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Research Article

Exploration of multivalent harmonic functions: Investigating essential properties

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Abstract

Within this manuscript, we introduce an innovative subclass of multivalent harmonic functions, encompassing higher-order derivatives within the confines of an open unit disk. Our investigation extends to the analysis of coefficient bounds, growth estimates, starlikeness, and convexity radii uniquely associated with this particular class. Furthermore, we scrutinize the property of closure under convolution operations for this subclass.

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1. Introduction

The exploration and analysis of complex-valued harmonic functions within the open unit disk $\mathbb{E}$ constitute the fundamental basis of this investigation, denoted as $\mathcal{H}$. These functions, expressed as $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where $\mathfrak{u}$ represents the analytic component and $\mathfrak{v}$ denotes its co-analytic counterpart, play an integral role in examining local univalent and sense-preserving properties within $\mathcal{H}$. The critical condition ensuring such properties is the inequality $|\mathfrak{v}'(z)| < |\mathfrak{u}'(z)|$, applicable to all $z \in \mathbb{E}$ (see [4]).

A more focused examination is directed towards a specific subset of functions within $\mathcal{H}$, denoted as $\mathcal{H}_\eta$. This subset encompasses functions of the form

$$\mathfrak{f}(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)} = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} u_{\mathfrak{m}} z^{\mathfrak{m}} + \overline{\sum_{\mathfrak{m}=\eta+1}^{\infty} v_{\mathfrak{m}} z^{\mathfrak{m}}} \tag{1.1}$$

where η is a positive integer. These functions manifest harmonic behavior within the open unit disk $\mathbb{E}$. Further refinement leads to the definition of the subclass $\mathcal{SH}_\eta$, comprising functions within $\mathcal{H}_\eta$ possessing the property of being sense-preserving and η-valent in $\mathbb{E}$. Functions in $\mathcal{SH}_\eta$ are denoted as η-valently harmonic functions in $\mathbb{E}$. Notably, the class $\mathcal{SH}_1 \equiv \mathcal{SH}^0$, consisting of sense-preserving and harmonic univalent functions. When $\mathfrak{v}(z)$ is identically zero, $\mathcal{H}_1$ contracts to class $\mathcal{A}$. Additionally, the subclasses $\mathcal{S}^*_\eta$ and $\mathcal{K}_\eta$ are introduced, representing functions within $\mathcal{H}_\eta$ that are mapped onto starlike and convex domains in $\mathbb{E}$, respectively.

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Al-Refai [3] delved into the subclass $\mathcal{R}_\eta(p,q,\gamma;\xi)$ for some γ ($\gamma < \eta![p+(\eta-\xi)q+(\eta-\xi)(\eta-\xi-1)(q-p)/2]/(\eta-\xi)!$) and $\xi = 0,1,...,\eta$, where $\eta+1-\xi+2p/(q-p) > 0$ or $p=q=1$ and $z \in \mathbb{E}$. This subclass consists of analytic functions

$$\mathfrak{u}(z) = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} u_\mathfrak{m} z^\mathfrak{m}$$

such that

$$\operatorname{Re}\left\{p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right\} > \gamma.$$

The class of multivalent harmonic functions that will now be discussed is the main subject of this paper:

Definition 1.1. The set $\mathcal{RH}_\eta(p,q,\gamma;\xi)$ is defined as the collection of functions $\mathfrak{f} = \mathfrak{u}+\overline{\mathfrak{v}} \in \mathcal{H}_\eta$ that adhere to the following inequality:

$$\operatorname{Re}\left\{p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right\} > \left|p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right| \tag{1.2}$$

for some γ (where $\gamma < \eta![p+(\eta-\xi)q+(\eta-\xi)(\eta-\xi-1)(q-p)/2]/(\eta-\xi)!$) and $\xi = 0,1,...,\eta$, with conditions $\eta+1-\xi+2p/(q-p) > 0$ or $p=q=1$ and $z \in \mathbb{E}$.

Remark 1.2. In the case where $\eta = 1$ and $\xi = 1$, the class $\mathcal{RH}_1(p,q,\gamma;1) \equiv \mathcal{RH}^0(p,q,\gamma)$ is defined as follows:

$$\operatorname{Re}\left[p\mathfrak{u}'(z) + qz\mathfrak{u}''(z) + \left(\frac{q-p}{2}\right)z^2\mathfrak{u}'''(z) - \gamma\right] > \left|p\mathfrak{v}'(z) + qz\mathfrak{v}''(z) + \left(\frac{q-p}{2}\right)z^2\mathfrak{v}'''(z)\right|$$

where $0 \le \gamma < p \le q$. This class has been investigated by Çakmak et al. (see [5]). Moreover, for $p = q$, we obtain the class $\mathcal{R}_\eta(p,p,\gamma;\xi) \equiv \mathcal{A}_\eta(p,p,\gamma;\xi)$ studied by Owa et al [11], the class $\mathcal{RH}_\eta(p,p,\gamma;\xi) \equiv GH_\eta(p,p,\gamma;\xi)$ studied by Oros et al. [10], the class $\mathcal{R}_1(1,\lambda,0;0) \equiv R^0_\eta(\lambda,\frac{\lambda-1}{2})$ studied by Rosihan et al. [12], and the class $\mathcal{RH}_1(1,\lambda,0;0) \equiv RH^0_\eta(\lambda,\frac{\lambda-1}{2})$ studied by Yasar et al. [14].

