
BENDING ANALYSIS OF COMPOSITE AND SANDWICH BEAMS USING RITZ METHOD

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ABSTRACT

In the present paper, the bending behaviour of laminated composite and sandwich beams subjected to various sets of boundary conditions which are simply supported (SS), clamped-simply supported (CS), clamped-clamped (CC) and clamped-free (CF) are investigated by using the Timoshenko beam theory and the Ritz method. In order to solve the problem, the shape functions for axial, transverse deflections and the rotation of the cross-section are presented in polynomial forms. The validation and convergence studies are performed by solving symmetric and anti-symmetric cross-ply composite beam problems with various boundary conditions and aspect ratios by adding auxiliary functions to the shape functions. The results in terms of mid-span deflections, axial and shear stresses are compared with those from previous studies to validate the accuracy of the present study. The effects of fiber angle, lay-up and aspect ratio on displacements and stresses are studied.

Keywords: Composite Beam, Bending, Ritz Method, Timoshenko Beam Theory

1. INTRODUCTION

In recent years, with the advance of the production technology for the composite materials, the use of composite beam structures has been increasing because of new demands in aerospace, marine, automotive, and civil engineering applications. Due to the attractive properties in strength, stiffness and lightness, various beam theories have been developed to understand the mechanical behavior of these structures during the last decade. In [1], the review of these theories can be found.

The kinematics, strain and stress relations of a beam can be represented by using various beam theories. These theories can be divided into three following categories: the Euler Bernoulli Beam Theory (EBT), the Timoshenko Beam Theory (TBT) and the Reddy-Bickford Beam Theory (RBT). Since the effect of the transverse shear deformation neglected in the EBT, it is only suitable for thin beams-. TBT overcomes this adverse by taking into account the shear deformation effect. However, the TBT requires the shear correction factor (SCF) to compensate the error due to the assumption of the constant transverse shear strain and shear stress through the beam thickness. The SCF depends on the geometric and material parameters of the beam but the loading and boundary conditions are also important to determine the SCF [2-3]. On the other hand, TBT cannot satisfy the zero traction boundary conditions on the top and bottom surfaces of the beam. Many higher order beam theories (HBT) including quasi-3D ones have been developed to study the bending behaviour of composite beams and only some of them [4-14] are referenced here. HBT does not require a shear correction factor, satisfies the zero traction boundary conditions and importantly allows having better prediction of static, dynamic and buckling responses of composite beams.

Analytical and numerical methods have been used to investigate the flexural behaviour of composite and sandwich beams. The finite element methods (FEMs) are the most popular ones for the analysis of composite beams [15-28]. There are few studies related to the flexure analysis of laminated composite and sandwich beams by employing a meshless method [29-33]. As an analytical approach, the Navier solution is the simplest one which can be used only for the solution of the problems with simply supported (SS) boundary condition [34-36]. In order to deal with arbitrary end conditions, many

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different methods have been developed. The most commonly used one is the Ritz method [37-41]. A critical review of literature regarding to the bending, buckling and free vibration of laminated composite and sandwich beams is given in [42].

As it is seen from the literature survey above, the studies related to flexure analysis of the laminated composite and sandwich beams by employing Ritz method are still limited. In [1], the trigonometric series solutions are presented for the static, buckling and free vibration responses of laminated composite beams. The vibration analysis of cross-ply laminated beams subjected to different sets of boundary conditions based on a three-degree-of-freedom shear deformable beam theory via Ritz method is presented in [37]. By using the same shear deformation theory given in [37], the vibration, buckling and thermal buckling of laminated composite beams are investigated in [38-40]. An analytical solution for the buckling and free vibration analysis of laminated beams by using a Quasi-3D theory and the Ritz method is given in [41]. Since the Ritz approach consists in permitting the analysis of any combination of boundary conditions, it is efficient to deal with static, buckling and vibration problems of composite beams. Furthermore, no restrictions on the stacking sequences exist, so that realistic configurations characterized by the presence of membrane and/or flexural anisotropy can be accounted for. The main scope of this work is to investigate the bending behaviour of the laminated composite and sandwich beams based on Timoshenko Beam Theory (TBT) by using Ritz method with polynomial shape functions. In the present paper, the static analysis of the laminated composite and sandwich beams are presented by considering various fiber angles, lay-ups, aspect ratios and sets of boundary conditions.

2. THEORY AND FORMULATION

In Figure 1, a laminated composite beam which is made of many plies of orthotropic materials in different orientations with respect to x-axis is presented. It is assumed that a lamina has no gaps or empty spaces, behaves like a linear elastic material and is bounded perfectly to each other.

