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ON THE FAMILY OF METRICS FOR SOME PLATONIC AND ARCHIMEDEAN POLYHEDRA

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Abstract. Convexity is an important property in mathematics and geometry. In geometry convexity is simply defined as; if every points of a line segment that connects any two points of the set are in the set then this set is convex. A polyhedra, when it is convex, is an extremely important solid in 3-dimensional analytical space. Polyhedra have interesting symmetries. Therefore they have attracted the attention of scientists and artists from past to present. Thus polyhedra are discussed in a lot of scientific and artistic works. There are many relationships between metrics and polyhedra. Some of them are given in previous studies. For example, in [7] the authors have shown that the unit sphere of Chinese Checkers 3-space is the deltoidal icositetrahedron. In this study, we introduce a family of metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are some well-known polyhedra.

1. INTRODUCTION

A polyhedron is a geometric solid bounded by polygons. Polygons form the faces of the solid; an edge of the solid is the intersection of two polygons, and a vertex of the solid is a point where three or more edges intersect. If all faces of a polyhedron are identical regular polygons and at every vertex same number of faces meet then it is called a regular polyhedron. A polyhedron is called semi-regular if all its faces are regular polygons and all its vertices are equal.

Polyhedra have very interesting symmetries. Therefore they have attracted the attention of scientists and artists from past to present. Thus mathematicians, geometers, physicists, chemists, artists have studied and continue to study on polyhedra. Consequently, polyhedra take place in many studies with respect to different fields. As it is stated in [3] and [6], polyhedra have been used for explaining the world around us in philosophical and scientific way. There are only five regular convex polyhedra known as the platonic solids. These regular polyhedra were known by the Ancient Greeks. They are generally known as the "Platonic" or "cosmic" solids

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because Plato mentioned them in his dialogue Timeous, where each is associated with one of the five elements - the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron with universe (or with ether, the material of the heavens). The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result.For more detailed knowledge, see [3] and [6].

The dual polyhedra of the Archimedean solids are called Catalan solids, and they are exactly thirteen just like Archimedean solids. Platonic solids are regular and convex polyhedra and Archimedean solids are semi-regular and convex polyhedra. The Catalan solids are all convex. They are face-transitive when all its faces are the same but not vertex-transitive. Unlike Platonic solids and Archimedean solids, the face of Catalan solids are not regular polygons.

As it is stated in [14], Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Thus, instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set. Some mathematicians studied and improved metric geometry in plane and space. (Some of these are [1, 4, 5, 8, 9, 10]) According to studies on polyhedra, there are some Minkowski geometries in which unit spheres of these spaces furnished by some metrics are associated with convex solids. For example, unit spheres of maximum space and taxicab space are cubes and octahedrons, respectively, which are Platonic Solids. And unit sphere of CC-space is a deltoidal icositetrahedron which is a Catalan solid. Therefore, there are some metrics in which unit spheres of space furnished by them are convex polyhedra. That is, convex polyhedra are associated with some metrics. When a metric is given we can find its unit sphere. Naturally a question can be asked; "Is it possible to find the metric when a convex polyhedron is given?". In this study, we introduce a family of metrics and show that spheres of 3-dimensional analytical space furnished by these metrics are some polyhedra. Then we give relationships between metrics and some of Platonic and Archimedean solids. Some results for these relationships are already known from previous studies. But we introduce three metrics and give three new relationships for cuboctahedron, truncated cube and truncated octahedron.

2. Archimedean Metric

As it is mentioned in introduction, there are some 3-dimensional Minkowski geometries which have distance function distinct from Euclidean distance and unit spheres of these geometries are convex polyhedrons. That is, convex polyhedra are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. When we started studying on this question, we firstly handled separately convex polyhedra. But we noticed a relationship between the metrics. Now, we introduce a family of distances which include Taxicab distance and maximum distance as special cases in $\mathbb{R}^3$.

Definition 2.1. Let $u \in [0,\infty)$, and $P_1 = (x_1, y_1 z_1)$, $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{AP} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ Archimedean polyhedral distance between P_1 and P_2 is defined by

$$d_{AP}(P_1,P_2)=\max\left\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|,u\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right)\right\}.$$

Clearly, there are infinitely many different distance functions in the family of distance functions defined above, depending on value of u. One can think the definition not to be well-defined since the Archimedean polyhedra distance between two points can also change according to value of u. To remove this confusion, supposing value of u is initially determined and fixed unless otherwise stated. We write $\mathbb{R}^3_{AP} = (\mathbb{R}^3, d_{AP})$ for the 3-dimensional analytical space furnished by Archimedean polyhedral distance defined above.