The primary objective of this comprehensive study is to systematically introduce a distinctive category of harmonic multivalent functions, distinguished by a higher-degree differential inequality, and thoroughly scrutinize the specific geometric properties inherently embedded within this novel class. In the second section of this meticulously crafted research, our focus shifts toward the derivation of rigorous coefficient bounds, meticulous growth estimates, and the determination of indispensable conditions for coefficients to rightfully belong to this distinguished class. Subsequently, the third section is intricately dedicated to the meticulous calculation and acquisition of the radii of starlikeness and convexity pertinent to this unique class of harmonic multivalent functions. Finally, the fourth section delves into establishing the closure properties of this exceptional class under the transformative actions of convex combinations and convolution operations meticulously applied to its esteemed members.

2. Sharp coefficient estimates and growth theorems

In this section, we delve into delineating the intricate relationship existing between the $\mathcal{RH}_\eta(p,q,\gamma;\xi)$ class and the $\mathcal{R}_\eta(p,q,\gamma;\xi)$ class. Furthermore, comprehensive efforts have been dedicated to acquiring profound insights into coefficient bounds and distortion theorems, enriching the depth of our understanding in this significant context. The careful examination and elucidation of these interconnections contribute significantly to the overarching narrative of our study, providing a nuanced perspective on the interplay between these essential mathematical classes.

Theorem 2.1. *The harmonic mapping* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$ *if and only if* $\mathfrak{F}_\delta = \mathfrak{u} + \delta\mathfrak{v} \in \mathcal{R}_\eta(p, q, \gamma; \xi)$ *for each* $\delta\,(|\delta| = 1)$.

Proof. Assume that $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$. For every δ with $|\delta| = 1$,

$$\begin{aligned}
&\operatorname{Re}\left[p\frac{\mathfrak{F}_\delta^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{F}_\delta^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{F}_\delta^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right] \\
&= \operatorname{Re}\left[p\frac{\mathfrak{u}^{(\xi)}(z) + \delta\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z) + \delta\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z) + \delta\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right] \\
&= \operatorname{Re}\left[p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right] \\
&+ \operatorname{Re}\left[\delta\left(p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right)\right] \\
&> \operatorname{Re}\left[p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right] \\
&- \left|p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right| \\
&> 0, z \in \mathbb{E}.
\end{aligned}$$

Hence, for each δ with $|\delta| = 1$, we have $\mathfrak{F}_\delta \in \mathcal{R}_\eta(p, q, \gamma; \xi)$. Conversely, consider $z \in \mathbb{E}$ and $\mathfrak{F}_\delta = \mathfrak{u} + \delta\mathfrak{v} \in \mathcal{R}_\eta(p, q, \gamma; \xi)$. In this case,

$$\begin{aligned}
&\operatorname{Re}\left[p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right] \\
&\qquad > \operatorname{Re}\left\{-\delta\left(p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right)\right\}.
\end{aligned}$$

The judicious selection of δ with $|\delta| = 1$ enables us to achieve

$$\begin{aligned}
&\operatorname{Re}\left\{p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right\} \\
&\qquad > \left|p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right|.
\end{aligned}$$

Consequently, we have $\mathfrak{f} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$.

□

Theorem 2.2. *Let* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$. *Then, for* $\mathfrak{m} \geq 2$, *the coefficients* $v_\mathfrak{m}$ *satisfy the inequality*

$$|v_\mathfrak{m}| \leq \frac{2\,(\sigma - \gamma)\,(\mathfrak{m} - \xi)!}{\mathfrak{m}!(\mathfrak{m} - \xi + 1)\,[2p + (q - p)\,(\mathfrak{m} - \xi)]}, \tag{2.1}$$

where $\sigma = \eta![p + q(\eta - \xi) + (\eta - \xi)(\eta - \xi - 1)(q - p)/2]/(\eta - \xi)!$. *The result is sharp, and equality is achieved for the function*

$$f(z) = z^\eta + \frac{2\,(\sigma - \gamma)\,(\mathfrak{m} - \xi)!}{\mathfrak{m}!(\mathfrak{m} - \xi + 1)\,[2p + (q - p)\,(\mathfrak{m} - \xi)]}\bar{z}^\mathfrak{m}.$$

Proof. Suppose that $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$. With $\mathfrak{v}(re^{i\theta})$ represented as a series, $0 \le \rho < 1$ and $\theta \in \mathbb{R}$, we get

$$\begin{aligned}
&\rho^{\mathfrak{m}-\eta}(\mathfrak{m}-\xi+1)\left[p+\left(\frac{q-p}{2}\right)(\mathfrak{m}-\xi)\right]\frac{\mathfrak{m}!}{(\mathfrak{m}-\xi)!}|v_{\mathfrak{m}}| \\
\le\ & \frac{1}{2\pi}\int_0^{2\pi}\left|p\frac{\mathfrak{v}^{(\xi)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^{\eta-\xi}}+q\frac{\mathfrak{v}^{(\xi+1)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^{\eta-\xi-2}}\right|d\theta \\
<\ & \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left[p\frac{\mathfrak{u}^{(\xi)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^{\eta-\xi}}+q\frac{\mathfrak{u}^{(\xi+1)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^{\eta-\xi-2}}-\gamma\right]d\theta \\
=\ & \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left[\sigma-\gamma+\sum_{\mathfrak{m}=\eta+1}^{\infty}(\mathfrak{m}-\xi+1)\left[p+\left(\frac{q-p}{2}\right)(\mathfrak{m}-\xi)\right]\frac{\mathfrak{m}!}{(\mathfrak{m}-\xi)!}u_{\mathfrak{m}}\rho^{\mathfrak{m}-\eta}e^{i(\mathfrak{m}-\eta)\theta}\right]d\theta \\
=\ & \sigma-\gamma,
\end{aligned}$$

where $\sigma = \eta![p + q(\eta-\xi) + (\eta-\xi)(\eta-\xi-1)(q-p)/2]/(\eta-\xi)!$. As ρ approaches 1 from the left, we obtain the desired bound. □

