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Research Article

Double phase variable exponent problems with nonlinear matrices diffusion

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Abstract

This work tackles a class of *double phase* elliptic problems with variable exponents and matrices diffusion. Under suitable assumptions on the data, we use critical point theory to establish both the existence and uniqueness of weak solutions to the *double phase* problem under consideration.

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1. Introduction

In this work, we are concerned with a nonlinear elliptic problem whose prototype reads as

$$\begin{cases} -\operatorname{div}\left(\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}} A\nabla u + \omega(x)\langle B\nabla u, \nabla u\rangle^{\frac{q(x)-2}{2}} B\nabla u\right) = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. The variable exponents $p(\cdot), q(\cdot) : \overline{\Omega} \to [2, +\infty)$ are two continuous functions satisfying $2 \le p(x) < N$ and $p(x) < q(x)$ for all $x \in \overline{\Omega}$. The function $\omega(\cdot)$ is nonnegative function belonging to a certain Lebesgue space. The source term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which belongs to a suitable Lebesgue type space to be discuss later. We assume that

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$A = (a_{i,j})_{1\le i,j\le N}$, $B = (b_{i,j})_{1\le i,j\le N}$ are two symmetric diffusion matrices, i.e. $a_{i,j} = a_{j,i}$ and $b_{i,j} = b_{j,i}$ satisfying

$$a_{i,j}, b_{i,j} \in L^\infty(\Omega) \cap \mathcal{C}^1(\Omega), \tag{1.2}$$

$$\langle A\xi, \xi\rangle = \sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \ge \lambda_A |\xi|^2, \tag{1.3}$$

$$\langle B\xi, \xi\rangle = \sum_{i,j=1}^{N} b_{i,j}(x)\xi_i\xi_j \ge \lambda_B |\xi|^2, \tag{1.4}$$

for all $\xi \in \mathbb{R}^N$ and for almost $x \in \Omega$, with λ_A and λ_B being nonnegative constants, and $\langle \cdot, \cdot \rangle$ stands for the scalar product on $\mathbb{R}^N$.

To effectively highlight the novelty and originality of our work, we aim to review pertinent existing literature on the same subject. We initiate this process by establishing the connection between the operator-driven problem (1.1) and certain well-known operators from earlier literature. It is worth noting that when $\omega \equiv 0$, the operator involved problem (1.1) reduces to

$$\mathcal{A}_{p(x)}(u) := -\operatorname{div}\left(\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}} A\nabla u \right), \tag{1.5}$$

which is recognized as the $\mathcal{A}_{p(x)}$-Laplace operator. It constitutes a generalization of the so-called $p(x)$-Laplace operator, i.e. $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ with the latter obtained as a special case when A equals the identity operator I_d in (1.5). The investigation of partial differential equations involving the $p(x)$-Laplace operator has garnered escalating attention in recent years [2, 3, 13–15, 17, 24]. The heightened interest in studying these operators has been spurred by their diverse applications in areas such as image restoration [1, 18], mathematical biology [26], elastic mechanics [46, 47], electrorheological and thermorheological fluids [35, 39, 41]. Conversely, partial differential equations featuring the $\mathcal{A}_p(x)$-Laplace operator have undergone thorough examination in recent years. In [4], Alvino et al. delved into the eigenvalue problem for an elliptic equation involving the $\mathcal{A}_p$-Laplace operator with constant exponents ($p(x) \equiv p$). The authors elucidated certain properties of the first eigenvalue and its corresponding eigenfunction by establishing a Payne-Rayner-type inequality in the process. For further exploration of PDEs governed by the $\mathcal{A}_p$-Laplace operator with constant exponent, we refer the readers to [23, 40], along with references provided therein. In a recent development, Mihăilescu and Repovš [36] have tackled a nonlinear elliptic equation with $\mathcal{A}_{p(x)}$-Laplace operator. Under a suitable assumption on the variable exponent $p(x)$, the authors combined Schauder's fixed point theorem with variational arguments to investigate the existence of solutions. In another scenario where $A \equiv B \equiv I_d$ and $\omega : \overline{\Omega} \to [0, \infty)$ is a measurable function, the problem (1.1) governed by the operator taking the form

$$\mathcal{L}(u) := -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \omega(x)|\nabla u|^{q(x)-2}\nabla u \right). \tag{1.6}$$

