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On Matrix Representations of Homeomorphism Classes

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Abstract: In this study, we investigate the matrix representations of homeomorphism classes. Considering well-known concepts such as the matrix of ones, column-sum, row-sum, one's complement, Hadamard product, and regular addition for matrices, we explore binary matrices' relationships with subsets of a finite set. The main results establish connections between matrix operations and set operations, providing insights into the structure of homeomorphism classes. The paper concludes with the formulation of a topology on a set based on specific matrix conditions.

Homeomorfizm Sınıflarının Matris Gösterimleri Hakkında

Anahtar Kelimeler
Matris gösterimleri,
Homeomorfizm sınıfları,
Topoloji

Öz: Bu çalışmada homeomorfizm sınıflarının matris temsillerini araştırdık. Birler matrisi, sütun toplamı, satır toplamı, birin tümleyeni, Hadamard çarpımı ve matrisler için düzenli toplama gibi iyi bilinen kavramları göz önünde bulundurarak ikili matrislerin sonlu bir kümenin alt kümeleriyle ilişkilerini araştırıyoruz. Ana sonuçlar, matris işlemleri ile küme işlemleri arasında bağlantılar kurarak homeomorfizm sınıflarının yapısına ilişkin bilgiler sağlar. Makale, belirli matris koşullarına dayalı bir küme üzerinde bir topolojinin formülasyonu ile sona ermektedir.

1. INTRODUCTION

R.E. Stong introduces the properties of topological spaces with a finite number of points [5]. He examines various aspects including homeomorphism classification, point-set topology properties, classification by homotopy type, and homotopy classes of mappings.

This article, which introduces a matrix representation that is completely different from that defined by R.E. Stong, aims to explore homeomorphism classes using matrix representations. Homeomorphism is a concept in mathematics that defines the transformability of topological structures, and this study investigates how these transformations can be understood through matrix representations.

We consider well-known fundamental matrix concepts such as the matrix of ones, column and row sums, and the one's complement will be introduced. These concepts will be elucidated in terms of their applicability to the analysis of homeomorphism classes. Additionally, the association of binary matrices with subsets of a set and the expression of this relationship through matrix operations will be examined. Finally, we examine which conditions the necessary and sufficient conditions for a family to be a topology on a set depend on in the corresponding incidence matrices.

2. PRELIMINARIES

Now we introduce some basic concepts (See [1-4,6] for more detailed information).

A matrix of ones, denoted $\mathbf{1}$, is a matrix whose all entries are 1. The column-sum of a $n \times m$-matrix $\mathbf{A}$ is a row matrix each entry of which is the sum of all entries in corresponding column of $\mathbf{A}$, and denoted by $\mathrm{sum}_c(\mathbf{A})$. Similarly, the row-sum of a $n \times m$-matrix $\mathbf{A}$ is a column matrix each entry of which is the sum of all entries in corresponding row of $\mathbf{A}$, and denoted by $\mathrm{sum}_r(\mathbf{A})$. Then it is easy to see that

$$\mathrm{sum}_c(\mathbf{A}) = \mathbf{1}^t\mathbf{A} \text{ and } \mathrm{sum}_r(\mathbf{A}) = \mathbf{A}\mathbf{1}$$

where $\mathbf{1}^t$ denotes the transpose of the $n \times m$-matrix of ones $\mathbf{1}$. The one's complement of a $n \times m$-matrix $\mathbf{A}$ is defined by $\mathbf{1}-\mathbf{A}$ and denoted by $\mathbf{A}^c$. We denote the maximum (the minimum) of all entries in a $n\times m$-matrix $\mathbf{A}$ by $\max\mathbf{A}(\min\mathbf{A})$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n\times m$-matrices. The Hadamard product $\mathbf{C}$ of $\mathbf{A}$ and $\mathbf{B}$ is defined by $\mathbf{C}_{ij}=\mathbf{A}_{ij}\mathbf{B}_{ij}$ for every $i\in\{1,2,\dots,n\}$ and every $j\in\{1,2,\dots,m\}$, and denoted by $\mathbf{A}\odot\mathbf{B}$, that is,

$$(\mathbf{A}\odot\mathbf{B})_{ij}=\mathbf{A}_{ij}\mathbf{B}_{ij}.$$

Furthermore, the regular addition $\mathbf{C}$ of two column matrices $\mathbf{A}$ and $\mathbf{B}$ is defined and denoted by

$$\mathbf{A}\oplus\mathbf{B}=(\mathbf{A}+\mathbf{B})-(\mathbf{A}\odot\mathbf{B})$$