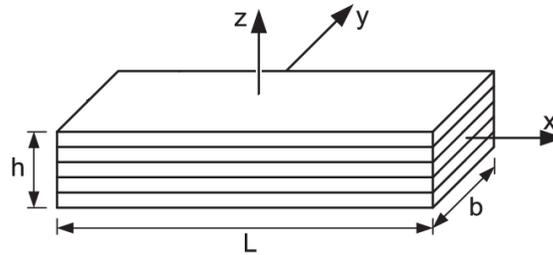


Figure 1. Geometry of a laminated composite beam

Where h is the height of the beam, b is the width and L is the length. The stress-strain relationship of the k^{th} orthotropic lamina in the material coordinate axes is given by [5]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} \quad (1)$$

where $(\sigma_{xx}, \sigma_{xz})$ are the axial and shear stresses and $(\varepsilon_{xx}, \gamma_{xz})$ are the axial strain and shear strain, respectively with respect to the laminate axes. Q_{ij} 's are the transformed elastic constants or stiffness matrix with respect to laminate axis x . The transformed elastic constants can be given by [8]:

$$\begin{aligned} Q_{11} &= C_{11} \cos^4 \theta + 2(C_{12} + 2C_{66}) \cos^2 \theta \sin^2 \theta + C_{22} \sin^4 \theta \\ Q_{55} &= C_{44} \sin^2 \theta + C_{55} \cos^2 \theta \end{aligned} \quad (2)$$

Where

$$C_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}; C_{12} = \frac{E_1\nu_{21}}{1 - \nu_{12}\nu_{21}}; C_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}; C_{66} = G_{12}; C_{55} = G_{13}; C_{44} = G_{23};$$

$E_1, E_2, G_{12}, G_{13}, G_{23}, \nu_{12}$ and ν_{21} are the six independent engineering constants. E is the Young's Modulus, G is the Shear Modulus and ν is the Poisson's ratio.

To describe the TBT, the following coordinate system is introduced. The x -coordinate is taken along the axis of the beam and the z -coordinate is taken through the height (thickness) of the beam. In the general beam theory, all the loads and the displacements (u, w, ϕ) along the coordinates (x, z) are only the functions of the x and z coordinates.

The following displacement field is given for the TBT [5],

$$\begin{aligned} U(x, z) &= u(x) + z\phi(x) \\ W(x, z) &= w(x) \end{aligned} \quad (3)$$

Here u and w are the axial and transverse displacements of any point on the neutral axis, ϕ is the rotation of the cross sections. The non zero strains can be given as

$$\begin{aligned} \varepsilon_{xx} &= \frac{dU}{dx} = \frac{du}{dx} + z \frac{d\phi}{dx} \\ \gamma_{xz} &= \frac{dU}{dz} + \frac{dW}{dx} = \phi + \frac{dw}{dx} \end{aligned} \quad (4)$$

The strain energy of the beam including the energy associated with the shearing strain can be written as,

$$U = \frac{1}{2} \int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz}) dV \quad (5)$$

where V is the volume of the beam. By substituting Eqs. (1) and (4) into Eq. (5), the strain energy can be obtained:

$$U = \frac{1}{2} \int_V \left[Q_{11} \left\{ \left(\frac{du}{dx} \right)^2 + z^2 \left(\frac{d\phi}{dx} \right)^2 + 2z \frac{du}{dx} \frac{d\phi}{dx} \right\} + Q_{55}\kappa_s \left\{ \phi^2 + \left(\frac{dw}{dx} \right)^2 + 2\phi \frac{dw}{dx} \right\} \right] dV \quad (6)$$

where κ_s is the shear correction factor to be used to compensate the error caused by the assumption of a constant transverse shear stress distribution along the beam thickness. The potential energy of the load $q(x)$ is given by

$$V = - \int_{-L/2}^{L/2} qwdx \quad (7)$$

The stiffness coefficients can be introduced as follows:

$$(A, B, D) = b \int_{-h/2}^{+h/2} Q_{11}(1, z, z^2) dz \quad (8)$$

$$A_s = b \int_{-h/2}^{+h/2} Q_{55} dz \quad (9)$$

By using Eqs. 6 to 9, the total potential energy (Π) can be written:

$$\Pi = U + V$$

$$\Pi = \frac{1}{2} \int_{-L/2}^{L/2} \left[A \left(\frac{du}{dx} \right)^2 + D \left(\frac{d\phi}{dx} \right)^2 + 2B \frac{du}{dx} \frac{d\phi}{dx} + \kappa_s A_s \phi^2 + \kappa_s A_s \left(\frac{dw}{dx} \right)^2 + 2\kappa_s A_s \phi \frac{dw}{dx} - 2qw \right] dx \quad (10)$$