We would like to emphasize that the operator $\mathcal{L}$ is commonly known as the *double phase* operator, a term that has gained significant attention in recent years. The term "*double phase*" was originally introduced in [19, 20], indicating that the flux function $\left(|\nabla u|^{p(x)-2} + \omega(x)|\nabla u|^{q(x)-2}\right)\nabla u$ oscillates between two distinct elliptic scenarios. To phrase it differently, its behavior depends on the values assumed by the function $\omega : \overline{\Omega} \to [0, \infty)$. Specifically, over the subset $\{x \in \Omega : \omega(x) = 0\}$, the operator is governed by the gradient of $p(x)$-growth, whereas in scenarios where $\{x \in \Omega : \omega(x) \ne 0\}$, it is influenced by the gradient of $q(x)$-growth. This dual behavior is the fundamental reason for designating it as the "*double phase*" operator.

This type of operator has demonstrated its efficacy in modeling a diverse range of complex physical phenomena, including those arising in nonlinear elasticity, highly anisotropic materials, and the Lavrentiev phenomenon. For a deeper understanding of the real-world applications of such operators, we encourage interested readers to explore the following works [5, 6, 46–48]. It is crucial to highlight that the impetus for investigating this operator's advantages can be traced back to Zhikov's work [46], where the author delved into the behavior of highly anisotropic materials through the analysis of the following functional:

$$u \longmapsto \int_{\Omega} (|\nabla u|^p + \omega(x)|\nabla u|^q)\, dx. \tag{1.7}$$

The inclusion of the modulating coefficient $\omega(\cdot)$ serves to regulate the balance between two distinct materials, each characterized by power hardening rates of p and q respectively. In the existing literature, comprehensive investigations have explored the well-posedness of the functional (1.7). This scrutiny is conducted in various works, examining the functional in its original form [9, 27, 28, 33, 34, 37, 44], as well as in its role as a differential operator with variable growth behaviors, characterized by functions $p(\cdot)$ and $q(\cdot)$ [8, 10, 29, 42, 43, 45].

An important line of research, led by Mingione and his collaborators, has delved into the regularity of minimizers in variational problems and solutions to differential equations involving *double phase* operators, as documented in [7, 19, 20]. In a study by Cencelj-Rădulescu-Repovš [11], a class of *double phase* variational integrals was investigated, governed by non-uniform potentials with variable exponents $p(\cdot)$ and $q(\cdot)$. Their analysis of the corresponding Euler equation revealed the existence of two distinct Rayleigh quotients. One of these quotients is linked to the existence of an infinite range of eigenvalues, while the second one is associated with the nonexistence of eigenvalues. The study [12] presented a novel parabolic *double phase* model which was applied in the field of image processing. The authors not only proved the existence and uniqueness of solutions but also illustrated the practicality of *double phase* operators in reducing image noise. In the subsequent work [16], the same authors extended their research from [12] by considering a reaction-diffusion parabolic system with variable exponents, governed by the *double phase* operator. They examined the existence and uniqueness of weak solutions to the proposed problem and subsequently validated the robustness of their model in the context of removing noise from Magnetic Resonance Images. Their research marks a significant milestone as it represents the first instance of utilizing parabolic *double phase* problems for image denoising and decomposition.

In the case where $p(\cdot) \equiv p$ and $q(\cdot) \equiv q$ are constants, Liu and Dai [31] successfully derived a sign-changing ground-state solution for a *double phase* problem, employing the Nehari manifold approach. Their subsequent work [32] extended the investigation, demonstrating the existence of at least three ground-state solutions through the application of the strong maximum principle. In a recent contribution [22], the authors established the first Schauder-type results for minima of nonuniformly elliptic integrals with nearly linear growth, encompassing the *double phase* integral operator as a special case. Furthermore, the paper by Ho and Winkert [28] introduced new embedding results for Musielak-Orlicz Sobolev spaces of *double phase* type. Their methodology involved the use of De Giorgi iteration and localization arguments, yielding appropriate boundedness results for corresponding weak solutions to two types of *double phase* problems, one with Dirichlet conditions and the other with nonlinear boundary conditions.

