w-v\|_{PC_{1-\gamma;\psi}}+\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}\|w-v\|_{PC_{1-\gamma;\psi}}\times(\psi(\iota)-\psi(0))^{p_1+p_2}\\
&+\frac{\Gamma(\gamma)|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2+\gamma)}\|w-v\|_{PC_{1-\gamma;\psi}}\times(\psi(\iota)-\psi(0))^{p_2+\gamma-1}\\
&\leq\left[m\mathscr{L}_I+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}+\frac{\Gamma(\gamma)|\lambda|(\psi(b)-\psi(0))^{p_2}}{\Gamma(p_2+\gamma)}\right]\|w-v\|_{PC_{1-\gamma;\psi}}.
\end{aligned}$$

Therefore,

$$\|\mathscr{A}w-\mathscr{A}v\|_{PC_{1-\gamma;\psi}}=\sup_{\iota\in J}\left|(\psi(\iota)-\psi(0))^{1-\gamma}((\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota))\right|\leq\mathscr{K}\,\|w-v\|_{PC_{1-\gamma;\psi}}. \tag{4.3}$$

From (4.2) and (4.3), we have

$$\begin{aligned}
\|\mathscr{A}w-\mathscr{A}v\|_{\Omega_{\gamma;\psi}}&=\max\left\{\|\mathscr{A}w-\mathscr{A}v\|_C,\|\mathscr{A}w-\mathscr{A}v\|_{PC_{1-\gamma;\psi}}\right\}\\
&\leq\mathscr{K}\max\left\{0,\|w-v\|_{PC_{1-\gamma;\psi}}\right\}\\
&\leq\mathscr{K}\,\|w-v\|_{\Omega_{\gamma;\psi}}.
\end{aligned}$$

As $\mathscr{K} < 1$, Banach's fixed-point theorem shows that the operator $\mathscr{A}$ has a fixed point, which is the unique solution to problem (1.1).

Part 2: Now, let us discuss the Ulam-Hyers-Rassias stability.
Take $v \in \Omega_{\gamma;\psi}$ as the solution to (2.4) and $w \in \Omega_{\gamma;\psi}$ as the unique solution to the problem:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota) = f\left(\iota, w(\iota), w(\sigma(\iota)), {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right), \quad \iota \in (0,b]\setminus\{\iota_1,\iota_2,\ldots,\iota_m\}, \\ \Delta w(\iota_i) = w\left(\iota_i^+\right) - w\left(\iota_i^-\right) = I_i\left(w\left(\iota_i^-\right)\right), \quad i=1,2,\ldots,m, \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0) = w_0, \quad \gamma = q+(p_1+p_2)(1-q), \\ w(\iota) = \varphi(\iota), \quad \iota \in [-r,0]. \end{cases} \tag{4.4}$$

According to Lemma 3.2, we have

$$w(\iota) = \begin{cases} \varphi(\iota), & \iota \in [-r,0], \\ \left[\frac{w_0}{\Gamma(\gamma)} + \sum_{i=1}^{m}\frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}_{\psi}^{\gamma}(\iota_i,0)}\right]\mathfrak{R}_{\psi}^{\gamma}(\iota,0) + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota) - \lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota), & \iota \in J. \end{cases} \tag{4.5}$$

By assuming that v is a solution of (2.4). Hence, based on Remark 2.7, the solution of

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)v(\iota) = f\left(\iota, v(\iota), v(\sigma(\iota)), {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right) + g(\iota), \quad \iota \in (0,b]\setminus\{\iota_1,\iota_2,\ldots,\iota_m\}, \\ \Delta v(\iota_i) = I_i\left(v\left(\iota_i^-\right)\right) + g_i, \quad i=1,2,\ldots,m, \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}v(0) = w_0, \quad \gamma = q+(p_1+p_2)(1-q), \\ v(\iota) = \varphi(\iota), \quad \iota \in [-r,0]. \end{cases}$$

It can be formulated as follows:

$$v(\iota) = \begin{cases} \varphi(\iota), & \iota \in [-r,0], \\ \left[\frac{w_0}{\Gamma(\gamma)} + \sum_{i=1}^{m}\frac{I_i\left(v\left(\iota_i^-\right)\right)+g_i}{\mathfrak{R}_{\psi}^{\gamma}(\iota_i,0)}\right]\mathfrak{R}_{\psi}^{\gamma}(\iota,0) + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_v(\iota) - \lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}v(\iota) + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}g(\iota), & \iota \in J, \end{cases} \tag{4.6}$$

where, $F_v(\iota) = f(\iota, v(\iota), v(\sigma(\iota)), F_v(\iota))$.