Theorem 2.3. *Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$. Afterward, for $\mathfrak{m} \ge 2$, we get*

i. $|u_{\mathfrak{m}}| + |v_{\mathfrak{m}}| \le \frac{4(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)[2p+(q-p)(\mathfrak{m}-\xi)]}$,

ii. $||u_{\mathfrak{m}}| - |v_{\mathfrak{m}}|| \le \frac{4(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)[2p+(q-p)(\mathfrak{m}-\xi)]}$,

iii. $|u_{\mathfrak{m}}| \le \frac{4(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)[2p+(q-p)(\mathfrak{m}-\xi)]}$.

where $\sigma = \eta![p + q(\eta-\xi) + (\eta-\xi)(\eta-\xi-1)(q-p)/2]/(\eta-\xi)!$. Each result is precise, and every equality is satisfied for the function $\mathfrak{f}(z) = z^\eta + \sum\limits_{\mathfrak{m}=\eta+1}^{\infty} \frac{4(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)[2p+(q-p)(\mathfrak{m}-\xi)]} z^{\mathfrak{m}}$.

Proof. (*i*) Suppose $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$. Then, from Theorem 2.1, $\mathfrak{F}_\delta = \mathfrak{u} + \delta\mathfrak{v} \in \mathcal{R}_\eta(p, q, \gamma; \xi)$ for each δ ($|\delta| = 1$). Hence, every δ ($|\delta| = 1$), we derive

$$\operatorname{Re}\left[p\frac{\mathfrak{u}^{(\xi)}(z)+\delta\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{u}^{(\xi+1)}(z)+\delta\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)+\delta\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}-\gamma\right]>0$$

for $z \in \mathbb{E}$. On the other hand, there is an analytic function $\Phi(z) = 1 + \sum_{\mathfrak{m}=1}^{\infty} \phi_{\mathfrak{m}} z^{\mathfrak{m}}$ in $\mathbb{E}$ whose real part is positive, satisfying

$$\begin{aligned}
&p\frac{\mathfrak{u}^{(\xi)}(z)+\delta\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{u}^{(\xi+1)}(z)+\delta\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} \\
&\quad+\left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)+\delta\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}=(\sigma-\gamma)\Phi(z)+\gamma
\end{aligned} \tag{2.2}$$

where $\sigma = \eta![p + q(\eta-\xi) + (\eta-\xi)(\eta-\xi-1)(q-p)/2]/(\eta-\xi)!$. Comparing coefficients on both sides of (2.2), we obtain

$$(\mathfrak{m}-\xi+1)\left[p+\left(\frac{q-p}{2}\right)(\mathfrak{m}-\xi)\right]\frac{\mathfrak{m}!}{(\mathfrak{m}-\xi)!}(u_{\mathfrak{m}}+\delta v_{\mathfrak{m}})=(\sigma-\gamma)\Phi_{\mathfrak{m}-1}\text{ for }\mathfrak{m}\ge\eta+1.$$

Since the real part of the function $\Phi(z)$ is positive, $|\phi_{\mathfrak{m}}| \le 2$ for $\mathfrak{m} \ge 1$ and δ ($|\delta| = 1$) is arbitrary, it follows that the proof for statement (i) is concluded. Proofs (ii) and (iii) can be obtained by using the proving techniques of (i). The function

$$f(z)=z^{\eta}+\sum_{\mathfrak{m}=\eta+1}^{\infty}\frac{4\left(\sigma-\gamma\right)\left(\mathfrak{m}-\xi\right)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)\left[2p+\left(q-p\right)\left(\mathfrak{m}-\xi\right)\right]}z^{\eta},$$

demonstrates the sharpness of every inequality. □

Theorem 2.4. *Let* $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{H}_{\eta}$ *with*

$$\sum_{\mathfrak{m}=\eta+1}^{\infty}\left(\mathfrak{m}-\xi+1\right)\left[2p+\left(q-p\right)\left(\mathfrak{m}-\xi\right)\right]\frac{\mathfrak{m}!}{\left(\mathfrak{m}-\xi\right)!}\left(|u_m|+|v_m|\right)\leq 2\left(\sigma-\gamma\right), \tag{2.3}$$

then $\mathfrak{f}\in\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$.

Proof. Suppose that $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{H}_{\eta}$. Then using (2.3),