The main contribution of this work is to study the existence and uniqueness of solutions to the innovative *double phase* problem (1.1). The key novelty of our paper lies in the integration of a *double phase* operator with matrices diffusion. To the best of our knowledge, this is the first instance of a study encompassing both of these concepts. Specifically, problem (1.1) encapsulates two noteworthy phenomena. Firstly, the operator involved in

(1.1) is the *double phase* operator, characterized by its ability to switch between two distinct elliptic situations. Secondly, our work introduces matrices A and B, which serve to generalize not only the *double phase* operator (1.6) but also the well-known $\mathcal{A}_{p(x)}$-Laplace operator (1.5).

The remainder of this paper is structured as follows: In Section 2, we commence by revisiting key definitions and notable properties of Lebesgue, Sobolev, and Musielak-Orlicz spaces with variable exponents. Section 3 is devoted to presenting our main findings. Using variational calculus theory, we establish two results concerning the existence and uniqueness of weak solutions to problem (1.1). The first pertains to the scenario when the source term is independent of the solution. Conversely, the second is dedicated to the case of strong nonlinear source terms.

2. Mathematical preliminaries

In this section, we offer a concise overview of fundamental properties of variable exponent Lebesgue-Sobolev spaces. For a more comprehensive understanding, we direct the reader to see the book [38]. Throughout this paper, we will consistently employ the following notation:

$$\mathcal{M}(\Omega) := \{u/\, u : \Omega \to \mathbb{R} \text{ measurable}\}\,.$$

2.1. Lebesgue and Sobolev spaces with variable exponent

The objective of this section is to establish the necessary foundation for our forthcoming analysis, centered around the Lebesgue and Sobolev spaces with variable exponents. Let us denote:

$$\mathcal{C}^+(\overline{\Omega}) = \left\{ p \in \mathcal{C}(\overline{\Omega}) : \inf_{x\in\overline{\Omega}} p(x) \geq 1 \right\}.$$

We define the Lebesgue space with a variable exponent as $L^{p(x)}(\Omega)$, where

$$L^{p(x)}(\Omega) = \{u \in \mathcal{M}(\Omega) \mid \quad \mathfrak{R}_{p(x)}(u) < \infty\},$$

here $\mathfrak{R}_{p(\cdot)}$ stands for the following convex modular

$$\mathfrak{R}_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

We endow the Lebesgue space $L^{p(x)}(\Omega)$ with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \delta > 0 \mid \mathfrak{R}_{p(x)}\left(\frac{u}{\delta}\right) \leq 1 \right\}.$$

For each $p \in \mathcal{C}^+(\overline{\Omega})$ given, we introduce the pair (p^-, p^+) as follows

$$p^- = \inf_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^+ = \sup_{x\in\overline{\Omega}} p(x).$$

Proposition 2.1 ([25], Theorems 1.2 and 1.3)**.** *Let $u \in L^{p(x)}(\Omega)$ and $\{u_n\} \subset L^{p(x)}(\Omega)$. Then, we have*

(i) *the inequalities are satisfied*

$$\min\left\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\right\} \leq \mathfrak{R}_{p(x)}(u) \leq \max\left\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\right\}, \tag{2.1}$$

$$\min\left\{\mathfrak{R}^{\frac{1}{p^-}}_{p(x)}(u), \mathfrak{R}^{\frac{1}{p^+}}_{p(x)}(u)\right\} \leq \|u\|_{L^{p(x)}(\Omega)} \leq \max\left\{\mathfrak{R}^{\frac{1}{p^-}}_{p(x)}(u), \mathfrak{R}^{\frac{1}{p^+}}_{p(x)}(u)\right\}. \tag{2.2}$$

(ii) *if $u_n \to u$ in $L^{p(x)}(\Omega)$, then the succeeding statements are equivalent:*

(a) $\lim\limits_{n\to+\infty} \|u_n - u\|_{L^{p(x)}(\Omega)} = 0.$

(b) $\lim_{n\to+\infty} \mathfrak{R}_{p(x)}(u_n - u) = 0.$

(c) $\lim_{n\to+\infty} \mathfrak{R}_{p(x)}(u_n) = \mathfrak{R}_{p(x)}(u)$ *and* $u_n \to u$ *in measure in* Ω.