The mid-span deflections, axial and shear stresses are obtained by using the Ritz method. The Ritz method is based on variational statements such as principles of virtual displacements or the principle of the minimum potential energy, which are equivalent to the governing equations as well as the natural boundary conditions. The displacement functions $u(x)$, $w(x)$ and the rotation of the cross section $\phi(x)$ are presented by using the following polynomial series which satisfy the kinematic boundary conditions given in Table 1,

$$u(x) = \sum_{j=1}^m a_j \theta_j(x), \quad \theta_j(x) = \left(x + \frac{L}{2} \right)^{p_u} \left(x - \frac{L}{2} \right)^{q_u} x^{j-1} \quad (11a)$$

$$w(x) = \sum_{j=1}^m b_j \varphi_j(x), \quad \varphi_j(x) = \left(x + \frac{L}{2} \right)^{p_w} \left(x - \frac{L}{2} \right)^{q_w} x^{j-1} \quad (11b)$$

$$\phi(x) = \sum_{j=1}^m c_j \psi_j(x), \quad \psi_j(x) = \left(x + \frac{L}{2} \right)^{p_\phi} \left(x - \frac{L}{2} \right)^{q_\phi} x^{j-1} \quad (11c)$$

where a_j , b_j and c_j are unknown values to be determined, $\theta_j(x)$, $\varphi_j(x)$ and $\psi_j(x)$ are the shape functions which are proposed for the boundary conditions (BC) to be studied within this paper, p_ξ and q_ξ ($\xi = u, w, \phi$) are the boundary exponents of auxiliary functions related with the boundary conditions given in Table 2. It has to be mentioned that the shape functions which do not satisfy the boundary conditions may cause slow convergence rates and numerical instabilities.

Table 1. Kinematic boundary conditions used for the numerical computations

BC	$x=-L/2$	$x=L/2$
S-S	$u = 0, w = 0$	$w = 0$
C-S	$u = 0, w = 0, \phi = 0, w' = 0$	$w = 0$
C-C	$u = 0, w = 0, \phi = 0, w' = 0$	$u = 0, w = 0, \phi = 0, w' = 0$
C-F	$u = 0, w = 0, \phi = 0, w' = 0$	

Table 2. Boundary exponents for various boundary conditions

BC	Left end			Right end		
	p_u	p_w	p_ϕ	q_u	q_w	q_ϕ
SS	1	1	0	0	1	0
CS	1	1	1	0	1	0
CC	1	1	1	1	1	1
CF	1	1	1	0	0	0

One can substitute Eq. (11) into Eq. (10) and then use the principle of the minimum potential energy given by Eq. (12) to obtain the system of equations and determine the values of a_j , b_j and c_j . As the number of parameters (m) is increased, the approximate solution converges to the true solution of the problem.

$$\frac{\partial \Pi}{\partial a_j} = 0, \quad \frac{\partial \Pi}{\partial b_j} = 0, \quad \frac{\partial \Pi}{\partial c_j} = 0, \quad j = 1, 2, 3, \dots, m \quad (12)$$

3. NUMERICAL RESULTS

This section is dedicated to understand the flexure behaviour of the composite beams based on the TBT formulation and Ritz method. The computed results which are obtained by employing different number of terms in the polynomial series expansions are used for convergence and verification studies. The results are presented in terms of displacements and stresses of composite beams considering various lay-ups, aspect ratios and boundary conditions. The results from previous studies [5,8] in terms of dimensionless mid-span deflections, axial and shear stresses are used for comparison purposes. Three different aspect ratios (L/h) 5, 10 and 50 are considered. The shear correction factor is set to $5/6$. The material properties of the problems studied within this paper are given in Table 3.

The following non-dimensional quantities are used for the representation of the results;
Non-dimensional maximum transverse deflection of the beam:

$$\bar{w} = \frac{100E_m b h^3}{q_0 L^4} w(0, z) \quad (13)$$

Non-dimensional axial and shear stresses of the beam:

$$\begin{aligned} \bar{\sigma}_x &= \frac{b h^2}{q_0 L^2} \sigma_x(0, z) \\ \bar{\sigma}_{xz} &= \frac{b h}{q_0 L} \sigma_{xz} \left(-\frac{L}{2}, z \right) \end{aligned} \quad (14)$$

Table 3. Material properties of composite and sandwich beams

Problem	Structure	Material Properties
1	Type A	$E_1/E_2 = 25; E_3 = E_2; G_{12} = G_{13} = 0.5E_2; G_{23} = 0.2E_2$ $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$
2	Type B	Face Layer: Type A
		Core Layer: $E_1/E_2 = 1; E_3 = E_2; G_{12} = G_{13} = 1.5E_2; G_{23} = 0.4E_2$ $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$

3.1. Verification, Comparison and Convergence Studies

The developed code is verified by solving symmetric and anti-symmetric cross-ply composite beams subjected to uniformly distributed load with various boundary conditions (SS, CF, CC and CS) and aspect ratios with respect to the different series number m .