Since proof is trivial by the definition of maximum function, we give following lemma without proof which is required to show that each of d_{AP} distances gives a metric.

Lemma 2.1. *Let $P_1 = (x_1, y_1 z_1)$ and $P_2 = (x_2, y_2, z_2)$ be any distinct points in $\mathbb{R}^3$. Then*

$$\begin{aligned}
&d_{AP}(P_1,P_2) \geq |x_1-x_2|,\\
&d_{AP}(P_1,P_2) \geq |y_1-y_2|,\\
&d_{AP}(P_1,P_2) \geq |z_1-z_2|,\\
&d_{AP}(P_1,P_2) \geq u\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right).
\end{aligned}$$

Theorem 2.1. *Every d_{AP} distance determines a metric in $\mathbb{R}^3$.*

Proof. Let $d_{AP} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is Archimedean polyhedral distance function, and $P_1=(x_1, y_1, z_1)$, $P_2=(x_2, y_2, z_2)$ and $P_3=(x_3, y_3, z_3)$ are distinct three points in $\mathbb{R}^3$. We have to show that d_{AP} is positive definite, symmetric, and the triangle inequality holds for d_P.

Absolute value gives always non-negative value and $u \geq 0$, then $d_{AP}(P_1, P_2) \geq 0$. Clearly, $d_{AP}(P_1, P_2) = 0$ iff $P_1 = P_2$. So d_P is positive definite.

Since $|a-b| = |b-a|$ for all a, $b \in \mathbb{R}$, obviously $d_{AP}(P_1, P_2) = d_{AP}(P_2, P_1)$. That is, d_{AP} is symmetric.

Now, we should prove that $d_P(P_1, P_3) \leq d_P(P_1, P_2) + d_P(P_2, P_3)$ for all P_1, P_2, $P_3 \in \mathbb{R}^3$.

$$\begin{aligned}
&d_P(P_1, P_3)\\
&= \max\{|x_1 - x_3|, |y_1 - y_3|, |z_1 - z_3|, u(|x_1 - x_3| + |y_1 - y_3| + |z_1 - z_3|)\}\\
&= \max\left\{\begin{array}{l} |x_3 - x_2 + x_2 - x_1|, |y_3 - y_2 + y_2 - y_1|, |z_3 - z_2 + z_2 - z_1|, \\ u(|x_3 - x_2 + x_2 - x_1| + |y_3 - y_2 + y_2 - y_1| + |z_3 - z_2 + z_2 - z_1|) \end{array}\right\}\\
&\leq \max\left\{\begin{array}{l} |x_3 - x_2| + |x_2 - x_1|, |y_3 - y_2| + |y_2 - y_1|, |z_3 - z_2| + |z_2 - z_1| \\ u(|x_3 - x_2| + |x_2 - x_1| + |y_3 - y_2| + |y_2 - y_1| + |z_3 - z_2| + |z_2 - z_1|) \end{array}\right\}\\
&= I
\end{aligned}$$

One can easily find that $I \leq d_{AP}(P_1, P_2) + d_{AP}(P_2, P_3)$ from Lemma 2.1. So $d_{AP}(P_1, P_3) \leq d_{AP}(P_1, P_2) + d_{AP}(P_2, P_3)$. Consequently, Archimedean polyhedral distance is a metric in 3-dimensional analytical space. □

According to Archimedean polyhedral metric, distance is one of quantities $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$ or u times sum of quantities $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$. Geometrically, there are two different paths between two points in $\mathbb{R}^3_{AP}$. If the line segment $\overline{P_1 P_2}$ is out of cones with apex P_1 and square bases which corner points are all permutations of the three axis components and all possible $+/-$ sign change of each axis component of $(\mp 1, \mp(1-u), 0)$, then

$$d_{AP}(P_1, P_2) = u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$$

,and the path between P_1 and P_2 is union of three line segments which is parallel to a coordinate axis. Otherwise, the path between P_1 and P_2 is a line segment which is parallel to a coordinate axis. Thus Archimedean polyhedral distance between P_1 and P_2 is u times sum of Euclidean lengths of these three line segments or the Euclidean length of line segment (See Figure 1).