More generally, the regular addition of a $n\times m$ matrix $\mathbf{A}$, denoted by $\oplus\mathbf{A}$, is defined by $\oplus\mathbf{A}=\oplus_j\mathbf{A}_{*j}$ where $\mathbf{A}_{*j}$ is the j-th subcolumn of $\mathbf{A}$. To put it more explicitly, $\oplus A=\mathbf{R}_m$ where $\mathbf{R}_1=\mathbf{A}_{*1}$ and $\mathbf{R}_k=\mathbf{R}_{k-1}\oplus\mathbf{A}_{*k}$ for $k>1$.
$\mathcal{M}_{n\times m}(\mathbb{Z})$ denotes the set of all $n\times m$-matrices over the ring $\mathbb{Z}$ of integers. We consider $\mathcal{M}_{n\times m}(\{0,1\})\subseteq\mathcal{M}_{n\times m}(\mathbb{Z})$, that is, the class of all matrices with entries 0 and 1. A matrix $\mathbf{M}\in\mathcal{M}_{n\times m}(\{0,1\})$ is called a binary matrix.
Let X be a non-empty finite set. Consider integer-indexed elements $x_1,x_2,\dots,x_n$ of X. Then, to a subset $A\subseteq X$, we can correspond the binary n-column matrix $\mathbf{A}$ with the entries defined by

$$\mathbf{A}_{i1}=\begin{cases}1 & \text{if } x_i\in A\\ 0 & \text{otherwise}\end{cases}$$

and called the incidence column matrix of A (with respect to the given integer-indexed set X).
Let $U_1,U_2,\dots U_{2^n}$ be integer-indexed elements of $P(X)$. To a subfamily $\mathcal{S}\subseteq P(X)$, we can correspond the binary $n\times m$-matrix $\mathcal{S}$ with the entries defined by

$$\mathcal{S}_{ij}=\begin{cases}1 & \text{if } x_i\in U_j\in\mathcal{S}\\ 0 & \text{otherwise}\end{cases}$$

and called the incidence matrix of $\mathcal{S}$ (with respect to the integer-indexed set X and the integer-indexed power set $P(X)$).
Since X can be integer-indexed in different ways, a subset A corresponds different incidence matrices which implies that a subset A of a non-indexed set X has different incidence matrices. Similarly, by different integer indexing of X and $P(X)$, we obtain different incidence matrices of a subfamily $\mathcal{A}\subseteq P(X)$. As a result of this, if both a set X and its power set $P(X)$ are not integer-indexed, then a subfamily of $\mathcal{A}\subseteq P(X)$ has different incidence matrices.

3. MAIN RESULTS

Proposition 1. Let A be a subset of a set X. If $\mathbf{A}$ is an incidence column matrix of A, then $\mathbf{A}^c$ is an incidence matrix of A^c.

Proof. Let $\mathbf{A}$ be an incidence column matrix of set A. Then we have $\mathbf{A}_{i1}=1$ if $x_i\in A$, otherwise $\mathbf{A}_{i1}=0$. From the definition of one's complement of a matrix, we get $\mathbf{A}^c_{i1}=1-\mathbf{A}_{i1}=1-1=0$ if $x_i\in A$, otherwise $\mathbf{A}^c_{i1}=1-\mathbf{A}_{i1}=1-0=1$; or equivalently, we have $\mathbf{A}^c_{i1}=1$ if $x_i\in A^c$, otherwise $\mathbf{A}^c_{i1}=0$. Thus $\mathbf{A}^c$ is an incidence column matrix of A^c.

Proposition 2. Let A,B be two subsets of a set X. Let $\mathbf{A},\mathbf{B}$ be incidence column matrices of A and B, respectively. Then the following are equivalent:

1. $A\cap B=\emptyset$
2. $\mathbf{A}^t\mathbf{B}=0$

Proof.

$(1\Rightarrow 2)$: Assume that $\mathbf{A}^t\mathbf{B}\neq 0$. Then

$$0\neq\mathbf{A}^t\mathbf{B}=\sum_{i=1}^{n}\mathbf{A}^t_{1i}\mathbf{B}_{i1}=\sum_{i=1}^{n}\mathbf{A}_{i1}\mathbf{B}_{i1}$$

and so $\mathbf{A}_{i1}=\mathbf{B}_{i1}=1$ for some $i\in\{1,2,\dots,n\}$. However, from the hypothesis $A\cap B=\emptyset$, for every $i\in\{1,2,\dots,n\},\mathbf{A}_{i1}\neq\mathbf{B}_{i1}$, which leads to a contradiction. This contradiction arises from our assumption $\mathbf{A}^t\mathbf{B}\neq 0$. Thus $\mathbf{A}^t\mathbf{B}=0$.