Proposition 2.2 ([25], Theorems 1.6 and 1.10). *Let* $p \in \mathcal{C}^+(\overline{\Omega})$. *Then the space* $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\right)$ *is separable. Moreover, when* $p^- > 1$, *the generalized Lebesgue space* $L^{p(x)}(\Omega)$ *becomes uniform convex and thus is reflexive.*

For any $p \in \mathcal{C}^+(\overline{\Omega})$, we define the conjugate exponent of p as p' which meets $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \overline{\Omega}$, with the convention that $\frac{1}{\infty} = 0$. The subsequent proposition is commonly known as the generalized Hölder inequality.

Proposition 2.3 ([30], Theorem 2.1). *Let* $p \in \mathcal{C}^+(\overline{\Omega})$ *with* $p^- > 1$. *For any* $(u, v) \in L^{p(x)}(\Omega) \times L^{p'(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\,dx\right| \leq 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

The following proposition emphasizes notable embedding results for Lebesgue spaces with variable exponents.

Proposition 2.4 ([25], Theorem 1.11). *Let* $p_1,\ p_2 \in \mathcal{C}^+(\overline{\Omega})$ *with* $1 \leq p_1(x) \leq p_2(x)$ *almost everywhere in* Ω. *Then, we have the following continuous embedding* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We define the Sobolev space with a variable exponent as $W^{1,p(x)}(\Omega)$, where

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

We equip $W^{1,p(x)}(\Omega)$ with the norm $\|\cdot\|_{W^{1,p(x)}(\Omega)}$ defined as

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

It is not hard to prove that the above norm is equivalent to the one

$$\|u\|_{W^{1,p(x)}(\Omega)} = \inf\left\{\delta > 0 \mid \int_\Omega \left(\left|\frac{\nabla u(x)}{\delta}\right|^{p(x)} + \left|\frac{u(x)}{\delta}\right|^{p(x)}\right) dx \leq 1\right\}.$$

Hereafter, we say that the function $p(x)$ satisfies the log-Hölder continuity condition, if there exists a constant $C > 0$ such that

$$|p(x_1) - p(x_2)| \leq \frac{C}{-\log|x_1 - x_2|}, \quad \text{for all } x_1, x_2 \in \Omega, \quad \text{with } |x_1 - x_2| < \frac{1}{2}. \tag{2.3}$$

Assumption (2.3) ensures the density of the space of smooth functions $\mathcal{C}_c^\infty(\Omega)$ within $W^{1,p(x)}(\Omega)$. For added convenience, we introduce the space $W_0^{1,p(x)}(\Omega) := \overline{\mathcal{C}_c^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$, with its dual space denoted as $\left(W_0^{1,p(x)}(\Omega)\right)^*$. Furthermore, for any $u \in W_0^{1,p(x)}(\Omega)$, the generalized Poincaré inequality holds:

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)}, \tag{2.4}$$

where C is a nonnegative constant depending only on $p(x)$ and Ω. Taking advantage of generalized Poincaré inequality (2.4), we can establish the following equivalent norm on the space $W_0^{1,p(x)}(\Omega)$:

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

In a manner akin to $L^{p(x)}(\Omega)$, when $p^- > 1$, both the Banach spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are also separable and reflexive. The ensuing propositions succinctly outline noteworthy embedding results that establish connections between the spaces $W^{1,p(x)}(\Omega)$ and $L^{p(x)}(\Omega)$.

