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On Equiform Rectifying, Normal and Osculating Curves in Minkowski Space-Time

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This article addresses the equiform rectifying, normal, and second type osculating curves within Minkowski space-time E_1^4. We expose the requisite and satisfactory criteria for a curve to qualify as a rectifying, normal, and second kind osculating curve with respect to equiform geometry within Minkowski space-time E_1^4. We derive the correlation between the curvatures of these curves to be congruent to a rectifying, normal and second kind osculating curve according to equiform geometry in Minkowski space-time E_1^4.

1. Introduction

In Euclidean space E^3, a curve is characterized through the Frenet frame $\{T, N, B\}$. The planes, spanned by the vectors, $\{T, B\}$, $\{N, B\}$ and $\{T, N\}$ are respectively called as the rectifying plane, normal plane, and osculating plane. A curve is called as a rectifying (resp. normal, osculating) curve if its position vector always lies in its rectifying (resp. normal, osculating) plane. In other words, the rectifying, normal, and osculating planes of these curves consistently include a specific point. A widely recognized principle states that when all the normal or osculating planes of a curve in three-dimensional Euclidean space E^3 intersect at a specific point, then the curve either lies on a sphere or is a planar curve, depending on the context. Furthermore, it is established knowledge that if all rectifying planes of a non-planar curve in E^3 intersect at a specific point, then the ratio between torsion and curvature for such a curve is accepted as a non-constant linear function [3]. Chen defined the concept of the rectifying curve in his study [2]. Given that the position vector of rectifying curve establishes instantaneous rotation axis at each point along the curve, the author also showed that these curves are necessary for mechanics, kinematics and differential geometry. In Minkowski 3-space E_1^3, rectifying curves exhibit analogous geometric properties to those in Euclidean 3-space E^3. Spacelike, timelike and null rectifying curves in E_1^3 were examined in [8]. Furthermore, the characterizations of rectifying curves were introduced in 4-dimensional Euclidean and Minkowski spaces in [1], [7], [9]. The equiform geometry of rectifying curves were studied in Galilean 4-space in [19].

Normal and osculating curves have been examined in various studies in three and four-dimensional spaces. The characterizations of the normal and osculating curves in 3-dimensional Euclidean and Minkowski spaces were introduced in [10], [11], [13]. In addition, the characterizations of the normal and osculating curves in 4-dimensional Minkowski space were obtained in [12], [14], [15], [16]. The equiform geometry of normal and osculating curves were introduced in Galilean 4-space in [20], [21].

This paper focuses on equiform rectifying, normal and second kind osculating curves in E_1^4. We examine the characteristics of rectifying, normal and second kind osculating curves based on their equiform curvature functions, establishing essential and sufficient conditions for any curve to qualify as

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rectifying, normal and second kind osculating curve, respectively.

2. Preliminaries

The Minkowski space E_1^4 is a Euclidean space provided with the indefinite flat metric given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

is called Lorentzian inner product. For any vector $\vartheta \in E_1^4$ can have one of three causal characters; it can be spacelike $\langle \vartheta, \vartheta\rangle > 0$ or $\vartheta = 0$, timelike if $\langle \vartheta, \vartheta\rangle < 0$ and null if $\langle \vartheta, \vartheta\rangle = 0$ and $\vartheta \neq 0$. The pseudo-norm of a vector non null ϑ is defined by $\|\vartheta\| = \sqrt{|\langle \vartheta, \vartheta\rangle|}$ [4], [6], [17].

Let $\alpha = \alpha(s)$ is a regular spacelike curve in E_1^4 with timelike vector field δ_3. The Frenet formulae in E_1^4 can be written as

$$\begin{bmatrix} t'(s) \\ \delta_1'(s) \\ \delta_2'(s) \\ \delta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & 0 & 0 \\ -k_1(s) & 0 & k_2(s) & 0 \\ 0 & -k_2(s) & 0 & k_3(s) \\ 0 & 0 & k_3(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ \delta_1(s) \\ \delta_2(s) \\ \delta_3(s) \end{bmatrix} \quad (2.1)$$

where k_1, k_2 and k_3 are curvature functions of spacelike curve α. Here, $\{t, \delta_1, \delta_2, \delta_3\}$ satisfy the following equations $\langle t,t\rangle = \langle \delta_1, \delta_1\rangle = \langle \delta_2, \delta_2\rangle = 1$ $\langle \delta_3, \delta_3\rangle = -1$ and $\langle t, \delta_1\rangle = \langle t, \delta_2\rangle = \langle t, \delta_3\rangle = \langle \delta_1, \delta_2\rangle = \langle \delta_1, \delta_3\rangle = \langle \delta_2, \delta_3\rangle = 0$ [6], [18].