Now let, $w, v \in \Omega_{\gamma;\psi}$. Then for $\iota \in [-r,0]$, we have:

$$\|w-v\|_C = 0. \tag{4.7}$$

Further, for any $\iota \in J$, we have

$$\begin{aligned} |w(\iota)-v(\iota)| \le & \mathfrak{R}_{\psi}^{\gamma}(\iota,0)\left[\sum_{i=1}^{m}\frac{\left|I_i\left(w\left(\iota_i^-\right)\right) - I_i\left(v\left(\iota_i^-\right)\right)\right|}{\mathfrak{R}_{\psi}^{\gamma}(\iota_i,0)}\right] + \mathfrak{R}_{\psi}^{\gamma}(\iota,0)\sum_{i=1}^{m}\frac{|g_i|}{\mathfrak{R}_{\psi}^{\gamma}(\iota_i,0)} + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}|g(\iota)| \\ & + \frac{1}{\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\left|F_w(s)-F_v(s)\right|ds + \frac{|\lambda|}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)\left|w(s)-v(s)\right|ds. \end{aligned}$$

Using (H1), (H2), and remark 2.7, we've obtained

$$\begin{aligned} |(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))| \le & \sum_{i=1}^{m}\mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right| + \sum_{i=1}^{m}\varepsilon\theta(\psi(\iota_i)-\psi(0))^{1-\gamma} \\ & + \frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma} \\ & \times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds \\ & + \frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1} \\ & \times(\psi(s)-\psi(0))^{1-\gamma}|w(s)-v(s)|\,ds + (\psi(\iota)-\psi(0))^{1-\gamma}\varepsilon\lambda_{\phi}\phi(\iota). \end{aligned}$$

Then, if $M = \max\left\{m(\psi(b)-\psi(0))^{1-\gamma}, \lambda_\phi(\psi(b)-\psi(0))^{1-\gamma}\right\}$, we get

$$\begin{aligned}|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))| \leq & M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{i=1}^{m}\mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right| \\ & +\frac{\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma} \\ & \times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds \\ & +\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1} \\ & \times(\psi(s)-\psi(0))^{1-\gamma}|w(s)-v(s)|\,ds.\end{aligned} \tag{4.8}$$

And now for every $z\in C([-r,b],\mathbb{R}_+)$, we define $\mathscr{T}: C([-r,b],\mathbb{R}_+)\to C([-r,b],\mathbb{R}_+)$ as

$$(\mathscr{T}z)(\iota)=\begin{cases}0, & \iota\in[-r,0],\\ M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{i=1}^{m}\mathscr{L}_{I_i}\left(z(\iota_i)\right) & \\ +\frac{\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{(1-\mathscr{L}_f)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\left(z(s)+z(\sigma(s))\right)ds & \\ +\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}z(s)ds, & \iota\in J.\end{cases} \tag{4.9}$$

We show that $\mathscr{T}$ is a Picard operator. Let $z,w\in C([-r,b],\mathbb{R}_+)$. Then,

$$\|\mathscr{T}z-\mathscr{T}w\|_C=0. \tag{4.10}$$

Further, for any $\iota\in J$, we have

$$\begin{aligned}|(\mathscr{T}z)(\iota)-(\mathscr{T}w)(\iota)| \leq & \sum_{i=1}^{m}\mathscr{L}_{I_i}\left|z\left(\iota_i^-\right)-w\left(\iota_i^-\right)\right| \\ & +\frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\times\{|z(s)-w(s)|+|z(\sigma(s))-w(\sigma(s))|\}\,ds \\ & +\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times|z(s)-w(s)|\,ds.\end{aligned}$$