The results in terms of non-dimensional mid-span deflections, axial and shear stresses are given in Tables 4-5 along with the results from previous studies. It is clear that the results obtained by using the Ritz method agree completely with those of previous papers [5,8]. It is found that for the static analysis, the responses converge quickly for all types of boundary conditions when m is set to 4 as it is seen from Tables 4 and 5. For the sake of accuracy, the extensive studies are performed by employing the series number m as 6.

Table 4. Verification and convergence studies, dimensionless mid-span deflections for different number of terms.

Theory	Reference	Symmetric (0°/90°/0°)			Anti-symmetric (0°/90°)		
		L/h=5	10	50	L/h=5	10	50
a. Simply Supported Beams (S-S)							
TBT	Khdeir and Reddy [5]	2.146	1.021	0.661	5.036	3.750	3.339
	2 terms	2.0171	0.8921	0.5321	4.3716	3.0858	2.6744
TBT	4 terms	2.1464	1.0214	0.6614	5.0359	3.7502	3.3387
	6 terms	2.1464	1.0214	0.6614	5.0359	3.7502	3.3387
	8 terms	2.1464	1.0214	0.6614	5.0359	3.7502	3.3387
b. Cantilever Beams (C-F)							
TBT	Khdeir and Reddy [5]	6.698	3.323	2.243	16.436	12.579	11.345
	2 terms	6.5018	2.9700	1.2103	14.5143	8.9728	5.5320
TBT	4 terms	6.6978	3.3228	2.2428	16.4362	12.5791	11.3448
	6 terms	6.6978	3.3228	2.2428	16.4362	12.5791	11.3448
	8 terms	6.6978	3.3228	2.2428	16.4362	12.5791	11.3448
c. Cantilever Beams (C-C)							
TBT	Khdeir and Reddy [5]	1.629	0.504	0.144	2.379	1.093	0.681
	2 terms	1.6005	0.4675	0.0412	2.1834	0.7756	0.0543
TBT	4 terms	1.6293	0.5043	0.1443	2.3786	1.0929	0.6815
	6 terms	1.6293	0.5043	0.1443	2.3786	1.0929	0.6815
	8 terms	1.6293	0.5043	0.1443	2.3786	1.0929	0.6815
d. Cantilever Beams (C-S)							
TBT	Khdeir and Reddy [5]	1.922	0.693	0.276	3.320	1.834	1.349
	2 terms	1.7923	0.5638	0.1471	2.6554	1.1701	0.6847
TBT	4 terms	1.9216	0.6931	0.2764	3.3197	1.8345	1.3490
	6 terms	1.9216	0.6931	0.2764	3.3197	1.8345	1.3490
	8 terms	1.9216	0.6931	0.2764	3.3197	1.8345	1.3490

Table 5. Verification and convergence studies, dimensionless axial $\bar{\sigma}_x \left(0, \frac{h}{2}\right)$ and shear $\bar{\sigma}_{xz} \left(-\frac{L}{2}, 0\right)$ stresses of S-S beams for different number of terms

Theory	Reference	Symmetric (0°/90°/0°)			Anti-symmetric (0°/90°)		
		L/h=5	10	50	L/h=5	10	50
a. Axial (Normal) Stress							
TBT	Zenkour [8]	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	2 terms	0.5184	0.5184	0.5184	0.1557	0.1557	0.1557
TBT	4 terms	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	6 terms	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	8 terms	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
b. Shear Stress							
TBT	Zenkour [8]	0.2994	0.2994	0.2994	0.8553	0.8553	0.8553
	2 terms	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571
TBT	4 terms	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571
	6 terms	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571
	8 terms	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571

3.2. Bending Analysis of Laminated Composite and Sandwich Beams

Four different boundary conditions, SS, CS, CC and CF are considered respectively for the bending analysis of laminated composite and sandwich beams subjected to uniformly distributed load. The mid-span deflections, axial and shear stresses are computed based on the TBT theories, lay-ups, fiber angles and aspect ratios.

3.2.1. Laminated Composite Beams: Type A

The symmetric [0°/θ/0°] and un-symmetric [0°/θ] composite beams are considered. In Tables 6 and 7, variations of mid-span displacements, axial and shear stresses respect to the fiber angle (θ) are given.

As the fiber angle increases, mid-span deflections and maximum axial stress values increase for all type of boundary conditions and aspect ratios.