Now we define rectifying, normal and osculating curves in E_1^4.

The curve α is termed as rectifying curve in E_1^4 whose position vector consistently resides in the orthogonal complement $\vec{\delta}_1^{\perp}$ of its principal normal vector field $\vec{\delta}_1$. Hence, position vector of rectifying spacelike curve can be expressed as:

$$\alpha(s) = \mu_1(s)t(s) + \mu_2(s)\delta_2(s) + \mu_3(s)\delta_3(s) \quad (2.2)$$

for some differentiable functions $\mu_1(s)$, $\mu_2(s)$ and $\mu_3(s)$ in arclength function s [7].

Subsequently, a normal curve in E_1^4 can be defined as a curve whose position vector consistently resides in its normal space. As a result, the position vector of normal spacelike curve can be written as:

$$\alpha(s) = \varsigma_1(s)\delta_1(s) + \varsigma_2(s)\delta_2(s) + \varsigma_3(s)\delta_3(s) \quad (2.3)$$

for some differentiable functions $\varsigma_1(s), \varsigma_2(s)$ and $\varsigma_3(s)$ in arclength function s [12].

Finally, we establish the definition of first kind or second kind osculating curve in E_1^4 where its position vector concerning a selected origin consistently lies in the orthogonal complement $\delta_3^{\perp}$ or $\delta_2^{\perp}$, respectively. Therefore, position vector of spacelike and timelike first kind osculating curve can be written as:

$$\alpha(s) = \lambda_1(s)t(s) + \lambda_2(s)\delta_1(s) + \lambda_3(s)\delta_2(s) \quad (2.4)$$

and position vector of spacelike and timelike second kind osculating curve can be written as:

$$\alpha(s) = \lambda_1(s)t(s) + \lambda_2(s)\delta_1(s) + \lambda_3(s)\delta_3(s) \quad (2.5)$$

where $\lambda_1(s), \lambda_2(s)$ and $\lambda_3(s)$ are differentiable functions in arclength function s [14].

For a regular spacelike curve $\alpha : I \to E_1^4$, let the equiform parameter of $\alpha(s)$ defined by $\sigma = \int k_1\, ds$ where $\rho = \frac{1}{k_1}$ is the radius of curvature of the curve α. Hence, it follows $\rho = \frac{ds}{d\sigma}$. Let $\{T, \eta, \zeta_1, \zeta_2\}$ be the equiform Frenet frame of the curve α, where $T(\sigma) = \rho t(s)$ is equiform tangent vector, $\eta(\sigma) = \rho\delta_1(s)$ is equiform principal normal vector, $\zeta_1(\sigma) = \rho\delta_2(s)$ is equiform first binormal vector and $\zeta_2(\sigma) = \rho\delta_3(s)$ is equiform second binormal vector. Moreover, equiform curvatures of the curve $\alpha = \alpha(\sigma)$ are defined by $K_1(\sigma) = \dot{\rho}$, $K_2(\sigma) = \frac{k_2}{k_1}$ and $K_3(\sigma) = \frac{k_3}{k_1}$. Thus, the equiform Frenet formulae in E_1^4 is written as follows:

$$\begin{bmatrix} T'(\sigma) \\ \eta'(\sigma) \\ \zeta_1'(\sigma) \\ \zeta_2'(\sigma) \end{bmatrix} = \begin{bmatrix} K_1(\sigma) & 1 & 0 & 0 \\ -1 & K_1(\sigma) & K_2(\sigma) & 0 \\ 0 & -K_2(\sigma) & K_1(\sigma) & K_3(\sigma) \\ 0 & 0 & K_3(\sigma) & K_1(\sigma) \end{bmatrix} \begin{bmatrix} T(\sigma) \\ \eta(\sigma) \\ \zeta_1(\sigma) \\ \zeta_2(\sigma) \end{bmatrix} \quad (2.6)$$

where $\left(' = \frac{d}{d\sigma} \right)$, $\langle T,T \rangle = \langle \eta,\eta \rangle = \langle \zeta_1,\zeta_1 \rangle = \rho^2$, $\langle \zeta_2,\zeta_2 \rangle = -\rho^2$ and $\langle T,\eta \rangle = \langle T,\zeta_1 \rangle = \langle T,\zeta_2 \rangle = \langle \eta,\zeta_1 \rangle = \langle \eta,\zeta_2 \rangle = \langle \zeta_1,\zeta_2 \rangle = 0$ [5].