Then

$$\begin{aligned}|(\mathscr{T}z)(\iota)-(\mathscr{T}w)(\iota)| \leq & m\mathscr{L}_I\|z-w\|_c \\ & +\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\|z-w\|_c\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)ds \\ & +\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\|z-w\|_c\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds \\ \leq & m\mathscr{L}_I\|z-w\|_c+\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}\|z-w\|_c\times(\psi(\iota)-\psi(0))^{p_1+p_2} \\ & +\frac{\Gamma(\gamma)|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2+\gamma)}\|z-w\|_c\times(\psi(\iota)-\psi(0))^{p_2+\gamma-1} \\ \leq & \left[m\mathscr{L}_I+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}+\frac{\Gamma(\gamma)|\lambda|(\psi(b)-\psi(0))^{p_2}}{\Gamma(p_2+\gamma)}\right]\|z-w\|_C.\end{aligned}$$

Therefore,

$$\|\mathscr{T}z-\mathscr{T}w\|_C=\sup_{\iota\in J}|(\mathscr{T}z)(\iota)-(\mathscr{T}w)(\iota)|\leq\mathscr{K}\|z-w\|_C.$$

By $\mathscr{K} < 1$, the operator $\mathscr{T}$ is a contraction mapping. According to [26, Theorem 2.1], we obtain that $\mathscr{T}$ is Picard operator and $\mathscr{F}_{\mathscr{T}} = z^*$. Then for all $\iota \in [-r, b]$,

$$\begin{aligned} z^*(\iota) =& M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{0<\iota_i<\iota}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\times\left(z^*(s)+z^*(\sigma(s))\right)ds \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds. \end{aligned} \tag{4.11}$$

Next, we prove that z^* is increasing. Take $\iota_1, \iota_2 \in [-r, b]$ with $\iota_1 < \iota_2$. Then for $\iota_1, \iota_2 \in [-r, 0]$, we have $z^*(\iota_2) - z^*(\iota_1) = 0$. Further, for $0 < \iota_1 < \iota_2 \le b$. Define $N_1 = \min_{s\in[0,b]}\left(z^*(s) + z^*(\sigma(s))\right)$ and $N_2 = \min_{s\in[0,b]} z^*(s)$, we have

$$\begin{aligned} z^*(\iota_2)-z^*(\iota_1) =& M\varepsilon\left(\theta+\phi(\iota_2)\right)-M\varepsilon\left(\theta+\phi(\iota_1)\right)+\sum_{0<\iota_i<\iota_2}\mathscr{L}_{I_i}z^*(\iota_i)-\sum_{0<\iota_i<\iota_1}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\left(z^*(s)+z^*(\sigma(s))\right)ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\left(z^*(s)+z^*(\sigma(s))\right)ds\right) \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds\right) \\ \geq& M\varepsilon\left(\phi(\iota_2)-\phi(\iota_1)\right)+\sum_{0<\iota_i<\iota_2-\iota_1}\mathscr{L}_{I_i}z^*(\iota_i)+\frac{N_1\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)ds\right) \\ &+\frac{N_2|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds\right) \\ \geq& \sum_{0<\iota_i<\iota_2-\iota_1}\mathscr{L}_{I_i}z^*(\iota_i)+\frac{N_1\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}\left((\psi(\iota_2)-\psi(0))^{p_1+p_2}-(\psi(\iota_1)-\psi(0))^{p_1+p_2}\right) \\ &+\frac{N_2|\Gamma(\gamma)\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2+\gamma)}\left((\psi(\iota_2)-\psi(0))^{p_2+\gamma-1}-(\psi(\iota_1)-\psi(0))^{p_2+\gamma-1}\right) \\ >& 0. \end{aligned}$$

Therefore, The operator z^* is increasing. Since $\sigma(\iota) \le \iota, z^*(\sigma(\iota)) \le z^*(\iota), \iota \in J$. By (4.11), we get

$$\begin{aligned} z^*(\iota) \leq& M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{0<\iota_i<\iota}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)z^*(s)ds \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds. \end{aligned} \tag{4.12}$$

As $0<\gamma<1$, then $(\psi(s)-\psi(0))^{\gamma-1}<1$. So, (4.12) reduce to

$$\begin{aligned} z^*(\iota) \leq & M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{0<\iota_i<\iota}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)z^*(s)ds \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)z^*(s)ds. \end{aligned} \tag{4.13}$$