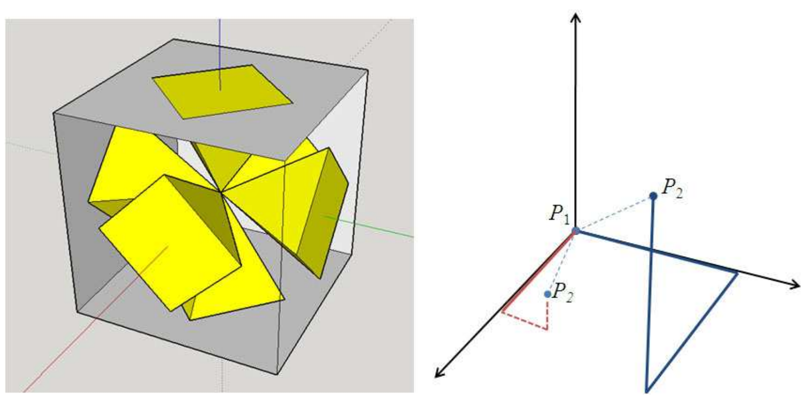

Figure 1: AP ways from P_1 to P_2

The following proposition gives an equation which relates the Euclidean distance to the Archimedean polyhedral distance between the points in $\mathbb{R}^3$:

Proposition 2.1. *Let l be the line through the points $P_1 = (x_1, y_1, z_1)$ and $P = (x_2, y_2, z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p, q, r), then*

$$d_{AP}(A, B) = \mu(AB) d_E(A, B)$$

where

$$\mu(AB) = \frac{\max\{|p|, |q|, |r|, u(|p| + |q| + |r|)\}}{\sqrt{p^2 + q^2 + r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R} \setminus \{0\}$. Thus,

$$d_{AP}(A,B) = |\lambda| \left(\max\{|p|, |q|, |r|, u(|p| + |q| + |r|)\}\right)$$

and $d_E(A,B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. □

The above lemma says that d_{AP}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 2.1. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{AP}(P_1, X) = d_{AP}(P_2, X)$.*

Corollary 2.2. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{AP}(X, P_1) \;/\; d_{AP}(X, P_2) = d_E(X, P_1) \;/\; d_E(X, P_2) \ .$$

That is, the ratios of the Euclidean and $d_{AP}-$distances along a line are the same.

3. Some relations about the Archimedean polyhedral distance and Polyhedra

The polyhedral metric gives a family of metrics and unit spheres in 3-dimensional analytical space furnished by Archimedean polyhedral metric which are some polyhedra. Of course, polyhedra varies depending on choice of u. Some results of relations between metrics and polyhedra are already known from previous studies. Here, we especially give three new relations between polyhedra and metrics by using Archimedean polyhedral metric. Now, according to choice of u, we give five cases for Archimedean polyhedral metric.

Case 1. Let $u \geq 1$. So $AP-$metric is u times taxicab metric. In particular, if $u = 1$, then $AP-$metric is taxicab metric. In this case the unit sphere is the octahedron.

Case 2. Set $u \in \left(0, \dfrac{1}{3}\right)$. Hence, $AP-$metric is the maximum metric. So the unit sphere is the hexahedron.

Case 3. Let $u = \dfrac{1}{2}$. Then Archimedean polyhedral metric gives a new result. In this case, the unit sphere is cuboctahedron. So we called cuboctahedron metric which is defined by

$$d_{AP}(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \frac{1}{2}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\}.$$

(see Figure 2a).

Case 4. Let $u \in \left(\dfrac{1}{3}, \dfrac{1}{2}\right)$. Then Archimedean polyhedral metric gives a new result. In particular, if $u = \sqrt{2} - 1$, then the unit sphere is truncated cube. So we called truncated cube metric which is defined by

$$d_{AP}(P_1, P_2)$$
$$= \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \left(\sqrt{2}\text{-}1\right)(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\}.$$

For $u \in \left(\frac{1}{3}, \frac{1}{2}\right)$ case, the unit sphere is like truncated cube. When $u \to \frac{1}{2}$ and $u \to \frac{1}{3}$, the unit sphere looks like cuboctahedron and cube, respectively. But for all values of u, unit sphere has 8-triangular faces and 6-octagonal faces (see Figure 2b).

Case 5. Let $u \in \left(\frac{1}{2}, 1\right)$. Then Archimedean polyhedral metric gives a new result. In particular, if $u = \frac{2}{3}$, then the unit sphere is truncated octahedron. So we called truncated octahedron metric which is defined by

$$d_{AP}(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|, \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\}.$$

For $u \in \left(\frac{1}{2}, 1\right)$ case, the unit sphere is like truncated octahedron. When $u \to 1$ and $u \to \frac{1}{2}$, the unit sphere looks like octahedron and cuboctahedron, respectively. But for all values of u, unit sphere has 6-square faces and 8-hexagonal faces (see Figure 2c).