$$\begin{aligned}
&\operatorname{Re}\left\{p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}-\gamma\right\}\\
=\;&\operatorname{Re}\left\{\sigma-\gamma+\sum_{\mathfrak{m}=\eta+1}^{\infty}\left(\mathfrak{m}-\xi+1\right)\left[p+\left(\frac{q-p}{2}\right)\left(\mathfrak{m}-\xi\right)\right]\frac{\mathfrak{m}!}{\left(\mathfrak{m}-\xi\right)!}u_{\mathfrak{m}}z^{\mathfrak{m}-1}\right\}\\
>\;&\sigma-\gamma-\sum_{\mathfrak{m}=2}^{\mathfrak{m}=\eta+1}\left(\mathfrak{m}-\xi+1\right)\left[p+\left(\frac{q-p}{2}\right)\left(\mathfrak{m}-\xi\right)\right]\frac{\mathfrak{m}!}{\left(\mathfrak{m}-\xi\right)!}|u_{\mathfrak{m}}|\\
\geq\;&\sum_{\mathfrak{m}=\eta+1}^{\infty}\left(\mathfrak{m}-\xi+1\right)\left[p+\left(\frac{q-p}{2}\right)\left(\mathfrak{m}-\xi\right)\right]\frac{\mathfrak{m}!}{\left(\mathfrak{m}-\xi\right)!}|v_{\mathfrak{m}}|\\
>\;&\left|\sum_{\mathfrak{m}=\eta+1}^{\infty}\left(\mathfrak{m}-\xi+1\right)\left[p+\left(\frac{q-p}{2}\right)\left(\mathfrak{m}-\xi\right)\right]\frac{\mathfrak{m}!}{\left(\mathfrak{m}-\xi\right)!}v_{\mathfrak{m}}z^{\mathfrak{m}-1}\right|\\
=\;&\left|p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right|.
\end{aligned}$$

Hence, $\mathfrak{f}\in\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$. □

The distortion theorem will now be presented using techniques introduced by Rosihan et al.[12]

Theorem 2.5. *Let* $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$. *Then,*

$$\frac{\eta!\,|z|^{\eta-\xi+1}}{(\eta-\xi+1)!}+4\left(\sigma-\gamma\right)\sum_{\mathfrak{m}=1}^{\infty}\frac{(-1)^{\mathfrak{m}}\,|z|^{\mathfrak{m}+\eta-\xi+1}}{(\mathfrak{m}+\eta-\xi+1)^{2}\left[2\gamma+\left(q-p\right)\left(\mathfrak{m}+\eta-\xi\right)\right]}\leq\left|\mathfrak{f}^{(\xi)}(z)\right|,$$

and

$$\left|\mathfrak{f}^{(\xi)}(z)\right|\leq\frac{\eta!\,|z|^{\eta-\xi+1}}{(\eta-\xi+1)!}+4\left(\sigma-\gamma\right)\sum_{\mathfrak{m}=1}^{\infty}\frac{|z|^{\mathfrak{m}+\eta-\xi+1}}{(\mathfrak{m}+\eta-\xi+1)^{2}\left[2\gamma+\left(\delta-\gamma\right)\left(\mathfrak{m}+\eta-\xi\right)\right]}.$$

Equality is satisfied for the function

$$\mathfrak{f}^{(\xi)}(z)=\frac{\eta!z^{\eta-\xi+1}}{(\eta-\xi+1)!}+4\left(\sigma-\gamma\right)\sum_{\mathfrak{m}=1}^{\infty}\frac{z^{\mathfrak{m}+\eta-\xi+1}}{(\mathfrak{m}+\eta-\xi+1)^{2}\left[2\gamma+\left(\delta-\gamma\right)\left(\mathfrak{m}+\eta-\xi\right)\right]}.$$

Proof. Consider $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$. Applying Theorem 2.1, $\mathfrak{F}_{\delta}\in\mathcal{R}_{\eta}(p,q,\gamma;\xi)$ and for every δ ($|\delta|=1$), we get $\operatorname{Re}\{F(z)\}>0$, where

$$F(z)=\frac{p\frac{\mathfrak{F}_{\delta}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{F}_{\delta}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{F}_{\delta}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}-\gamma}{\sigma-\gamma}.$$

This leads to the equation

$$p\mathfrak{F}_{\delta}^{(\xi)}(z) + qz\mathfrak{F}_{\delta}^{(\xi+1)}(z) + \left(\frac{q-p}{2}\right) z^2\mathfrak{F}_{\delta}^{(\xi+2)}(z) = \gamma z^{\eta-\xi} + (\sigma-\gamma)z^{\eta-\xi}F(z). \tag{2.4}$$

Multiplying both sides of 2.4 by $\frac{2}{q-p}$, we get

$$\frac{2p}{q-p}\left(z\mathfrak{F}_{\delta}^{(\xi)}(z)\right)' + \left(z^2\mathfrak{F}_{\delta}^{(\xi+1)}(z)\right)' = \frac{2}{q-p}\{\gamma z^{\eta-\xi} + (\sigma-\gamma)z^{\eta-\xi}F(z)\}. \tag{2.5}$$

Integrating the expression 2.5 from 0 to z yields

$$\frac{2p}{q-p}z\mathfrak{F}_{\delta}^{(\xi)}(z) + z^2\mathfrak{F}_{\delta}^{(\xi+1)}(z) = \frac{2}{q-p}\left\{\frac{\gamma}{\eta-\xi+1}z^{\eta-\xi+1} + (\sigma-\gamma)\int_0^z t^{\eta-\xi}F(t)dt\right\} \tag{2.6}$$

Substituting $t = s^{\frac{1}{\eta-\xi+1}}z$ into 2.6 and simplifying, we get

$$\frac{2p}{q-p}z\mathfrak{F}_{\delta}^{(\xi)}(z) + z^2\mathfrak{F}_{\delta}^{(\xi+1)}(z) = \frac{2}{q-p}\left\{\frac{\gamma}{\eta-\xi+1}z^{\eta-\xi+1} + \frac{\sigma-\gamma}{\eta-\xi+1}z^{\eta-\xi+1}\int_0^1 F(s^{\frac{1}{\eta-\xi+1}}z)ds\right\} \tag{2.7}$$