Proposition 2.5 ([25], Theorem 2.3)**.** *Let $(p(\cdot), q(\cdot)) \in \mathcal{C}^+(\overline{\Omega}) \times \mathcal{C}^+(\overline{\Omega})$ with $p^- > 1$ and $1 \leq q(x) < p^\star(x)$ for almost $x \in \overline{\Omega}$. Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact, where*

$$p^\star(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if} \quad p(x) < N \\ +\infty, & \text{if} \quad p(x) \geq N. \end{cases}$$

2.2. Orlicz-Sobolev spaces with variable exponent

This subsection provides a basic overview of Orlicz-Sobolev spaces, introducing key concepts. For a more in-depth exploration, readers are encouraged to see the work [21]. Let $(p(\cdot), q(\cdot)) \in \mathcal{C}^+(\overline{\Omega}) \times \mathcal{C}^+(\overline{\Omega})$ be such that $1 < p(x) < N$ and $p(x) < q(x)$ for all $x \in \overline{\Omega}$, we define $\mathcal{H} : \Omega \times [0, \infty) \to [0, \infty)$ as

$$\mathcal{H}(x, r) := r^{p(x)} + \omega(x) r^{q(x)}, \qquad \text{for all } (x, r) \in \Omega \times [0, \infty),$$

where ω is a nonnegative function belonging to $L^\infty(\Omega)$. It is straightforward to confirm that $\mathcal{H}$ is a generalized N-function and is locally integrable. We proceed to introduce the Musielak-Orlicz space, denoted as $L^{\mathcal{H}}(\Omega)$

$$L^{\mathcal{H}}(\Omega) = \{u \in \mathcal{M}(\Omega) \mid \mathfrak{R}_{\mathcal{H}}(u) < \infty\},$$

where the convex modular $\mathfrak{R}_{\mathcal{H}}$ is introduced as

$$\mathfrak{R}_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, |u|)\, dx.$$

and endowed $L^{\mathcal{H}}(\Omega)$ with the norm

$$\|u\|_{L^{\mathcal{H}}(\Omega)} = \inf\left\{\delta > 0 \mid \mathfrak{R}_{\mathcal{H}}\left(\frac{u}{\delta}\right) \leq 1\right\}.$$

Also, we introduce the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ as follows

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \mid |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}.$$

equipped $W^{1,\mathcal{H}}(\Omega)$ with the norm

$$\|u\|_{W^{1,\mathcal{H}}(\Omega)} = \|u\|_{L^{\mathcal{H}}(\Omega)} + \|\nabla u\|_{L^{\mathcal{H}}(\Omega)}.$$

Moreover, we denote by $W_0^{1,\mathcal{H}}(\Omega)$ the Musielak-Orlicz Sobolev spaces with zero traces, which, actually, represents the closure of $\mathcal{C}_c^\infty(\Omega)$ with respect to $W^{1,\mathcal{H}}(\Omega)$ norm, that is

$$W_0^{1,\mathcal{H}}(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{W^{1,\mathcal{H}}(\Omega)}.$$

With these norms, the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$, and $W_0^{1,\mathcal{H}}(\Omega)$ attain the properties of being separable, reflexive, and uniformly convex Banach spaces. The subsequent proposition elucidates the relationships between the modular $\mathfrak{R}_{\mathcal{H}}(\cdot)$ and the norm $\|\cdot\|_{L^{\mathcal{H}}(\Omega)}$.

Proposition 2.6 ([21])**.** *Let u be an element of $L^{\mathcal{H}}(\Omega)$. Then, we have*

(i) *the inequalities hold*

$$\min\left\{\|u\|_{L^{\mathcal{H}}(\Omega)}^{p^-}, \|u\|_{L^{\mathcal{H}}(\Omega)}^{q^+}\right\} \leq \mathfrak{R}_{\mathcal{H}}(u) \leq \max\left\{\|u\|_{L^{\mathcal{H}}(\Omega)}^{p^-}, \|u\|_{L^{\mathcal{H}}(\Omega)}^{q^+}\right\}. \tag{2.5}$$

(ii) *if* $p(x) < q(x) < p^\star(x)$ *for all* $x \in \Omega$ *and* $0 \le \omega(\cdot) \in L^\infty(\Omega)$*, then, we have*

$$W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \text{ is compact.}$$

(iii) *there exists a nonnegative constant* C *such that*

$$\|u\|_{L^{\mathcal{H}}(\Omega)} \le C\|\nabla u\|_{L^{\mathcal{H}}(\Omega)}, \qquad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega). \tag{2.6}$$

From (2.6), we can see that the following norm is an equivalent norm of $W_0^{1,\mathcal{H}}(\Omega)$

$$\|u\|_{W_0^{1,\mathcal{H}}(\Omega)} = \|\nabla u\|_{L^{\mathcal{H}}(\Omega)}, \qquad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

Notably, it is worth observing that the Lebesgue space with a variable exponent $L^{p(x)}(\Omega)$ is a particular instance of the Musielak-Orlicz space. In relation to the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$, we present the following set of embedding theorems.