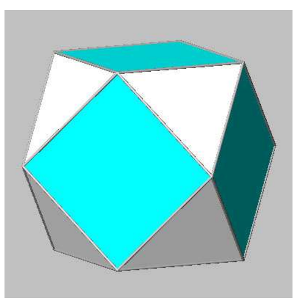

Figure 2a Cuboctahedron

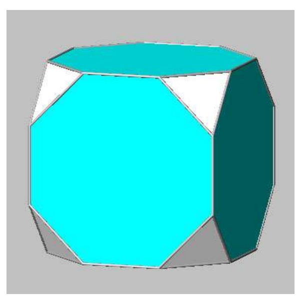

Figure 2b Truncated cube

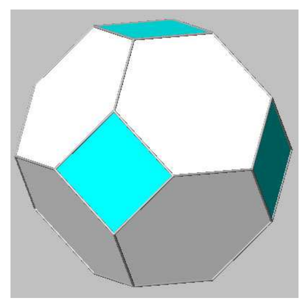

Figure 2c Truncated octahedron

One can observe that the Archimedean metric has two parts, one is $\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ and the other is $u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$. In fact, $\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ and $u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$ indicate the hexahedron and the octahedron, respectively. Thus sphere of Archimedean polyhedral metric is intersection of hexahedron and octahedron. The cases which defined above are explicated by this way.

One can take $d_{AP}(O, P) = r$. then gets $\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} = r$ and $u(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|) = r$. That is, these are the cube with vertices such that all permutations of $(\mp r, \mp r, \mp r)$ and the octahedron with vertices such that all permutations of $\left(\mp\frac{r}{u}, 0, 0\right)$, respectively. The faces of the cube are on the planes with equations $|x| = r$, $|y| = r$ and $|z| = r$, and the faces of octahedron are on the planes with equations $|x| + |y| + |z| = \frac{r}{u}$. The intersection of the faces of the cube and the octahedron are found by solving the systems of linear equations

$$\begin{cases} |x| + |y| + |z| = \frac{r}{u} \\ |x| = r \end{cases}, \quad \begin{cases} |x| + |y| + |z| = \frac{r}{u} \\ |y| = r \end{cases}, \quad \begin{cases} |x| + |y| + |z| = \frac{r}{u} \\ |z| = r \end{cases}.$$

For example, we handle the system of equations $\begin{cases} |x|+|y|+|z| = \dfrac{r}{u} \\ |x| = r \end{cases}$. Since $|x|=r$, it is obtained that $|y|+|z| = \dfrac{r}{u} - r$. The solution is the taxicab circles with the center $(\mp r, 0, 0)$ and radius $\dfrac{r}{u} - r$ on planes $|x| = r$. If $u \in \left[\dfrac{1}{2}, 1\right]$, then the circle is completely on face of the cube. Thus intersection consist of squares and hexagons. If $u \in \left(\dfrac{1}{3}, \dfrac{1}{2}\right)$, then the circle is not completely on face of the cube. Therefore intersection consist of triangles and octagons. If $u = \dfrac{1}{2}$, then intersection consist of squares and triangles. Figure 3a,3b,3c illustrate these cases.

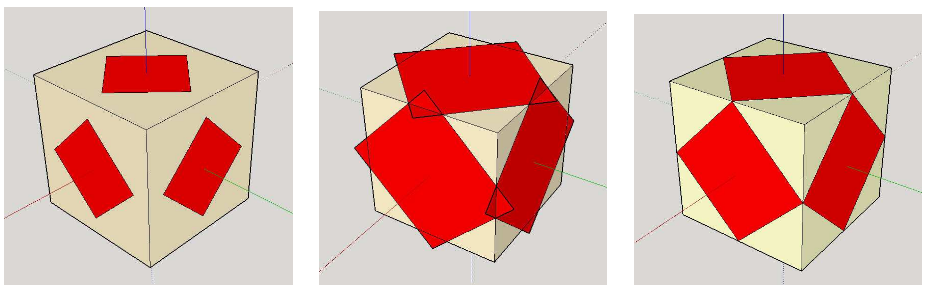

Figure 3a Figure 3b Figure 3c 3

Now, we can give some new results:

The truncated cube, or truncated hexahedron, is an Archimedean solid. It has 14 regular faces (6 octagonal and 8 triangular), 36 edges, and 24 vertices (See [16]).

The cuboctahedron is an archimedean solid with eight triangular faces and six square faces. It has 12 identical vertices, with two triangles and two squares meeting at each, and 24 identical edges, each separating a triangle from a square (See [15]).

The truncated octahedron is an archimedean solid which has 14 faces (8 regular hexagonal and 6 square), 36 edges, and 24 vertices. Since each of its faces has point symmetry the truncated octahedron is a zonohedron (See [17]).