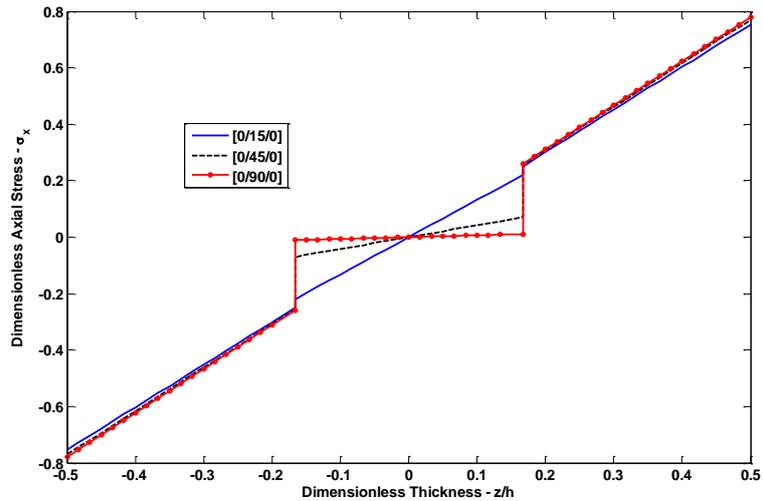
As the aspect ratio increases the mid-span deflection decreases. One can easily observe that the axial and shear stresses remain same as the aspect ratio increases. It is clear that from Figures 2 and 3, as the fiber angle increases, the dimensionless axial and shear stresses increase for all type of boundary conditions and aspect ratios. The discontinuities are visible for all types of composite beam structures.

Table 6. Dimensionless mid-span deflections of $[0^\circ/\theta/0^\circ]$ and $[0^\circ/\theta]$ beams for various boundary conditions under a uniformly distributed load, Type A.

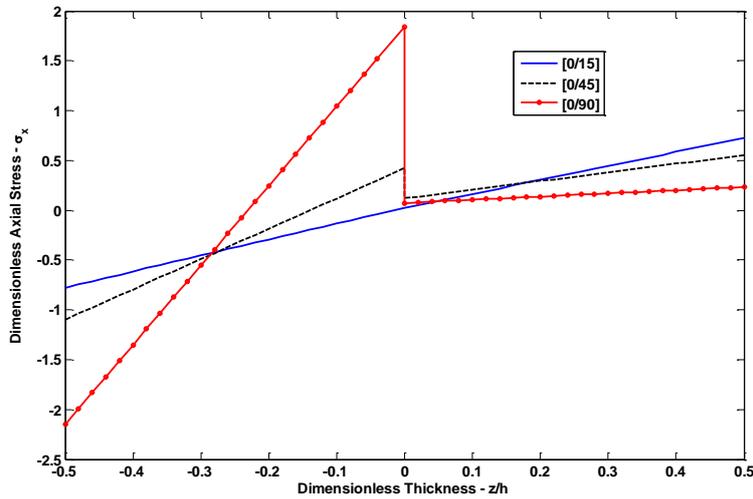
Aspect Ratio (L/h)	Theory	Lay-ups	0°	15°	30°	45°	60°	75°	90°	
a. Simply supported beams (S-S)										
5	TBT	$[0^\circ/\theta]$	1.8234	1.8910	2.1276	2.6757	3.7836	4.8467	5.0359	
		$[0^\circ/\theta/0^\circ]$	1.8234	1.8426	1.8964	1.9737	2.0566	2.1216	2.1464	
10		$[0^\circ/\theta]$	0.9234	0.9726	1.1547	1.6169	2.6223	3.5968	3.7502	
		$[0^\circ/\theta/0^\circ]$	0.9234	0.9304	0.9490	0.9737	0.9978	1.0151	1.0214	
50		$[0^\circ/\theta]$	0.6354	0.6787	0.8433	1.2780	2.2507	3.1969	3.3387	
		$[0^\circ/\theta/0^\circ]$	0.6354	0.6385	0.6458	0.6537	0.6590	0.6610	0.6614	
b. Clamped simply supported beams (C-S)										
5		TBT	$[0^\circ/\theta]$	1.5899	1.6371	1.7929	2.1135	2.6811	3.2070	3.3197
			$[0^\circ/\theta/0^\circ]$	1.5899	1.6087	1.6623	1.7409	1.8269	1.8953	1.9216
10			$[0^\circ/\theta]$	0.5983	0.6229	0.7107	0.9194	1.3500	1.7637	1.8345
	$[0^\circ/\theta/0^\circ]$		0.5983	0.6041	0.6203	0.6432	0.6674	0.6860	0.6931	
50	$[0^\circ/\theta]$		0.2636	0.2811	0.3475	0.5223	0.9125	1.2919	1.3490	
	$[0^\circ/\theta/0^\circ]$		0.2636	0.2649	0.2683	0.2720	0.2747	0.2760	0.2764	
c. Cantilever beams (C-F)										
5	TBT		$[0^\circ/\theta]$	5.7197	5.9398	6.7150	8.5326	12.2449	15.8121	16.4363
			$[0^\circ/\theta/0^\circ]$	5.7197	5.7783	5.9424	6.1774	6.4278	6.6232	6.6978
10			$[0^\circ/\theta]$	3.0197	3.1844	3.7961	5.3561	8.7611	12.0626	12.5791
		$[0^\circ/\theta/0^\circ]$	3.0197	3.0416	3.1002	3.1774	3.2513	3.3038	3.3228	
50		$[0^\circ/\theta]$	2.1557	2.3027	2.8621	4.3397	7.6462	10.8628	11.3448	
		$[0^\circ/\theta/0^\circ]$	2.1557	2.1659	2.1908	2.2174	2.2348	2.2416	2.2428	
d. Clamped clamped beams (C-C)										
5		TBT	$[0^\circ/\theta]$	1.3247	1.3579	1.4634	1.6645	1.9954	2.3025	2.3786
			$[0^\circ/\theta/0^\circ]$	1.3247	1.3416	1.3898	1.4614	1.5407	1.6046	1.6293
10			$[0^\circ/\theta]$	0.4247	0.4394	0.4904	0.6057	0.8341	1.0527	1.0929
	$[0^\circ/\theta/0^\circ]$		0.4247	0.4293	0.4424	0.4614	0.4819	0.4981	0.5043	
50	$[0^\circ/\theta]$		0.1367	0.1455	0.1790	0.2669	0.4625	0.6527	0.6815	
	$[0^\circ/\theta/0^\circ]$		0.1367	0.1374	0.1393	0.1414	0.1431	0.1440	0.1443	