$(2\Rightarrow 1)$: Assume that $A\cap B\neq\emptyset$. Then, for some $i\in\{1,2,\dots,n\}$, we have $\mathbf{A}_{i1}=1$ and $\mathbf{B}_{i1}=1$. On the other hand, from the hypothesis $\mathbf{A}^t\mathbf{B}=0$, we have $\sum_{i=1}^{n}\mathbf{A}_{i1}\mathbf{B}_{i1}=0$. Then, there exists no $i\in\{1,2,\dots,n\}$ such that $\mathbf{A}_{i1}=1=\mathbf{B}_{i1}$ which leads to a contradiction. This contradiction arises from our assumption $A\cap B\neq\emptyset$. Thus $A\cap B=\emptyset$.

Conclusion 3. Let A, B be incidence column matrices of subsets A and B of a set X, respectively. Then $A\cap B\neq\emptyset$ if and only if $\mathbf{A}^t\mathbf{B}\geq 1$.

Proposition 4. Let A, B be incidence column matrices of subsets A and B of a set X, respectively. Then the Hadamard product $\mathbf{A}\odot\mathbf{B}$ is an incidence column matrix of the intersection $A\cap B$.

Proof. Let A, B be incidence column matrices of subsets A and B of a set X, respectively. If $(\mathbf{A}\odot\mathbf{B})_{i1}=1$, then the member of X corresponding $(\mathbf{A}\odot\mathbf{B})_{i1}$ belongs to both A and B and so belongs to $A\cap B$. Otherwise, it does not belong to at least one of A and B and so does not belong to $A\cap B$. Thus, the proof is completed.

Proposition 5. Let $\mathbf{A},\mathbf{B}$ be incidence column matrices of subsets A and B of a set X, respectively. Then an incidence column matrix of the union $A\cup B$ is the regular addition $\mathbf{A}\oplus\mathbf{B}$.

Proof. Let $\mathbf{A},\mathbf{B}$ be incidence column matrices of subsets A and B of a set X, respectively. If $(\mathbf{A}\oplus\mathbf{B})_{i1}=0$, then the member of X corresponding $(\mathbf{A}\oplus\mathbf{B})_{i1}$ belongs to neither A nor B and so does not belong to $A\cup B$.

Otherwise, it belongs to at least one of A and B and so belongs to $A \cup B$. Thus, the proof is completed.

Proposition 6. Let $\mathcal{S}$ be incidence matrix of a family $\mathcal{S}$ of subsets of a set X. Then an incidence column matrix of the union $\bigcup\mathcal{S}$ is the regular addition $\oplus\mathcal{S}$.

Proof. Let $\mathcal{S}$ be incidence matrix of a family $\mathcal{S}$ of subsets of a set X. If $(\oplus\mathcal{S})_{i1} = 0$, then the member of X corresponding $(\oplus\mathcal{S})_{i1}$ belongs to no member of $\mathcal{S}$ and so does not belong to $\bigcup\mathcal{S}$. Otherwise, it belongs to at least one member of $\mathcal{S}$ and so belongs to $\bigcup\mathcal{S}$. Thus, the proof is completed.

Proposition 7. Let A, B be two subsets of a set X. Let $\mathbf{A}, \mathbf{B}$ incidence column matrices of A and B, respectively. Then the following are equivalent:

1. $A \subseteq B$

Proof.

$(1 \Rightarrow 2)$: Let A be a subset of B. Then we have $A \cap B^c = \emptyset$. From Proposition 1 and Proposition 2, we have $\mathbf{A}^t\mathbf{B}^c = 0$.

$(2 \Rightarrow 1)$: Let $\mathbf{A}^t\mathbf{B}^c = 0$. Then From Proposition 1 and Proposition 2, we yield $A \cap B^c = \emptyset$, or equivalently, $A \subseteq B$.

Proposition 8. Let $\mathbf{A}, \mathbf{B}$ be incidence column matrices of subsets A and B of a set X, respectively. Then $A = B$ if and only if $\mathbf{1}^t(\mathbf{A} - \mathbf{B}) = 0$.