The following corollaries are direct consequences of Proposition 2.1, Corollary 2.1 and Corollary 2.2

Corollary 3.1. *The equations of cuboctahedron, truncated cube and truncated octahedron with center $C = (x_0, y_0, z_0)$ and radius r are*

$$\max\left\{|x-x_0|, |y-y_0|, |z-z_0|, \frac{1}{2}\left(|x-x_0|+|y-y_0|+|z-z_0|\right)\right\} = r$$

$$\max\left\{|x-x_0|, |y-y_0|, |z-z_0|, \left(\sqrt{2}-1\right)\left(|x-x_0|+|y-y_0|+|z-z_0|\right)\right\} = r$$

$$\max\left\{|x-x_0|, |y-y_0|, |z-z_0|, \frac{2}{3}\left(|x-x_0|+|y-y_0|+|z-z_0|\right)\right\} = r$$

,respectively. The the cuboctahedron, truncated cube and the truncated octahedron have 14- regular faces with vertices such that all permutations of the three axis components and all possible +/- sign changes of each axis component of $\left(r, r, \left(\sqrt{2}-1\right) r\right)$, $(r, r, 0)$ *and* $(r/2, r, 0)$, *respectively (See Figure 4a,4b,4c).*

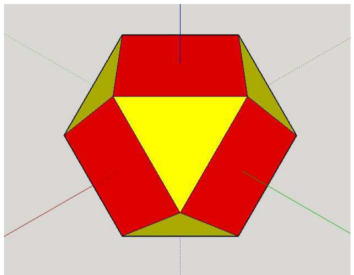

Figure 4a

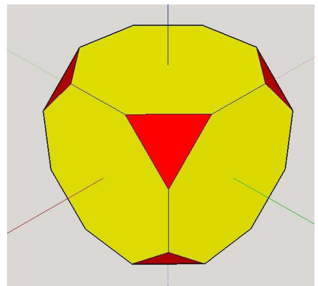

Figure 4b

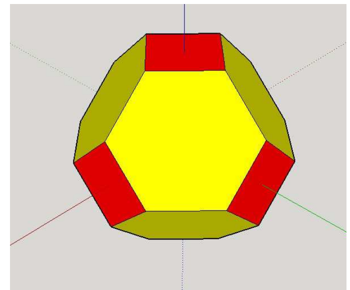

Figure 4c

Lemma 3.1. *Let l be the line through the points $P_1 = (x_1, y_1 z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the analytical 3−dimensional space and d_E, d_{TC}, d_{CO} and d_{TO} denote the Euclidean metric, the truncated metric, the cuboctahedron metric and the truncated metric respectively. If l has direction vector (p, q, r), then*

$$\begin{aligned} d_{CO}(P_1, P_2) &= \frac{\max\left\{|p|, |q|, |r|, \frac{1}{2}(|p| + |q| + |r|)\right\}}{\sqrt{p^2 + q^2 + r^2}} d_E(P_1, P_2) \\ d_{TC}(P_1, P_2) &= \frac{\max\left\{|p|, |q|, |r|, \left(\sqrt{2} - 1\right)(|p| + |q| + |r|)\right\}}{\sqrt{p^2 + q^2 + r^2}} d_E(P_1, P_2) \\ d_{TO}(P_1, P_2) &= \frac{\max\left\{|p|, |q|, |r|, \frac{2}{3}(|p| + |q| + |r|)\right\}}{\sqrt{p^2 + q^2 + r^2}} d_E(P_1, P_2). \end{aligned}$$

Corollary 3.2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$\begin{aligned} d_E(P_1, X) &= d_E(P_2, X) \text{ if and only if } d_{CO}(P_1, X) = d_{CO}(P_2, X) \\ d_E(P_1, X) &= d_E(P_2, X) \text{ if and only if } d_{TC}(P_1, X) = d_{TC}(P_2, X) \\ d_E(P_1, X) &= d_E(P_2, X) \text{ if and only if } d_{TO}(P_1, X) = d_{TO}(P_2, X). \end{aligned}$$

Corollary 3.3. *If P_1, P_2 and X are any distinct collinear points in $\mathbb{R}^3$, then*

$$\frac{d_E(P_1, X)}{d_E(P_2, X)} = \frac{d_{CO}(P_1, X)}{d_{CO}(P_2, X)} = \frac{d_{TC}(P_1, X)}{d_{TC}(P_2, X)} = \frac{d_{TO}(P_1, X)}{d_{TO}(P_2, X)}.$$

That is, the ratios of the Euclidean, the cuboctahedron, the truncated cube and the truncated octahedron distances along a line are the same.

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