Using lemma 2.11, with

$$\begin{aligned} &w(\iota)=z^*(\iota), \quad v(\iota)=M\varepsilon\left(\theta+\phi(\iota)\right) \\ &g_1(\iota)=\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}, \quad g_2(\iota)=\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}, \quad \beta=\mathscr{L}_I, \end{aligned}$$

we obtain

$$\begin{aligned} z^*(\iota) &\leq M\varepsilon\left(\theta+\phi(\iota)\right) \\ &\times\left(1+\mathscr{L}_I\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)}(\psi(\iota)-\psi(0))^{p_1+p_2}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma}(\psi(\iota)-\psi(0))^{p_2}\right)\right]\right)^k \\ &\times\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)}(\psi(\iota)-\psi(0))^{p_1+p_2}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma}(\psi(\iota)-\psi(0))^{p_2}\right)\right] \\ &\leq M\varepsilon\left(\theta+\phi(\iota)\right)\times\left(1+\mathscr{L}_I\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right]\right)^k \\ &\times\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right] \\ &\leq c_{f,m,\phi}\varepsilon\left(\theta+\phi(\iota)\right), \quad \iota\in J, \end{aligned}$$

where

$$\begin{aligned} c_{f,m,\phi} =& M\left(1+\mathscr{L}_I\left[E_{p_1+p_2,\psi}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2,\psi}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right]\right)^k \\ &\times\left[E_{p_1+p_2,\psi}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2,\psi}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right]. \end{aligned}$$

Specifically, when $z(\iota)=(\psi(\iota)-\psi(0))^{1-\gamma}|w(\iota)-v(\iota)|$, using (4.8), we obtain $z\leq\mathscr{T}(z)$, where the Picard operator $\mathscr{T}$ is increasing. Next, applying Lemma 2.9, we get to $z\leq z^*$.Therefore, it follows

$$\left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right|\leq c_{f,m,\phi}\varepsilon(\phi(\iota)+\theta), \quad \iota\in J. \tag{4.14}$$

Thus, problem (1.1) is Ulam-Hyers-Rassias stable. □

Remark 4.2. *As a consequences of theorem 4.1, we can obtain the Ulam–Hyers stability (U-H). While ϕ is increasing function for any $\iota\in J$, the inequality (4.14) reduce to*

$$\left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right|\leq c_{f,m,\phi}\varepsilon(\phi(b)+\theta), \quad \iota\in J.$$

Therefore

$$\left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right|\leq c_f\varepsilon, \quad \iota\in J,$$

where $c_f=c_{f,m,\phi}(\phi(b)+\theta)$, and problem (1.1) is Ulam-Hyers stable.

5. Example

Example 5.1. *Taking the following problem:*

$$\begin{cases} {}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)=\dfrac{(\psi(\iota)-\psi(0))^2\left(1+|w(\iota)|+|w(\iota-\frac{1}{2})|\right)}{30e^{(\psi(\iota)-\psi(0))^2+2}(|w(\iota)|+|w(\iota-\frac{1}{2})|)} \\ +\dfrac{1}{43\left(1+\left|{}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)\right|\right)}, \quad \iota\in(0,1]\backslash\left\{\dfrac{1}{2}\right\}, \\ I_1\left(w\left(\dfrac{1}{2}^-\right)\right)=\dfrac{1+|w\left(\frac{1}{2}^-\right)|}{11|w\left(\frac{1}{2}^-\right)|}, \\ \mathfrak{I}^{1-\gamma;\psi}_{0^+,\iota}w(0)=1, \quad \gamma=\beta+(p_1+p_2)(1-q), \\ w(\iota)=0, \quad \iota\in[-1,0]. \end{cases} \tag{5.1}$$