The paper focuses on rectifying, normal and second kind osculating curves within the context of equiform geometry in E_1^4, characterizing these curves based on their equiform curvature functions.

3. Spacelike Equiform Rectifying Curves in E_1^4

Theorem 3.1. Assume that α is a unit speed spacelike curve in E_1^4 with spacelike vector fields T,ζ_1 and timelike vector field ζ_2. Then α is a rectifying curve if and only if

$$\left(\frac{1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right) \right)' = \frac{(c+s)K_3}{\rho K_2} - \frac{K_1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right) \tag{3.1}$$

where K_1, K_2, K_3 are non-zero equiform curvatures and c is non-zero constant.

Proof, Let α be a unit speed spacelike rectifying curve in E_1^4 with equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Hence, by definition we have

$$\alpha(\sigma) = \mu_1(\sigma)T(\sigma) + \mu_2(\sigma)\zeta_1(\sigma) + \mu_3(\sigma)\zeta_2(\sigma) \tag{3.2}$$

for some differentiable functions μ_1, μ_2 and μ_3 in E_1^4. Differentiating (3.2) with respect to σ and using Eq. (2.6), the following equation is gathered

$$T = \left(\mu_1' + \mu_1 K_1 \right)T + (\mu_1 - \mu_2 K_2)\eta + \left(\mu_2' + \mu_2 K_1 + \mu_3 K_3 \right)\zeta_1 + \left(\mu_3' + \mu_3 K_1 + \mu_2 K_3 \right)\zeta_2.$$

It follows that

$$\begin{cases} \mu_1' + \mu_1 K_1 - 1 = 0, \\ \mu_1 - \mu_2 K_2 = 0, \\ \mu_2' + \mu_2 K_1 + \mu_3 K_3 = 0, \\ \mu_3' + \mu_2 K_3 + \mu_3 K_1 = 0 \end{cases} \tag{3.3}$$

and therefore

$$\begin{cases} \mu_1 = \frac{c+s}{\rho}, \qquad \mu_2 = \frac{c+s}{\rho K_2}, \\ \mu_3 = -\frac{1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right). \end{cases} \tag{3.4}$$

where c is non-zero constant. In this way, the functions $\mu_1(\sigma)$, $\mu_2(\sigma)$ and $\mu_3(\sigma)$ can be expressed in terms of the equiform curvatures K_1, K_2 and K_3. Thus

$$\alpha(\sigma) = \frac{c+s}{\rho}T(\sigma) + \frac{c+s}{\rho K_2}\zeta_1(\sigma) - \left(\frac{1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right) \right)\zeta_2(\sigma). \tag{3.5}$$

Furthermore, employing the last equation in (3.3) and the relation (3.4), it can be deduced that the equiform curvature functions K_1, K_2 and K_3 satisfy the equation

$$\left(\frac{1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right) \right)' = \frac{(c+s)K_3}{\rho K_2} - \frac{K_1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right). \tag{3.6}$$

Consider the vector $X \in E_1^4$ expressed as

$$X(\sigma) = \alpha(\sigma) - \frac{c+s}{\rho}T(\sigma) - \frac{c+s}{\rho K_2}\zeta_1(\sigma) + \left(\frac{1}{K_3}\left(\frac{1}{K_2} + (c+s)\left(\frac{1}{K_2} \right)' \right) \right)\zeta_2(\sigma).$$

Then, we can easily find $X' = 0$, that is, X is a constant vector. Hence, α is rectifying curve.

Theorem 3.2. There are no spacelike rectifying curves in E_1^4 with constant equiform curvatures K_1, K_2 and K_3.

Proof. The clarity of the statement is evident from Theorem 3.1.

Theorem 3.3. Assume that α is a unit speed spacelike curve in E_1^4 with non-zero equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Hence, α is rectifying curve if

$$K_2 = \text{constant}, \quad K_3 = \text{constant}, \quad K_1 = \frac{(c+s)}{\rho} K_3^2 \tag{3.7}$$

where c is non-zero constant.