Proposition 2.7 ([21])**.** *Let* $s(\cdot) \in \mathcal{C}(\overline{\Omega})$*. Then, we have*

(i) *if* $1 \le s(x) \le p(x)$*, for all* $x \in \Omega$*, then the embedding* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,s(x)}(\Omega)$ *is continuous.*

(ii) *if* $1 \le s(x) \le p^\star(x)$*, for all* $x \in \Omega$*, then the embedding* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ *is continuous.*

(iii) *if* $1 \le s(x) < p^\star(x)$*, for all* $x \in \Omega$*, then the embedding* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ *is compact.*

Throughout this paper, we assume that $(p,q) \in \mathcal{C}^+(\overline{\Omega}) \times \mathcal{C}^+(\overline{\Omega})$ take values within the intervals (p^-,p^+) and (q^-,q^+) respectively, satisfying log-Hölder continuity condition (2.3) such that

$$2 \le p(x) < N \qquad \text{and} \qquad p(x) < q(x). \tag{2.7}$$

Assumption (2.7) leads to employing the aforementioned properties of Lebesgue and Sobolev spaces with variable exponent. Furthermore, we assume that $\omega : \Omega \to [0,+\infty)$ is a measurable function belonging to $L^\infty(\Omega)$.

3. Main results

We begin this section by clarifying the notion of weak solutions used to solve problem (1.1).

Definition 3.1. A measurable function $u : \Omega \to \mathbb{R}$ is said to be a *weak* solution to problem (1.1) if it satisfies

$$\int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}} \langle A\nabla u, \nabla\varphi\rangle\, dx + \int_\Omega \omega(x)\, \langle B\nabla u, \nabla u\rangle^{\frac{q(x)-2}{2}} \langle B\nabla u, \nabla\varphi\rangle\, dx = \int_\Omega f(x,u)\varphi\, dx, \tag{3.1}$$

for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$.

In the following theorem, we present our first existence result, which pertains to the case where the source term $f(x,u)$ is independent of the steady-state solution.

Theorem 3.2. *Under the assumptions* (1.2)*,* (1.3)*,* (1.4) *and* (2.7)*, we assume that* $f(x,r) = f(x)$ *with* $f \in L^{\theta(x)}(\Omega)$*, where* $\theta \in \mathcal{C}^+(\overline{\Omega})$ *is such that* $\frac{1}{\theta(x)} + \frac{1}{p^\star(x)} = 1$*. Then, problem* (1.1) *admits a unique weak solution.*

Proof. Let us start by introducing the energy functional $\mathcal{J} : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ associated to problem (1.1) as follows

$$\mathcal{J}(u) = \int_\Omega \frac{1}{p(x)} \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\, dx + \int_\Omega \frac{\omega(x)}{q(x)} \langle B\nabla u, \nabla u\rangle^{\frac{q(x)}{2}}\, dx - \int_\Omega f(x)u\, dx. \tag{3.2}$$

By employing standard arguments, we can ensure that $\mathcal{J}$ belongs to $\mathcal{C}^1(W_0^{1,\mathcal{H}}(\Omega);\mathbb{R})$ and its derivative in the weak sense is given by

$$\left\langle \mathcal{J}'(u),\varphi\right\rangle = \int\limits_\Omega \langle A\nabla u,\nabla u\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u,\nabla\varphi\rangle\,dx + \int\limits_\Omega \omega(x)\langle B\nabla u,\nabla u\rangle^{\frac{q(x)-2}{2}}\langle B\nabla u,\nabla\varphi\rangle\,dx - \int\limits_\Omega f(x)\varphi\,dx,$$

for all $\varphi\in W_0^{1,\mathcal{H}}(\Omega)$. Therefore, the critical points of $\mathcal{J}$ precisely correspond to the weak solutions of problem (1.1). By analyzing (3.2) and taking into account (1.3) and (1.4), it follows that