Table 7. Dimensionless axial $\bar{\sigma}_x \left(0, \frac{h}{2}\right)$ and shear $\bar{\sigma}_{xz} \left(-\frac{L}{2}, 0\right)$ stresses of $[0^\circ/\theta/0^\circ]$ and $[0^\circ/\theta]$ S-S beams under a uniformly distributed load, Type A.

Aspect Ratio (L/h)	Theory	Lay-ups	0°	15°	30°	45°	60°	75°	90°	
a. Axial stress										
5	TBT	$[0^\circ/\theta]$	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336	
		$[0^\circ/\theta/0^\circ]$	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776	
10		$[0^\circ/\theta]$	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336	
		$[0^\circ/\theta/0^\circ]$	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776	
50		$[0^\circ/\theta]$	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336	
		$[0^\circ/\theta/0^\circ]$	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776	
e. Shear stress										
5		TBT	$[0^\circ/\theta]$	0.6000	0.6123	0.6486	0.7059	0.7742	0.8332	0.8571
			$[0^\circ/\theta/0^\circ]$	0.6000	0.5837	0.5368	0.4667	0.3882	0.3247	0.3000
10			$[0^\circ/\theta]$	0.6000	0.6123	0.6486	0.7059	0.7742	0.8332	0.8571
	$[0^\circ/\theta/0^\circ]$		0.6000	0.5837	0.5368	0.4667	0.3882	0.3247	0.3000	
50	$[0^\circ/\theta]$		0.6000	0.6123	0.6486	0.7059	0.7742	0.8332	0.8571	
	$[0^\circ/\theta/0^\circ]$		0.6000	0.5837	0.5368	0.4667	0.3882	0.3247	0.3000	



a. Symmetric lay-up



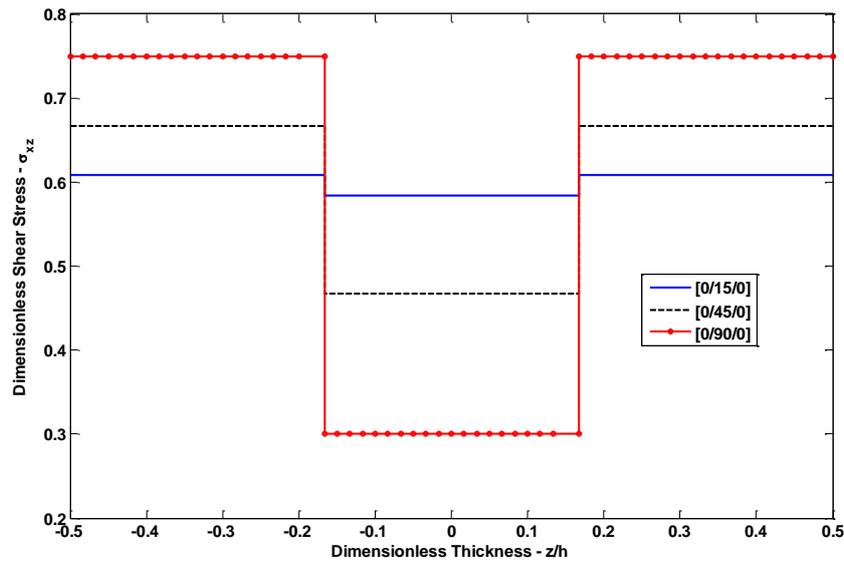
b. Anti-symmetric lay-up

Figure 2. Axial stress distribution through the thickness of symmetric and anti-symmetric beams with S-S boundary condition based on TBT, Type A, $L/h=5$.