Define $f:(0,1]\times\mathbb{R}^3\to\mathbb{R}$ *by*

$$f(\iota,w,v,u)=\frac{(\psi(\iota)-\psi(0))^2(1+|w|+|v|)}{30e^{(\psi(\iota)-\psi(0))^2+2}(|w|+|v|)}+\frac{1}{43(1+|u|)},$$

and $I_1:\mathbb{R}\to\mathbb{R}$ *by*

$$I_1(u)=\frac{1+|u|}{11|u|}.$$

For $\iota\in(0,1]$, *we have*

$$\begin{aligned} &|f(\iota,w_1,v_1,u_1)-f(\iota,w_2,v_2,u_2)| \\ &\le \frac{(\psi(\iota)-\psi(0))^2}{30e^{(\psi(\iota)-\psi(0))^2+2}}\left|\frac{1+|w_1|+|v_1|}{|w_1|+|v_1|}-\frac{1+|w_2|+|v_2|}{|w_2|+|v_2|}\right|+\frac{1}{43}\left|\frac{1}{1+|u_1|}-\frac{1}{1+|u_2|}\right| \\ &\le \frac{(\psi(\iota)-\psi(0))^2}{30e^2}(|w_1-w_2|+|v_1-v_2|)+\frac{1}{43}|u_1-u_2| \\ &\le \frac{(\psi(\iota)-\psi(0))^{\gamma+1}}{30e^2}(\psi(\iota)-\psi(0))^{1-\gamma}(|w_1-w_2|+|v_1-v_2|)+\frac{1}{43}|u_1-u_2| \\ &\le \frac{(\psi(1)-\psi(0))^{\gamma+1}}{30e^2}(\psi(\iota)-\psi(0))^{1-\gamma}(|w_1-w_2|+|v_1-v_2|)+\frac{1}{43}|u_1-u_2|. \end{aligned}$$

This implies that f *satisfies* (H_1) *with* $\mathscr{L}=\frac{(\psi(1)-\psi(0))^{\gamma+1}}{30e^2}$, *and* $\mathscr{L}_f=\frac{1}{43}$.
Also,

$$|I_1(w)-I_1(v)|=\frac{1}{11}\left(\left|\frac{1+|w|}{|w|}-\frac{1+|v|}{|v|}\right|\right)\le\frac{1}{11}|w-v|.$$

Therfore, (H_2) *is satisfied with* $\mathscr{L}_{I_1}=\mathscr{L}_I=\frac{1}{11}$.
Now, take $p_1=\frac{1}{2}, p_2=\frac{1}{4}, q=1, \lambda=\frac{1}{2}$ *and* $\psi(\iota)=\iota^2$. *Then* $\gamma=1$, *and* $\mathscr{L}=\frac{1}{30e^2}$.
As $m=1$, *we have* $\mathscr{K}:=\frac{1}{11}+\frac{2\times\frac{1}{30e^2}}{(1-\frac{1}{43})\Gamma(\frac{3}{4}+1)}+\frac{\frac{1}{2}}{\Gamma(\frac{1}{4}+1)}=0,652593<1$ *and* (H_3) *is satisfied.*

Furthermore, by selecting $\phi(\iota)=\iota^2$, *for any* $\iota\in(0,1]$, *we have*

$$I^{p_1+p_2;\psi}_{0+,\iota}\phi(\iota)=I^{\frac{3}{4};\iota^2}_{0+,\iota}\phi(\iota)=\frac{16}{21\Gamma(\frac{3}{4})}\iota^{\frac{7}{2}}=\frac{16}{21\Gamma(\frac{3}{4})}\iota^{\frac{3}{2}}\phi(\iota)\le\frac{16}{21\Gamma(\frac{3}{4})}\phi(\iota).$$

By setting $\lambda_\phi=\frac{16}{21\Gamma(\frac{3}{4})}$, *we get* (H_4). *So all conditions of theorem 4.1 are satisfied. Hence, (5.1) has a unique solution and is Ulam-Hyers-Rassias stable.*

6. Conclusion

During this paper, we have examined the existence and uniqueness of a class of fractional implicit ψ-Hilfer Langevin equations with time delay and impulsive. The obtained results are proven using Banach's fixed-point theorem. Additionally, the Ulam-Heyrs-Rassias stability for problem (1.1) is considered via a novel form of generalized Gronwall inequality and Picard operator theory. Finally, we provide an example to show how the theoretical findings stated previously are valid.

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References

[1] J. V. D. C. Sousa, E. C. Capelas de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul., **60** (2018), 72–91.

[2] A. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science Limited, **204** (2006), 7–10.

[3] I. Podlubny, *Fractional Differential equation*, Academic Press, San Diego, 1999.

[4] C. Beck, G. Roepstorff, *From dynamical systems to the Langevin equation*, Phys. A, **145**(1-2) (1987), 1-14.