$$\begin{aligned}\mathcal{J}(u) &\geq \int\limits_\Omega \frac{\lambda_A}{p(x)}|\nabla u|^{p(x)}\,dx + \int\limits_\Omega \frac{\lambda_B}{q(x)}\omega(x)|\nabla u|^{q(x)}\,dx - \|f\|_{L^{\theta(x)}(\Omega)}\|u\|_{L^{p^\star(x)}(\Omega)},\\ &\geq \frac{\lambda_A}{p^+}\int\limits_\Omega |\nabla u|^{p(x)}\,dx + \frac{\lambda_B}{q^+}\int\limits_\Omega \omega(x)|\nabla u|^{q(x)}\,dx - C\|f\|_{L^{\theta(x)}(\Omega)}\|u\|_{W_0^{1,\mathcal{H}}(\Omega)},\end{aligned}$$

where C is the Sobolev embedding constant of $W_0^{1,\mathcal{H}}(\Omega)\hookrightarrow L^{p^\star(x)}(\Omega)$. We mention that the later inequality is obtained via the use of Hölder inequality with the help of Proposition (2.7). Then, we have

$$\mathcal{J}(u)\geq \min\left\{\frac{\lambda_A}{p^+},\frac{\lambda_B}{q^+}\right\}\int\limits_\Omega\left(|\nabla u|^{p(x)}+\omega(x)|\nabla u|^{q(x)}\right)dx - C\|f\|_{L^{\theta(x)}(\Omega)}\|u\|_{W_0^{1,\mathcal{H}}(\Omega)}. \quad (3.3)$$

By taking advantage of (2.5), it results from (3.3) that for any $u\in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\|_{W_0^{1,\mathcal{H}}(\Omega)}>1$, one has

$$\mathcal{J}(u)\geq \min\left\{\frac{\lambda_A}{p^+},\frac{\lambda_B}{q^+}\right\}\|u\|^{p^-}_{W_0^{1,\mathcal{H}}(\Omega)} - C\|f\|_{L^{\theta(x)}(\Omega)}\|u\|_{W_0^{1,\mathcal{H}}(\Omega)}. \quad (3.4)$$

Therefore, inequality (3.4) proves that $\mathcal{J}$ is coercive on $W_0^{1,\mathcal{H}}(\Omega)$. But, from the definition of $\mathcal{J}$, it can see that $\mathcal{J}$ is bounded and continuous. In addition, with the help of the Clarkson's type inequality (see p. 449 of [4]), it indicates that for all $r,s\geq 2$ and for all $\xi_1,\xi_2\in\mathbb{R}^N$, one has

$$\frac{\langle A\xi_1,\xi_1\rangle^{\frac{r}{2}}+\langle A\xi_2,\xi_2\rangle^{\frac{r}{2}}}{2}\geq\left\langle A\left(\frac{\xi_1+\xi_2}{2}\right),\frac{\xi_1+\xi_2}{2}\right\rangle^{\frac{r}{2}}+\left\langle A\left(\frac{\xi_1-\xi_2}{2}\right),\frac{\xi_1-\xi_2}{2}\right\rangle^{\frac{r}{2}}, \quad (3.5)$$

$$\frac{\langle B\xi_1,\xi_1\rangle^{\frac{s}{2}}+\langle B\xi_2,\xi_2\rangle^{\frac{s}{2}}}{2}\geq\left\langle B\left(\frac{\xi_1+\xi_2}{2}\right),\frac{\xi_1+\xi_2}{2}\right\rangle^{\frac{s}{2}}+\left\langle B\left(\frac{\xi_1-\xi_2}{2}\right),\frac{\xi_1-\xi_2}{2}\right\rangle^{\frac{s}{2}}. \quad (3.6)$$

The estimates (3.5) and (3.6) illustrates that $\mathcal{J}$ is convex and therefore weakly lower semi-continuous. Consequently, through the application of classical critical point theory, we establish the existence of $u\in W_0^{1,\mathcal{H}}(\Omega)$ as a minimizer of $\mathcal{J}$. This, in turn, implies that u constitutes a weak solution to problem (1.1). Furthermore, the strict convexity property of $\mathcal{J}$ allows us to affirm the uniqueness of u. This completes the proof of Theorem 3.2. □

Now, let us comeback to study the existence of solutions to (1.1) when the source term is nonlinear. To this end, we shall assume that there are two nonnegative constants C_1 and C_2 such that

$$|f(x,r)|\leq C_1+C_2|r|^{\delta-1}, \quad (3.7)$$

for a.e $x\in\Omega$ and for all $r\in\mathbb{R}$, where $1\leq\delta<p^-$. We have the following existence result.