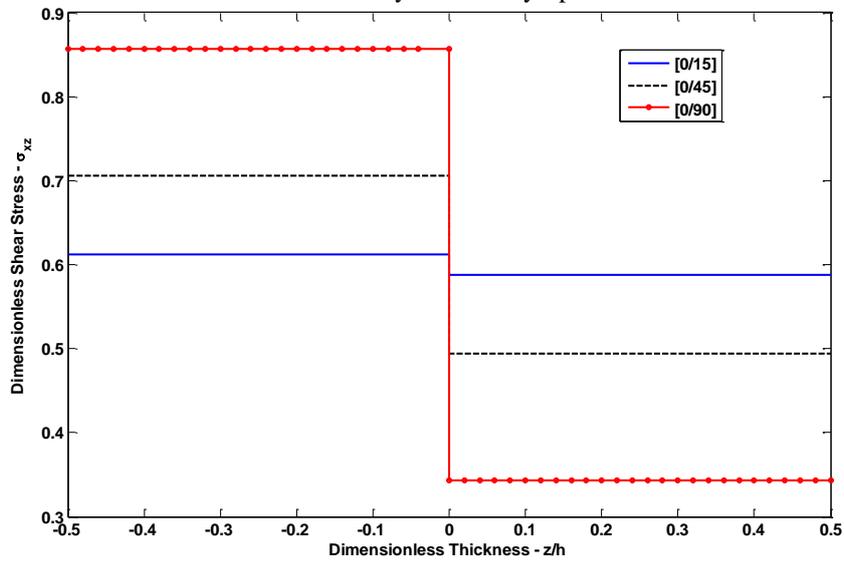
3.2.2. Laminated Composite Sandwich Beams: Type B

In this example, the elasto-static analysis of cross-ply sandwich beams (Type B) under uniformly distributed load with the top and bottom face thickness (h_1) and core thickness (h_2) are studied. Based on the various thickness and aspect ratios, the dimensionless mid-span deflections and stresses are presented by using Ritz method in Tables 8 and 9. It is clear that the dimensionless mid-span deflections increase as the thickness ratio increases.

The dimensionless axial and shear stress variations through thickness of the sandwich beams are plotted in Figure 4 for different thickness ratios. It is found that the stresses increase as the thickness ratio changes from 3 to 8. The maximum dimensionless axial and shear stresses are obtained for the thickness value at 8.



a. Symmetric lay-up



b. Anti-symmetric lay-up

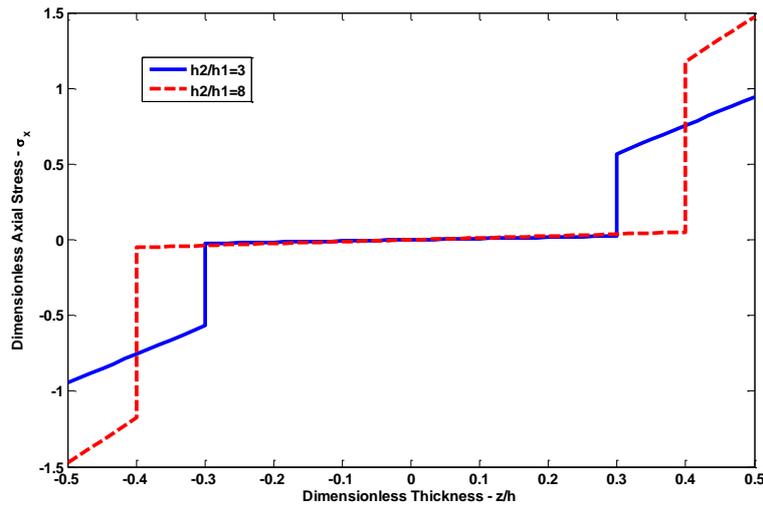
Figure 3. Shear stress distribution through the thickness of symmetric and anti-symmetric beams with S-S boundary condition based on TBT, Type A, $L/h=5$.

Table 8. Dimensionless mid-span deflections of $[0^\circ/90^\circ/0^\circ]$ beams for various boundary conditions under a uniformly distributed load, Type B.