Multiplying both sides of 2.7 by $z^{\frac{2p}{q-p}-2}$, we get

$$\left(z^{\frac{2p}{q-p}}\mathfrak{F}_{\delta}^{(\xi)}(z)\right)' = \frac{2}{(q-p)(\eta-\xi+1)}\left\{\gamma z^{\frac{2p}{q-p}+\eta-\xi-1} + (\sigma-\gamma)z^{\frac{2p}{q-p}+\eta-\xi-1}\int_0^1 F(s^{\frac{1}{\eta-\xi+1}}z)ds\right\} \tag{2.8}$$

Integrating the expression 2.8 from 0 to z and substituting $s = c^{\frac{q-p}{2p+(\eta-\xi)(q-p)}}z$, we obtain

$$\mathfrak{F}_{\delta}^{(\xi)}(z) = \frac{2z^{\eta-\xi}}{(\eta-\xi+1)\left[2p+(\eta-\xi)(q-p)\right]} \times\left\{\gamma + (\sigma-\gamma)\int_0^1\int_0^1 F(s^{\frac{1}{\eta-\xi+1}}c^{\frac{q-p}{2p+(\eta-\xi)(q-p)}}z)\,ds\,dc\right\}. \tag{2.9}$$

However, since $\operatorname{Re}\{F(z)\} > 0$, $F(z) \prec \frac{1+z}{1-z}$, where $\prec$ signifies the subordination [7]. Define

$$A(z) = \int_0^1\int_0^1 \frac{dsdc}{1 - s^{\frac{1}{\eta-\xi+1}}c^{\frac{q-p}{2p+(\eta-\xi)(q-p)}}z} = 1 + \sum_{\mathfrak{m}=1}^{\infty}\frac{z^{\mathfrak{m}}}{(1+\frac{1}{\eta-\xi+1}\mathfrak{m})\left(1+\frac{q-p}{2p+(\eta-\xi)(q-p)}\mathfrak{m}\right)}$$

and

$$B(z) = \frac{1+z}{1-z} = 1 + \sum_{\mathfrak{m}=1}^{\infty}2z^{\mathfrak{m}}.$$

Afterwards, from (2.9) we get

$$\begin{aligned}
\mathfrak{F}_{\delta}^{(\xi)}(z) &\prec \frac{2z^{\eta-\xi}}{(\eta-\xi+1)\left[2p+(\eta-\xi)(q-p)\right]}\left\{\gamma + (\sigma-\gamma)\Big((A*B)(z)\Big)\right\} \\
&= \frac{\eta!}{(\eta-\xi)!}z^{\eta-\xi} + \sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)z^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}
\end{aligned}$$

Since

$$\begin{aligned}
\left|\mathfrak{F}_{\delta}^{(\xi)}(z)\right| &= |\mathfrak{u}'(z) + \delta\mathfrak{v}'(z)| \\
&\le \frac{\eta!}{(\eta-\xi)!}|z|^{\eta-\xi} + \sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)\,|z|^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}
\end{aligned}$$

and

$$\begin{aligned}\left|\mathfrak{F}_{\delta}^{(\xi)}(z)\right| &= |\mathfrak{u}'(z)+\delta\mathfrak{v}'(z)| \\ &\geq \frac{\eta!}{(\eta-\xi)!}|z|^{\eta-\xi}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)(-1)^{\mathfrak{m}}|z|^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}\end{aligned}$$

Especially, we have

$$|\mathfrak{u}'(z)|+|\mathfrak{v}'(z)|\leq\frac{\eta!}{(\eta-\xi)!}|z|^{\eta-\xi}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)|z|^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}$$

and

$$|\mathfrak{u}'(z)|-|\mathfrak{v}'(z)|\geq\frac{\eta!}{(\eta-\xi)!}|z|^{\eta-\xi}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)(-1)^{\mathfrak{m}}|z|^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}$$

Assume Γ is the radial segment extending from 0 to z

$$\begin{aligned}|\mathfrak{f}(z)| &= \left|\int_{\Gamma}\frac{\partial\mathfrak{f}}{\partial t}dt+\frac{\partial\mathfrak{f}}{\partial\bar{t}}d\bar{t}\right|\leq\int_{\Gamma}(|\mathfrak{u}'(t)|+|\mathfrak{v}'(t)|)\,|dt| \\ &\leq \int_{0}^{|z|}\left(\frac{\eta!}{(\eta-\xi)!}|\tau|^{\eta-\xi}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)|\tau|^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}\right)d\tau \\ &= \frac{\eta!}{(\eta-\xi+1)!}|z|^{\eta-\xi+1}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)|z|^{\mathfrak{m}+\eta-\xi+1}}{(\mathfrak{m}+\eta-\xi+1)^2[2p+(q-p)(\eta-\xi+\mathfrak{m})]}\end{aligned}$$

and

$$\begin{aligned}|\mathfrak{f}(z)| &\geq \int_{\Gamma}(|\mathfrak{u}'(t)|-|\mathfrak{v}'(t)|)\,|dt| \\ &\geq \int_{0}^{|z|}\left(\frac{\eta!}{(\eta-\xi)!}|z|^{\eta-\xi}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)(-1)^{\mathfrak{m}-1}|\tau|^{\mathfrak{m}+\eta-\xi}}{(\mathfrak{m}+\eta-\xi+1)[2p+(q-p)(\mathfrak{m}+\eta-\xi)]}\right)d\tau \\ &= \frac{\eta!}{(\eta-\xi+1)!}|z|^{\eta-\xi+1}+\sum_{\mathfrak{m}=1}^{\infty}\frac{4(\sigma-\gamma)(-1)^{\mathfrak{m}-1}|z|^{\mathfrak{m}+\eta-\xi+1}}{(\mathfrak{m}+\eta-\xi+1)^2[2p+(q-p)(\eta-\xi+\mathfrak{m})]}\end{aligned}$$

□

3. Radius of η-valent starlikeness and convexity in the class $\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$

In this section, our paramount focus revolves around furnishing comprehensive insights into the determination of the radius of η-valent starlikeness and η-valent convexity for the functions encapsulated within the distinguished class $\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$. Through rigorous analysis and calculation, we aim to provide a thorough understanding of the radii associated with these essential characteristics within the mathematical framework of $\mathcal{RH}_{\eta}(p,q,\gamma;\xi)$.