[5] B. Ahmad, J. J. Nieto, A. Alsaedi, M. El-Shahed, *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Real World Appl., **13**(2) (2012), 599–606.

[6] S. Harikrishnan, K. Kanagarajan, E. M. Elsayed, *Existence and stability results for Langevin equations with Hilfer fractional derivative*, Res. Fixed Point Theory Appl., **20183** (2018).

[7] K. Hilal, A. Kajouni, H. Lmou, *Boundary value problem for the Langevin equation and inclusion with the Hilfer fractional derivative*, Int. J. Differ. Equ., **2022** (2022) 1–12.

[8] M. Aydin, N. I. Mahmudov, *ψ-Caputo type time-delay Langevin equations with two general fractional orders*, Math. Methods Appl. Sci., **46**(8) (2023), 9187-9204.

[9] I. T. Huseynov, N. I. Mahmudov, *A class of Langevin time-delay differential equations with general fractional orders and their applications to vibration theory*, J. King Saud Univ. Sci., **33**(8) (2021), 101596.

[10] M. Aydin, *Langevin delayed equations with Prabhakar derivatives involving two generalized fractional distinct orders*, Turkish J. Math., **48** (2024), 144-162

[11] N. I. Mahmudov, A. Ahmadova, I. T. Huseynov, *A novel technique for solving Sobolev type fractional multi-order evolution equations*, Comput. Appl. Math., **41**(2) (2022), 1–35.

[12] M. Aydin, N. I. Mahmudov, *Some applications of the generalized Laplace transform and the representation of a solution to Sobolev-type evolution equations with the generalized Caputo derivative*, Bull. Polish Acad. Sci. Tech. Sci., **72**(2) (2024).

[13] M. Aydin, N. I. Mahmudov, *The Sequential Conformable Langevin-Type Differential Equations and Their Applications to the RLC Electric Circuit Problems*, J. Appl. Math., **2024** (2024), 1–14.

[14] Z. Shi, Y. Li, H. Cheng, *Dynamic analysis of a pest management smith model with impulsive state feedback control and continuous delay*, Mathematics, **7**(7) (2019), 591.

[15] U. Forys, J. Poleszczuk, T. Liu, *Logistic tumor growth with delay and impulsive treatment*, Math. Population Stud., **21** (2014), 146–158.

[16] S. M. Ulam, *A collection of mathematical problems*, Interscience Publishers, 1960.

[17] T. M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.

[18] R. Rizwan, A. Zada, H. Waheed, U. Riaz, *Switched coupled system of nonlinear impulsive Langevin equations involving Hilfer fractional-order derivatives*, Int. J. Nonlin. Sci. Num., **24**(6) (2023), 2405–2423.

[19] M. S. Abdo, S. K. Panchal, H.A. Wahash, *Ulam–Hyers–Mittag-Leffler stability for a ψ-Hilfer problem with fractional order and infinite delay*, Results Appl. Math., **7**(100) (2020), 115.

[20] K. B. Lima, J. V. D. C. Sousa, E. C. Capelas de Oliveira, *Ulam–Hyers type stability for ψ-Hilfer fractional differential equations with impulses and delay*, Comput. Appl. Math., **40**(293) (2021).

[21] J. V. C. Sousa, E.C. Capelas de Oliveira, *A Gronwall inequality and the Cauchy type problem by means of Hilfer operator*, Diff. Equ. and Appl., **11**(1) (2019) 87-106.

[22] R. Rizwan, J. R. Lee, C. Park, A. Zada, *Existence, uniqueness and Ulam's stabilities for a class of impulsive Langevin equation with Hilfer fractional derivatives*, AIMS Mathematics, **7**(4) (2022), 6204–6217

[23] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, 2014.

[24] I. A. Rus, *Gronwall lemmas: Ten open problems*, Sci. Math. Jpn., **70** (2019), 221-228.

[25] J. Alzabut, Y. Adjabi, W. Sudsutad, M. ur Rehman, *New generalizations for Gronwall type inequalities involving a ψ-fractional operator and their applications*, AIMS Math., **6** (2021), 5053–5077.

[26] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **94** (2012). https://doi.org/10.1186/1687-1812-2012-94.