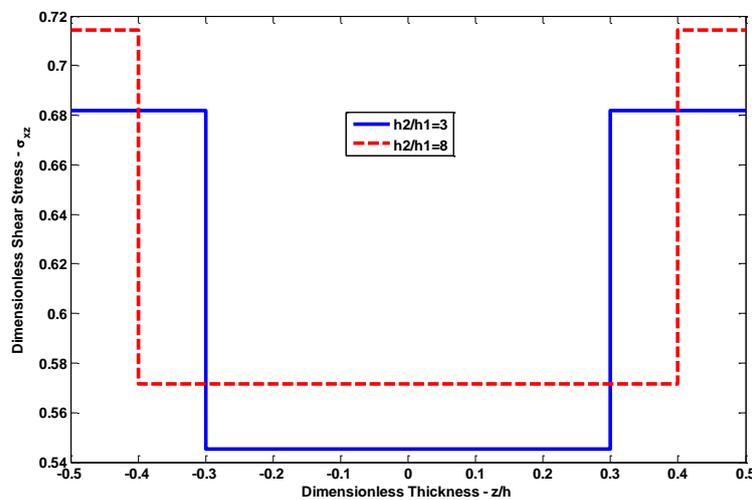
Theory	$h_2/h_1 = 3$				$h_2/h_1 = 8$			
	$L/h=5$	10	20	50	$L/h=5$	10	20	50
a. Simply supported beams (S-S)								
TBT	2.1496	1.1269	0.8712	0.7996	2.6515	1.5801	1.3122	1.2372
b. Clamped simply supported beams (C-S)								
TBT	1.8438	0.7116	0.4151	0.3306	2.1140	0.9077	0.5948	0.5061
c. Cantilever beams (C-F)								
TBT	6.7634	3.6951	2.9280	2.7133	8.4437	5.2294	4.4258	4.2008
d. Clamped clamped beams (C-C)								
TBT	1.5208	0.4981	0.2424	0.1708	1.6732	0.6017	0.3339	0.2589

Table 9. Dimensionless axial $\bar{\sigma}_x\left(0, \frac{h}{2}\right)$ and shear $\bar{\sigma}_{xz}\left(-\frac{L}{2}, 0\right)$ stresses of $[0^\circ/90^\circ/0^\circ]$ S-S beams under a uniformly distributed load, Type B.

Theory	$h_2/h_1 = 3$				$h_2/h_1 = 8$			
	L/h=5	10	20	50	L/h=5	10	20	50
a. Axial stress								
TBT	0.9455	0.9455	0.9455	0.9455	1.4712	1.4712	1.4712	1.4712
b. Shear stress								
TBT	0.5455	0.5455	0.5455	0.5455	0.5714	0.5714	0.5714	0.5714



a. Axial Stress



b. Shear Stress

Figure 4. Axial and shear stress distribution through the thickness of symmetric sandwich beams with S-S boundary condition based on TBT, Type B, L/h=5.

4. CONCLUSION

The polynomial series solution for the flexure behaviour of the laminated composite and sandwich beams using TBT formulation and Ritz method is presented. Various composite and sandwich beam configurations are considered based on different aspect ratios, lay-ups, fiber angles and boundary conditions. The shape functions for axial and transverse deflections and the rotation of the cross

sections are assumed to be in polynomial forms to obtain the approximate solution. The computed results are compared with the calculations obtained by other authors.

The following results can be drawn from the computed results based on the TBT:

- Since the all coupling effects from material vanish for the fiber angle value 0° , the axial displacement u cannot be obtained.
- Flexural behaviour of the laminated composite beams can be controlled to meet the desired goals by choosing suitable fiber angle.
- Regarding to the anti-symmetric laminated composite beams (Type A), the importance of the shear effect increases while the fiber angle increases for all type of boundary conditions.
- The fiber angles has a significant effect on the mid-span deflections, axial and shear stresses of the symmetric and anti-symmetric laminated composite beams (Type A) for all type of boundary conditions and aspect ratios.
- It is explicit that the difference between the computed results in terms of mid-span deflections, axial and shear stresses for the symmetric and anti-symmetric laminated composite beams (Type A) increase as the fiber angle increases.
- It is worth noting that C-F laminated composite sandwich beam (Type B) is much more sensitive to the thickness ratio change than the other sandwich beam models.
- According to the analysis for S-S laminated composite sandwich beam (Type B), the difference in terms of axial stress values is larger than those of obtained by using shear stress values with respect to thickness ratio change.

It is found that using the Ritz method with polynomial shape functions including auxiliary functions is simple to implement, efficient and provides quick convergence rates and expected results for the static analysis of laminated composite and sandwich beams.

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