The main conclusions are illustrated using the two lemmas that follow:

Lemma 3.1. ([1]) Suppose $\mathfrak{f}\in\mathcal{H}\eta$. If the series $\sum_{\mathfrak{m}=\eta+1}^{\infty}\mathfrak{m}\left(|u_{\mathfrak{m}}|+|v_{\mathfrak{m}}|\right)\leq\eta$, then $\mathfrak{f}$, then it follows that $\mathfrak{f}$ is η-valently starlike in the domain $\mathbb{E}$.

Lemma 3.2. ([2]) Suppose $\mathfrak{f}\in\mathcal{H}\eta$. If the series $\sum_{\mathfrak{m}=\eta+1}^{\infty}\mathfrak{m}^2\left(|u_{\mathfrak{m}}|+|v_{\mathfrak{m}}|\right)\leq\eta^2$, then $\mathfrak{f}$, then it follows that $\mathfrak{f}$ is η-valently convex in the domain $\mathbb{E}$.

The following theorem examines the first outcome regarding the radius of starlikeness.

Theorem 3.3. *Let $\mathfrak{f} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$ be a sense-preserving harmonic mapping defined in $\mathbb{E}$. Then, in the open disk $|z| < \rho_\star$, where*

$$\rho_\star = \inf_{\mathfrak{m} \geq \eta+1} \left(\frac{\eta^2(\mathfrak{m}-2)!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}{4\mathfrak{m}(\sigma-\gamma)(\mathfrak{m}-\xi)!} \right)^{\frac{1}{\mathfrak{m}-\eta}},$$

the function $\mathfrak{f}$ is η-valently starlike.

Proof. Let $\mathfrak{f} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$, and let $\rho, 0 < \rho < 1$, be fixed. Then

$$\mathfrak{f}_\rho(z) = \rho^{-\eta}\mathfrak{f}(\rho z) = \rho^{-\eta}\mathfrak{u}(\rho z) + \rho^{-\eta}\overline{\mathfrak{v}(\rho z)} \in \mathcal{RH}_\eta(p, q, \gamma; \xi)$$

and

$$f_\rho(z) = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} u_\mathfrak{m}\rho^{\mathfrak{m}-\eta}z^\mathfrak{m} + \overline{\sum_{\mathfrak{m}=\eta+1}^{\infty} v_\mathfrak{m}\rho^{\mathfrak{m}-\eta}z^\mathfrak{m}}, \quad z \in \mathbb{E}.$$

As per Lemma 3.1, it is sufficient to demonstrate that

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{m}\left(|u_\mathfrak{m}| + |v_\mathfrak{m}|\right)\rho^{\mathfrak{m}-\eta} \leq \eta$$

for $\rho < \rho_\star$. According to Theorem 2.3 (i),

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{m}\left(|u_\mathfrak{m}| + |v_\mathfrak{m}|\right)\rho^{\mathfrak{m}-\eta} \leq \sum_{\mathfrak{m}=\eta+1}^{\infty} \frac{4\mathfrak{m}(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}\rho^{\mathfrak{m}-\eta} \tag{3.1}$$

Moreover considering that

$$\eta = \eta^2 \sum_{\mathfrak{m}=\eta+1}^{\infty} \frac{1}{\mathfrak{m}(\mathfrak{m}-1)},$$

As is well known, the inequality (3.1) can be expressed as

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \frac{4\mathfrak{m}(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}\rho^{\mathfrak{m}-\eta} \leq \eta^2 \sum_{\mathfrak{m}=\eta+1}^{\infty} \frac{1}{\mathfrak{m}(\mathfrak{m}-1)}$$

Thus, if

$$\rho^{\mathfrak{m}-\eta} \leq \frac{\eta^2(\mathfrak{m}-2)!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}{4\mathfrak{m}(\sigma-\gamma)(\mathfrak{m}-\xi)!}$$

for all $\mathfrak{m} \geq \eta + 1$, then

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{m}\left(|u_\mathfrak{m}| + |v_\mathfrak{m}|\right)\rho^{\mathfrak{m}-\eta} \leq \eta$$

Therefore, we obtain

$$\rho_\star = \inf_{\mathfrak{m} \geq \eta+1} \left(\frac{\eta^2(\mathfrak{m}-2)!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}{4\mathfrak{m}(\sigma-\gamma)(\mathfrak{m}-\xi)!} \right)^{\frac{1}{\mathfrak{m}-\eta}}.$$

□

The succeeding theorem determines the convexity radius for the class $\mathcal{RH}_\eta(p, q, \gamma; \xi)$.

Theorem 3.4. *Assume that* $\mathfrak{f} \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$ *is a harmonic mapping in* $\mathbb{E}$ *that preserves sense. For a given constant*

$$\rho_c = \inf_{\mathfrak{m}\geq\eta+1}\left(\frac{\eta^3(\mathfrak{m}-2)!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}{4\mathfrak{m}^2(\sigma-\gamma)(\mathfrak{m}-\xi)!}\right)^{\frac{1}{\mathfrak{m}-\eta}}$$

it holds that $\mathfrak{f}$ *exhibits* η*-valent convexity within the domain* $|z| < \rho_c$.