Proof.

$$
\begin{aligned}
A = B &\Leftrightarrow A \subseteq B \wedge B \subseteq A \\
&\Leftrightarrow \mathbf{A}^t\mathbf{B}^c = 0 \wedge \mathbf{B}^t\mathbf{A}^c = 0 \\
&\Leftrightarrow \mathbf{A}^t(\mathbf{1} - \mathbf{B}) = 0 \wedge \mathbf{B}^t(\mathbf{1} - \mathbf{A}) = 0 \\
&\Leftrightarrow \mathbf{A}^t\mathbf{1} - \mathbf{A}^t\mathbf{B} = 0 \wedge \mathbf{B}^t\mathbf{1} - \mathbf{B}^t\mathbf{A} = 0 \\
&\Leftrightarrow \mathbf{A}^t\mathbf{1} = \mathbf{A}^t\mathbf{B} \wedge \mathbf{B}^t\mathbf{1} = \mathbf{B}^t\mathbf{A} \\
&\Leftrightarrow \mathbf{A}^t\mathbf{1} = \mathbf{A}^t\mathbf{B} = \mathbf{B}^t\mathbf{A} = \mathbf{B}^t\mathbf{1} \\
&\Leftrightarrow \mathbf{A}^t\mathbf{1} = \mathbf{B}^t\mathbf{1} \\
&\Leftrightarrow (\mathbf{A}^t - \mathbf{B}^t)\mathbf{1} = 0 \\
&\Leftrightarrow [(\mathbf{A}^t - \mathbf{B}^t)\mathbf{1}]^t = 0 \\
&\Leftrightarrow \mathbf{1}^t(\mathbf{A}^t - \mathbf{B}^t)^t = 0 \\
&\Leftrightarrow \mathbf{1}^t(\mathbf{A} - \mathbf{B}) = 0.
\end{aligned}
$$

Proposition 9. Let A be a subset of a set X. Let $\mathbf{A}$ be an incidence matrix of A, and let $\mathcal{S}$ be an incidence matrix of a family $\mathcal{S}$ of subsets of X. Then the following are equivalent:

1. $A \in \mathcal{S}$
2. There exists $U \in \mathcal{S}$ such that $\mathbf{1}^t(\mathbf{A} - \mathbf{U}) = 0$ where $\mathbf{U}$ is an incident matrix of U.

Proof.

$(1 \Rightarrow 2)$: Let $\mathbf{A}$ be the incidence matrix of a member A of $\mathcal{S}$. Set $U = A$. Let $\mathbf{U}$ be an incident matrix of U. Then, from Proposition 8, we have $\mathbf{1}^t(\mathbf{A} - \mathbf{U}) = 0$.

$(2 \Rightarrow 1)$: Let $\mathbf{A}$ be the incidence matrix of a subset A of a set X. Consider a member U of $\mathcal{S}$ such that $\mathbf{1}^t(\mathbf{A} - \mathbf{U}) = 0$ where $\mathbf{U}$ is an incident matrix of U. Then, from Proposition 8, we obtain $A = U \in \mathcal{S}$.

Theorem 10. Given a subfamily $\mathcal{T}$ of subsets of a set X. Let $\mathcal{T}$ be an incidence matrix of $\mathcal{T}$. Then $\mathcal{T}$ is a topology on X if and only if the following hold:

1. There exists $G \in \mathcal{T}$ with an incidence column matrix $\mathbf{G}$ such that $\mathbf{1}^t\mathbf{G} = 0$.
2. There exists $G \in \mathcal{T}$ with an incidence column matrix $\mathbf{G}$ such that $\mathbf{1}^t\mathbf{G}^c = 0$.
3. If $\mathbf{G}, \mathbf{H}$ is incidence column matrices of members $G, H \in \mathcal{T}$, respectively, then there exists $U \in \mathcal{T}$ with an incident matrix $\mathbf{U}$ such that $\mathbf{1}^t(\mathbf{G} \odot \mathbf{H} - \mathbf{U}) = 0$.
4. If $\mathcal{G}$ is an incidence matrix of a subfamily $\mathcal{G} \subseteq \mathcal{T}$, then there exists $U \in \mathcal{T}$ with an incident matrix $\mathbf{U}$ such that $\mathbf{1}^t[\oplus\mathbf{G} - \mathbf{U}] = 0$.

Proof. Let $\mathcal{T}$ be an incidence matrix of a subfamily $\mathcal{T}$ of subsets of a set X.

$(\Rightarrow)$: Let $\mathcal{T}$ be a topology on a set X.