Theorem 3.3. *Assuming that* (1.2), (1.3), (1.4), (2.7) *and* (3.7) *are satisfied. Then, problem* (1.1) *has a weak solution* u *in the sense of Definition 3.1.*

Proof. We consider the energy functional $\mathcal{K} : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ associated to problem (1.1) as follows

$$\mathcal{K}(u) = \int_\Omega \frac{1}{p(x)} \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}} \, dx + \int_\Omega \frac{\omega(x)}{q(x)} \langle B\nabla u, \nabla u\rangle^{\frac{q(x)}{2}} \, dx - \int_\Omega F(x,u)\, dx, \tag{3.8}$$

where $F(x,r) = \int_0^r f(x,s)\, ds$. From (3.7), we can see that

$$|F(x,r)| \le C\left(1 + |r|^\delta\right), \tag{3.9}$$

for a.e $x \in \Omega$ and for all $r \in \mathbb{R}$. Furthermore, by setting $\Psi(u) = \int_\Omega F(x,u)\, dx$, it follows that $\Psi' : W_0^{1,\mathcal{H}}(\Omega) \to \left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$ is completely continuous, which means that if $(u_n) \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$, then $\Psi(u_n) \to \Psi(u)$. Consequently, the functional Ψ is weakly continuous. It is obvious that $\mathcal{K}$ belongs to $\mathcal{C}^1(W_0^{1,\mathcal{H}}(\Omega); \mathbb{R})$. Hence, the weak solutions of (1.1) correspond precisely to the critical points of $\mathcal{K}$. Analyzing (3.8) while considering (1.3), (1.4) and (3.9), we can deduce that

$$\begin{aligned} \mathcal{K}(u) &\ge \min\left\{\frac{\lambda_A}{p^+}, \frac{\lambda_B}{q^+}\right\} \int_\Omega \left(|\nabla u|^{p(x)} + \omega(x)|\nabla u|^{q(x)}\right) dx - C\int_\Omega |u|^\delta \, dx - C_3 \\ &\ge \min\left\{\frac{\lambda_A}{p^+}, \frac{\lambda_B}{q^+}\right\} \int_\Omega \left(|\nabla u|^{p(x)} + \omega(x)|\nabla u|^{q(x)}\right) dx - C_4\|u\|^\delta_{W_0^{1,\mathcal{H}}(\Omega)} - C_3. \end{aligned} \tag{3.10}$$

The latter inequality is derived through the use of Proposition (2.7). Employing (2.5), it follows from (3.10) that for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\|_{W_0^{1,\mathcal{H}}(\Omega)} > 1$, the following relationship holds:

$$\mathcal{K}(u) \ge \min\left\{\frac{\lambda_A}{p^+}, \frac{\lambda_B}{q^+}\right\} \|u\|^{p^-}_{W_0^{1,\mathcal{H}}(\Omega)} - C_4\|u\|^\delta_{W_0^{1,\mathcal{H}}(\Omega)} - C_3.$$

The fact that $1 \le \delta < p^-$ enables us to establish that

$$\mathcal{K}(u) \to +\infty \quad \text{as} \quad \|u\|_{W_0^{1,\mathcal{H}}(\Omega)} \to +\infty.$$

Thus, $\mathcal{J}$ is coercive on $W_0^{1,\mathcal{H}}(\Omega)$. As $\mathcal{K}$ is weakly lower semicontinuous, we conclude that $\mathcal{K}$ possesses a minimum point, denoted as u in $W_0^{1,\mathcal{H}}(\Omega)$. In consequence, we infer that u is a weak solution to (1.1). This completes the proof. □

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