Proof. Let $f \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$, and let $\rho,\ 0<\rho<1$, be fixed. Then

$$\mathfrak{f}_\rho(z) = \rho^{-\eta}\mathfrak{f}(\rho z) = \rho^{-\eta}\mathfrak{u}(\rho z) + \rho^{-\eta}\overline{\mathfrak{v}(\rho z)} \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$$

and

$$f_\rho(z) = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} u_\mathfrak{m}\rho^{\mathfrak{m}-\eta}z^\mathfrak{m} + \overline{\sum_{\mathfrak{m}=\eta+1}^{\infty} v_\mathfrak{m}\rho^{\mathfrak{m}-\eta}z^\mathfrak{m}}, \quad z\in\mathbb{E}.$$

Lemma 3.2 states that it is sufficient to demonstrate that

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{m}^2\left(|u_\mathfrak{m}|+|v_\mathfrak{m}|\right)\rho^{\mathfrak{m}-\eta} \leq \eta^2$$

for $\rho<\rho_c$. According to Theorem 2.3 (i),

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{m}^2\left(|u_\mathfrak{m}|+|v_\mathfrak{m}|\right)\rho^{\mathfrak{m}-\eta} \leq \sum_{\mathfrak{m}=\eta+1}^{\infty}\frac{4\mathfrak{m}^2(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}\rho^{\mathfrak{m}-\eta} \qquad (3.2)$$

Additionally, taking into account that

$$\eta^2 = \eta^3\sum_{\mathfrak{m}=\eta+1}^{\infty}\frac{1}{\mathfrak{m}(\mathfrak{m}-1)},$$

It is known that the inequality (3.2) can be expressed as

$$\sum_{\mathfrak{m}=\eta+1}^{\infty}\frac{4\mathfrak{m}^2(\sigma-\gamma)(\mathfrak{m}-\xi)!}{\mathfrak{m}!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}\rho^{\mathfrak{m}-\eta} \leq \eta^3\sum_{\mathfrak{m}=\eta+1}^{\infty}\frac{1}{\mathfrak{m}(\mathfrak{m}-1)}$$

Consequently, if

$$\rho^{\mathfrak{m}-\eta} \leq \frac{(\mathfrak{m}-2)!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}{4\mathfrak{m}^2(\sigma-\gamma)(\mathfrak{m}-\xi)!}$$

for all $\mathfrak{m}\geq\eta+1$, then

$$\sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{m}^2\left(|u_\mathfrak{m}|+|v_\mathfrak{m}|\right)\rho^{\mathfrak{m}-\eta} \leq \eta^2.$$

As a result, we get

$$\rho_c = \inf_{\mathfrak{m}\geq\eta+1}\left(\frac{\eta^3(\mathfrak{m}-2)!(\mathfrak{m}-\xi+1)\left[2p+(q-p)(\mathfrak{m}-\xi)\right]}{4\mathfrak{m}^2(\sigma-\gamma)(\mathfrak{m}-\xi)!}\right)^{\frac{1}{\mathfrak{m}-\eta}}.$$

□

4. Exploring closure properties of the class $\mathcal{RH}_\eta(p,q,\gamma;\xi)$

The closure properties of the class $\mathcal{RH}_\eta(p,q,\gamma;\xi)$ under convex combinations and convolutions are shown in this section.

Theorem 4.1. *The class $\mathcal{RH}_\eta(p,q,\gamma;\xi)$ is closed under convex combinations.*

Proof. Let $\mathfrak{f}_k=\mathfrak{u}_k+\overline{\mathfrak{v}_k}\in\mathcal{RH}_\eta(p,q,\gamma;\xi)$ for $k=1,2,...,n$, and suppose $\sum\limits_{k=1}^{n}\varphi_k=1$ with $0\le\varphi_k\le 1$. The convex combination of functions $\mathfrak{f}_k$ $(k=1,2,...,n)$ can be expressed as:

$$\mathfrak{f}(z)=\sum_{k=1}^{n}\varphi_k\mathfrak{f}_k(z)=\mathfrak{u}(z)+\overline{\mathfrak{v}(z)},$$

where

$$\mathfrak{u}(z)=\sum_{k=1}^{n}\varphi_k\mathfrak{u}_k\left(z\right)\ \text{ and }\ \mathfrak{v}(z)=\sum_{k=1}^{n}\varphi_k\mathfrak{v}_k\left(z\right).$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic within the open unit disk $\mathbb{E}$, satisfying initial conditions $\mathfrak{u}(0)=\mathfrak{v}(0)=\mathfrak{u}'(0)=\mathfrak{v}'(0)=\ldots=\mathfrak{u}^{(\eta-1)}(0)=\mathfrak{v}^{(\eta-1)}(0)=\mathfrak{u}^{(\eta)}(0)-\eta!=\mathfrak{v}^{\eta}(0)=0$ and

$$\begin{aligned}
&\operatorname{Re}\left\{p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}-\gamma\right\}\\
=\ &\operatorname{Re}\left\{\sum_{k=1}^{n}\varphi_k\left(p\frac{\mathfrak{u}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{u}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{u}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}-\gamma\right)\right\}\\
>\ &\sum_{k=1}^{n}\varphi_k\left|p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right|\\
\ge\ &\left|p\frac{\mathfrak{v}^{(\xi)}(z)}{z^{\eta-\xi}}+q\frac{\mathfrak{v}^{(\xi+1)}(z)}{z^{\eta-\xi-1}}+\left(\frac{q-p}{2}\right)\frac{\mathfrak{v}^{(\xi+2)}(z)}{z^{\eta-\xi-2}}\right|
\end{aligned}$$