1. Since $\emptyset \in \mathcal{T}$, we have $\mathbf{1}^t\mathbf{0} = 0$ for the incidence column matrix $\mathbf{0}$ of the empty set $\emptyset$.
2. Since $X \in \mathcal{T}$, we get $\mathbf{1}^t\mathbf{1}^c = \mathbf{1}^t\mathbf{0} = 0$ for the incidence column matrix $\mathbf{1}$ of the whole set X.
3. Let $G, H \in \mathcal{T}$ have incidence column matrices $\mathbf{G}, \mathbf{H}$, respectively. Then, from Proposition 4, $\mathbf{G} \odot \mathbf{H}$ is an incidence column matrix of $G \cap H$. Since $G \cap H \in \mathcal{T}$, from Proposition 9, there exists $U \in \mathcal{T}$ with an incidence column matrix $\mathbf{U}$ such that $\mathbf{1}^t(\mathbf{G} \odot \mathbf{H} - \mathbf{U}) = 0$.
4. Let $\mathcal{G}$ be an incidence matrix of a subfamily $\mathcal{G} \subseteq \mathcal{T}$. From Proposition 6, $\oplus\mathcal{G}$ is an incident column matrix of the union $\cup\mathcal{G}$. Since $\cup\mathcal{G} \in \mathcal{T}$, by Proposition 9, there exists $U \in \mathcal{T}$ with an incident matrix $\mathbf{U}$ such that $\mathbf{1}^t[\oplus\, \mathbf{G} - \mathbf{U}] = 0$.

$(\Leftarrow)$: (O_1) From the first item of the hypothesis, there exists $G \in \mathcal{T}$, say G_0, with an incidence column matrix $\mathbf{G}$ such that $\mathbf{1}^t\mathbf{G} = 0$. It is clear that G_0 is the empty set $\emptyset$. By Proposition 9, we have $\emptyset \in \mathcal{T}$. From the second item of the hypothesis, there exists $G \in \mathcal{T}$, say G_0, with an incidence column matrix $\mathbf{G}$ such that $\mathbf{1}^t\mathbf{G}^c = 0$. It is clear that G_0 is the whole set X. By Proposition 9, we have $X \in \mathcal{T}$.

(O_2) Let $\mathbf{G}, \mathbf{H}$ be incidence column matrices of members $G, H \in \mathcal{T}$, respectively. From the third item of the hypothesis, there exists $U \in \mathcal{T}$ with an incident matrix $\mathbf{U}$ such that $\mathbf{1}^t(\mathbf{G} \odot \mathbf{H} - \mathbf{U}) = 0$. Then, from Proposition 4 and Proposition 9, $\mathbf{G} \odot \mathbf{H}$ is an incidence column matrix of the intersection $G \cap H$ and so we have $G \cap H \in \mathcal{T}$.

(O_3) Let $\mathcal{G}$ be an incidence matrix of a subfamily $\mathcal{G} \subseteq \mathcal{T}$. From the fourth item of the hypothesis, there exists $U \in \mathcal{T}$ with an incident matrix $\mathbf{U}$ such that $\mathbf{1}^t[\oplus\mathbf{G} - \mathbf{U}] = 0$. Then, from Proposition 6 and Proposition 9, $\oplus\, \mathcal{G}$ is an incident column matrix of the union $\bigcup\mathcal{G}$ and so we have $\cup\, \mathcal{G} \in \mathcal{T}$.

4. CONCLUSION

We have presented a comprehensive exploration of various concepts related to matrices and subsets of a set X.

We show that it can be used the notion of incidence matrix to represent the membership relations between elements of X and subsets of X. Through propositions and theorems, we established relationships between these matrices and fundamental set operations such as intersection, union, complement, and subset relationships.

Moreover, we extended our analysis to consider families of subsets and their properties in the context of forming a topology on X. Our results provide insights into the structural properties of matrices representing subsets and lay the groundwork for further investigation into combinatorial and topological aspects of finite sets.

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REFERENCES

[1] Brualdi RA, Ryser HJ, et al. Combinatorial matrix theory, volume 39. Springer; 1991.
[2] Eves HW. Elementary matrix theory. Courier Corporation; 1980.
[3] Franklin JN. Matrix theory. Courier Corporation; 2012.
[4] Ortega JM. Matrix theory: A second course. Springer Science & Business Media; 2013.
[5] Stong RE. Finite topological spaces. Transactions of the American Mathematical Society. 1966;123(2):325–340.
[6] Zhang F. Matrix theory: basic results and techniques. Springer Science & Business Media; 2011.