Proof. Suppose that $K_2 = \text{constant}$ and $K_3 = \text{constant}$. By using the equation (3.6), we get (3.7).

Corollary 3.1. Let α be a unit speed spacelike curve with non-zero equiform curvatures K_1, K_2, K_3 and vector fields T, ζ_1 and ζ_2 in E_1^4. If α is a rectifying curve then the following relations satisfied:

(i) The principal tangent component of position vector of rectifying curve is expressed as

$$\langle \alpha, T \rangle = \rho(c+s).$$

(ii) The first and second binormal components of position vector of rectifying curve can be expressed as

$$\langle \alpha, \zeta_1 \rangle = \rho \frac{c+s}{K_2},$$

$$\langle \alpha, \zeta_2 \rangle = \frac{\rho^2}{K_3} \left(\frac{1}{K_2} + (c+s) \left(\frac{1}{K_2} \right)' \right).$$

4. Spacelike Equiform Normal Curves in E_1^4

Theorem 4.1. Assume that α is a unit speed spacelike curve in E_1^4 with spacelike vector fields η, ζ_1 and timelike vector field ζ_2. Then α is a normal curve if and only if

$$\left(\frac{1}{K_3} \left(K_2 + \left(\frac{K_1}{K_2} \right)' + \frac{K_1^2}{K_2} \right) \right)' = \frac{K_1 K_3}{K_2} - \frac{K_1}{K_3} \left(K_2 + \left(\frac{K_1}{K_2} \right)' + \frac{K_1^2}{K_2} \right) \tag{4.1}$$

where K_1, K_2 and K_3 are non-zero equiform curvatures.

Proof Let α be a unit speed spacelike curve in E_1^4 with non-zero equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Hence, by definition we have

$$\alpha(\sigma) = \varsigma_1(\sigma)\eta(\sigma) + \varsigma_2(\sigma)\zeta_1(\sigma) + \varsigma_3(\sigma)\zeta_2(\sigma) \tag{4.2}$$

for some differentiable functions ς_1, ς_2 and ς_3 in E_1^4. Differentiating (4.2) with respect to σ and using the Eq. (2.6), the following equation is gathered

$$\begin{aligned} T = & -\varsigma_1 T + (\varsigma_1' + \varsigma_1 K_1 - \varsigma_2 K_2)\eta \\ & + (\varsigma_1 K_2 + \varsigma_2' + \varsigma_2 K_1 + \varsigma_3 K_3)\zeta_1 \\ & + (\varsigma_2 K_3 + \varsigma_3 K_1 + \varsigma_3')\zeta_2. \end{aligned}$$

It follows that

$$\begin{cases} -\varsigma_1 - 1 = 0, \\ \varsigma_1' + \varsigma_1 K_1 - \varsigma_2 K_2 = 0, \\ \varsigma_1 K_2 + \varsigma_2' + \varsigma_2 K_1 + \varsigma_3 K_3 = 0, \\ \varsigma_2 K_3 + \varsigma_3 K_1 + \varsigma_3' = 0 \end{cases} \tag{4.3}$$

and therefore

$$\begin{cases} \varsigma_1 = -1, \qquad \varsigma_2 = -\dfrac{K_1}{K_2}, \\ \varsigma_3 = \dfrac{1}{K_3} \left(K_2 + \left(\dfrac{K_1}{K_2} \right)' + \dfrac{K_1^2}{K_2} \right). \end{cases} \tag{4.4}$$

In this way, the functions $\varsigma_1(\sigma), \varsigma_2(\sigma)$ and $\varsigma_3(\sigma)$ can be expressed in terms of the equiform curvatures K_1, K_2 and K_3. Thus

$$\begin{aligned} \alpha(\sigma) = & -\eta(\sigma) - \frac{K_1}{K_2}\zeta_1(\sigma) \\ & + \frac{1}{K_3} \left(K_2 + \left(\frac{K_1}{K_2} \right)' - \frac{K_1^2}{K_2} \right) \zeta_2(\sigma). \end{aligned} \tag{4.5}$$

Additionally, employing the last equation in (4.3) and the relation (4.4), it is straightforward to determine that equiform curvature functions K_1, K_2 and K_3 satisfy the equation

$$\frac{K_1 K_3}{K_2} - \frac{K_1}{K_3} \left(K_2 + \left(\frac{K_1}{K_2} \right)' + \frac{K_1^2}{K_2} \right) = \left(\frac{1}{K_3} \left(K_2 + \left(\frac{K_1}{K_2} \right)' + \frac{K_1^2}{K_2} \right) \right)'. \tag{4.6}$$

Consider the vector $X \in E_1^4$ expressed as

$$X(\sigma)=\alpha(\sigma)+\eta(\sigma)+\frac{K_1}{K_2}\zeta_1(\sigma) -\frac{1}{K_3}\left(K_2+\left(\frac{K_1}{K_2}\right)'+\frac{K_1^2}{K_2}\right)\zeta_2(\sigma)$$

gives $X'=0$, and X is a constant vector. Hence, α is a normal curve.

Theorem 4.2. Assume that α is a unit speed spacelike curve in E_1^4 with non-zero constant equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Then α is normal curve if $K_1^2+K_2^2=K_3^2$.

Proof. The clarity of the statement is evident from Theorem 4.1.

Theorem 4.3. Assume that α is a unit speed spacelike curve in E_1^4 with non-zero equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Then α is a normal curve if

$$K_1=\text{constant},\quad K_2=\text{constant},$$

$$K_3=\pm e^{K_1(\sigma+c)}\sqrt{\frac{1+K_1^2}{e^{2K_1(\sigma+c)}\pm\left(1+K_1^2\right)}} \tag{4.7}$$

where c is non-zero constant.

Proof. Suppose that $K_1=a_1$, $K_2=a_2$ and $K_2+\frac{K_1^2}{K_2}=a_3$, where a_1,a_2,a_3 are non-zero constants. By using the equation (4.6), we get

$$K_3'=a_1K_3-\frac{a_1}{a_2a_3}K_3^3. \tag{4.8}$$

Then the solution of differential equation (4.8) gives (4.7).

Corollary 4.1. Let α be a unit speed spacelike curve with non-zero equiform curvatures K_1,K_2,K_3 and vector fields η, ζ_1, ζ_2 in E_1^4. If α is a normal curve, then the following relations are satisfied:

(i) The principal normal component of position vector of normal curve can be expressed as

$$\langle\alpha,\eta\rangle=-\rho^2.$$

(ii) The first and second binormal components of position vector of normal curve can be expressed as

$$\langle\alpha,\zeta_1\rangle=-\rho^2\frac{K_1}{K_2},$$

$$\langle\alpha,\zeta_2\rangle=-\frac{\rho^2}{K_3}\left(K_2+\left(\frac{K_1}{K_2}\right)'+\frac{K_1^2}{K_2}\right).$$

5. Spacelike Equiform Second Kind Osculating Curves in E_1^4

Theorem 5.1. Assume that α is a unit speed spacelike curve in E_1^4 with spacelike vector fields T, η and timelike vector field ζ_2. Hence, α is a second kind osculating curve if and only if

$$\left(\frac{K_3}{K_2}\right)''=-\frac{K_3}{K_2}+\frac{1}{c}e^{\int K_1 d\sigma},\quad c\in R. \tag{5.1}$$

Proof Let α be a unit speed spacelike osculating curve in E_1^4 with non-zero equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Hence, by definition we have

$$\alpha(\sigma)=\lambda_1(\sigma)T(\sigma)+\lambda_2(\sigma)\eta(\sigma)+\lambda_3(\sigma)\zeta_2(\sigma) \tag{5.2}$$

for some differentiable functions λ_1,λ_2 and λ_3 in E_1^4. Differentiating (5.2) with respect to σ and using the Eq. (2.6), the following equation is gathered

$$T=(\lambda_1'+\lambda_1K_1-\lambda_2)T+(\lambda_1+\lambda_2'+K_1\lambda_2)\eta +(\lambda_2K_2+\lambda_3K_3)\zeta_1+(\lambda_3'+\lambda_3K_1)\zeta_2.$$

It follows that

$$\begin{cases}\lambda_1'+\lambda_1K_1-\lambda_2-1=0,\\ \lambda_1+\lambda_2'+K_1\lambda_2=0,\\ \lambda_2K_2+\lambda_3K_3=0,\\ \lambda_3'+\lambda_3K_1=0\end{cases} \tag{5.3}$$

and therefore

$$\begin{cases}\lambda_1=c\left(\frac{K_3}{K_2}\right)'e^{-\int K_1 d\sigma},\\ \lambda_2=-c\left(\frac{K_3}{K_2}\right)e^{-\int K_1 d\sigma},\\ \lambda_3=ce^{-\int K_1 d\sigma}\end{cases} \tag{5.4}$$

where c is non-zero constant. Accordingly, the functions $\lambda_1(\sigma)$, $\lambda_2(\sigma)$ and $\lambda_3(\sigma)$ can be expressed in terms of the equiform curvatures K_1, K_2 and K_3. Thus

$$\alpha(\sigma)=ce^{-\int K_1 d\sigma}\begin{bmatrix}\left(\frac{K_3}{K_2}\right)'T(\sigma)-\left(\frac{K_3}{K_2}\right)\eta(\sigma)\\ +\zeta_2(\sigma)\end{bmatrix}. \quad (5.5)$$

Furthermore, utilizing the first equation in (5.3) and the Eq. (5.4), the equiform curvature functions K_1, K_2 and K_3 satisfy the equation

$$\left(\frac{K_3}{K_2}\right)''=-\frac{K_3}{K_2}+\frac{1}{c}e^{\int K_1 d\sigma}, \qquad c\in\mathbb{R}.$$

(5.6)

Consider the vector $X\in E_1^4$ expressed as

$$X(\sigma)=\alpha(\sigma)-c\left(\frac{K_3}{K_2}\right)'e^{-\int K_1 d\sigma}T(\sigma)+c\frac{K_3}{K_2}e^{-\int K_1 d\sigma}\eta(\sigma)-ce^{-\int K_1 d\sigma}\zeta_2(\sigma)$$

gives $X'=0$, and X is a constant vector. Hence, α is a second kind osculating curve.

Theorem 5.2. There are no spacelike second kind osculating curves in E_1^4 with non-zero constant equiform curvatures K_1, K_2 and K_3.

Proof. The clarity of the statement is evident from Theorem 5.1.

Theorem 5.3. Let α be a unit speed spacelike second kind osculating curve in E_1^4 with non-zero equiform curvatures K_1, K_2 and K_3 with respect to the equiform invariant parameter σ. Hence, α is a second kind osculating curve if

$$K_1=\text{constant},$$

$$\frac{K_3}{K_2}=-\int\left(\int\frac{K_3}{K_2}d\sigma\right)d\sigma+\frac{1}{cK_1^2}e^{K_1\sigma}+d_1\sigma+d_2 \quad (5.7)$$

where $c,d_1,d_2\in R$.

Proof. Suppose that $K_1=\text{constant}$. By utilizing the equation (5.6), we obtain differential equation

$$\left(\frac{K_3}{K_2}\right)''=-\frac{K_3}{K_2}+\frac{1}{c}e^{K_1\int d\sigma}, \quad c\in R. \quad (5.8)$$

Solution of the differential equation (5.8) can be formulated as

$$\frac{K_3}{K_2}=-\int\left(\int\frac{K_3}{K_2}d\sigma\right)d\sigma+\frac{1}{cK_1^2}e^{K_1\sigma}+d_1\sigma+d_2,$$

where $c,d_1,d_2\in R$.

Corollary 5.1. Let α be a unit speed spacelike curve with non-zero equiform curvatures K_1, K_2 and K_3 and vector fields T,η and ζ_2 in E_1^4. If α is a second kind osculating curve then the following relations are satisfied:

(i) The tangential and principal normal components of position vector of second kind osculating curve are expressed as respectively

$$\langle\alpha,T\rangle=c\rho^2\left(\frac{K_3}{K_2}\right)'e^{-\int K_1 d\sigma},$$

$$\langle\alpha,\eta\rangle=-c\rho^2\frac{K_3}{K_2}e^{-\int K_1 d\sigma}.$$

(ii) The second binormal component of position vector of second kind osculating curve is expressed as

$$\langle\alpha,\zeta_2\rangle=-c\rho^2e^{-\int K_1 d\sigma}.$$

6. Conclusion

This study gives rectifying, normal and second kind osculating curves according to equiform geometry in E_1^4. We examine rectifying, normal, and second-kind osculating curves based on their curvature functions and identify the essential and sufficient conditions for rectifying, normal and second kind osculating curves to be congruent to rectifying, normal and second kind osculating curves according to equiform geometry in E_1^4, respectively.

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Statement of Research and Publication Ethics

The study adheres to research and publication ethics.

References

[1] A. T. Ali and M. Önder, "Some characterizations of spacelike rectifying curves in the Minkowski space-time", *Glob J Sci Front Res Math Decision Sci*, vol. 12, no. 1, pp. 57-64, 2012.

[2] B. Y. Chen, "When does the position vector of a space curve always lie in its rectifying plane?", *Amer. Math. Monthly*, vol. 110, pp. 147-152, 2003.

[3] B. Y. Chen and F. Dillen, "Rectifying curves as centrodes and extremal curves", *Bull. Inst. Math. Academia Sinica*, vol. 2, pp. 77-90, 2005.

[4] B. O'Neill, "*Semi-Riemannian geometry with applications to relativity*", Academic Press, London: 1983.

[5] H.S. Abdel-Aziz, M.K. Khalifa Saad, and A.A. Abdel-Salam, "Equiform Differential Geometry of Curves in Minkowski Space-Time", 2015, arXiv:1501.02283v1 [math DG] January.

[6] J. Walrave, "Curves and surfaces in Minkowski space", Ph.D. dissertation, Leuven University,1995.

[7] K. İlarslan and E. Nešovic, "Some characterizations of rectifying curves in the Euclidean space E^4", *Turkish. J. Math.*, vol. 32, no. 1, pp. 21-30, 2008.

[8] K. İlarslan, E. Nešovic, and M. Petrovic-Torgasev, "Some characterizations of rectifying curves in the Minkowski 3-space", *Novi Sad J. Math.*, vol. 33, no. 2, pp. 23-32, 2003.

[9] K. İlarslan and E. Nešovic, "Some characterizations of null, pseudo null and partially null rectifying curves in Minkowski space-time", *Taiwanese J. Math.*, vol. 12, no. 5, pp. 1035-1044, 2008.

[10] K. İlarslan, "Spacelike normal curves in Minkowski space E_1^3", *Turkish J. Math.*, vol. 29, no. 1, 53-63, 2005.

[11] K. İlarslan and E. Nešovic, "Timelike and null normal curves in Minkowski space E_1^3", *Indian J. Pure Appl. Math.*, vol. 35, no. 7, 881-888, 2004.

[12] K. İlarslan and E. Nešovic, "Spacelike and timelike normal curves in Minkowski space-time", *Publ. Inst. Math. (Belgrad) (N.S.)*, vol. 85, no. 99, pp. 111-118, 2009.

[13] K. İlarslan and E. Nešovic, "Some characterizations of osculating curves in the Euclidean spaces", *Demonstratio Mathematica*, vol. 16, pp. 931-939, 2008.

[14] K. İlarslan and E. Nešovic, "The first kind and the second kind osculating curves in Minkowski space-time", *Compt. Rend. Acad. Bulg. Sci.*, vol. 62, no. 6, pp. 677-686, 2009.

[15] K. İlarslan and E. Nešovic, "Some characterizations of null osculating curves in the Minkowski space-time", *Proceedings of the Estonian Academy of Sciences*, vol. 6, no. 1, pp. 1-8, 2012.

[16] K. İlarslan, N. Kılıç, and H. Altın Erdem, "Osculating curves in 4-dimensional semi-Euclidean space with index 2", *Open Mathematics*, vol. 15, pp. 562-567, 2017.

[17] R. Lopez, "Differential geometry of curves and surfaces in Lorentz-Minkowski space", *Int. Electron. J. Geom.*, vol. 3, pp. 67-101, 2010.

[18] S. Yılmaz and M. Turgut, "On the Differential Geometry of the curves in Minkowski space-time", *Int. J. Contemp. Math. Sci.*, vol. 3, no. 7, pp. 1343-1349, 2008.

[19] M. Elzawy and S. Mosa, "Equiform rectifying curves in Galilean space G^4", *Scientific African,* vol 22, e01931, 2023.

[20] M. Fakharany, A. El-Abed, M. Elzawy and S. Mosa, "On the geometry of equiform normal curves in the Galilean space G^4", *Inf. Sci. Lett.*, vol. 11, no. 5, pp. 1711-1715, 2022.

[21] D. W. Yoon, J. W. Lee and C. W Lee, "Osculating curves in the Galilean 4-space", *International Journal of Pure and Applied Mathematics*, vol. 100, no. 4, pp. 497-506, 2015.