showing that $\mathfrak{f}\in\mathcal{RH}_\eta(p,q,\gamma;\xi)$. □

If a sequence $\{a_\mathfrak{m}\}_{\mathfrak{m}=0}^{\infty}$ of non-negative real numbers meets the following criteria, it is said to be a convex null sequence: as $\mathfrak{m}\to\infty$, $a_\mathfrak{m}$ approaches 0, and the inequality

$$a_0-a_1\ge a_1-a_2\ge a_2-a_3\ge\ldots\ge a_{\mathfrak{m}-1}-a_\mathfrak{m}\ge\ldots\ge 0$$

holds. In order to derive results for convolution, the forthcoming proofs rely on Lemma 4.2 and Lemma 4.3.

Lemma 4.2. (see [8]) When $\{a_\mathfrak{m}\}_{\mathfrak{m}=0}^{\infty}$ is a convex null sequence, indicates that the function

$$q(z)=\frac{a_0}{2}+\sum_{\mathfrak{m}=1}^{\infty}a_\mathfrak{m}z^\mathfrak{m}$$

is analytic, and the real part of $q(z)$ is positive within the open unit disk $\mathbb{E}$.

Lemma 4.3. (see [13]) Suppose the function $\Phi(z)$ is analytic within the domain $\mathbb{E}$, satisfying $\Phi(0)=1$ and $\operatorname{Re}\{\Phi(z)\}>1/2$ throughout $\mathbb{E}$. For any analytic function F defined in $\mathbb{E}$, it follows that the function $\Phi*F$ maps to values within the convex hull of the image of $\mathbb{E}$ under F.

Lemma 4.4. *Let $F\in\mathcal{R}_\eta(p,q,\gamma;\xi)$, then Re* $\left\{\dfrac{F(z)}{z^\eta}\right\}>\dfrac{1}{2}$.

Proof. Consider F belonging to the class $\mathcal{R}\eta(p,q,\gamma;\xi)$, defined as $F(z)=z^\eta+\sum_{\mathfrak{m}=\eta+1}^{\infty}\mathfrak{Um}z^\mathfrak{m}$. Then, the inequality

$$\operatorname{Re}\left\{\sigma+\sum_{\mathfrak{m}=\eta+1}^{\infty}(\mathfrak{m}-\xi+1)\left[p+\left(\frac{q-p}{2}\right)(\mathfrak{m}-\xi)\right]\frac{\mathfrak{m}!}{(\mathfrak{m}-\xi)!}\right\}>\gamma\quad(z\in\mathbb{E}),$$

can be equivalently expressed as $\mathrm{Re}\{\Phi(z)\} > \frac{1}{2}$ within the open unit disk $\mathbb{E}$, where

$$\Phi(z) = 1 + \frac{1}{4(\sigma-\gamma)} \sum_{\mathfrak{m}=\eta+1}^{\infty} (\mathfrak{m}-\xi+1)\left[2\gamma + (q-p)(\mathfrak{m}-\xi)\right] \frac{\mathfrak{m}!}{(\mathfrak{m}-\xi)!} \mathfrak{U}_{\mathfrak{m}} z^{\mathfrak{m}-\eta}.$$

Consider a sequence $\{a_{\mathfrak{m}}\}_{\mathfrak{m}=0}^{\infty}$ defined by

$$a_0 = 1 \text{ and } a_{\mathfrak{m}-1} = \frac{4(\sigma-\gamma)(\eta+\mathfrak{m}-\xi)!}{(\eta+\mathfrak{m})!(\eta+\mathfrak{m}-\xi+1)\left[2\gamma+(q-p)(\mathfrak{m}+\eta-\xi)\right]} \quad \text{for } \mathfrak{m} \geq 2.$$

The fact that the sequence $\{a_{\mathfrak{m}}\}_{\mathfrak{m}=0}^{\infty}$ forms a convex null sequence is apparent. By making use of Lemma 4.2, we can infer that the function

$$q(z) = \frac{1}{2} + \sum_{\mathfrak{m}=1}^{\infty} \frac{4(\sigma-\gamma)(\eta+\mathfrak{m}-\xi)!}{(\eta+\mathfrak{m})!(\eta+\mathfrak{m}-\xi+1)\left[2\gamma+(q-p)(\mathfrak{m}+\eta-\xi)\right]} z^{\mathfrak{m}}$$

is analytic, and $\mathrm{Re}\{q(z)\} > 0$ holds in $\mathbb{E}$. Expressing

$$\frac{F(z)}{z^\eta} = \Phi(z) * \left(1 + \sum_{\mathfrak{m}=1}^{\infty} \frac{4(\sigma-\gamma)(\eta+\mathfrak{m}-\xi)!}{(\eta+\mathfrak{m})!(\eta+\mathfrak{m}-\xi+1)\left[2\gamma+(q-p)(\mathfrak{m}+\eta-\xi)\right]} z^{\mathfrak{m}}\right),$$

and utilizing Lemma 4.3, we conclude that $\mathrm{Re}\left\{\frac{F(z)}{z^\eta}\right\} > \frac{1}{2}$ holds for $z \in \mathbb{E}$. □

Lemma 4.5. *Let $F_k \in \mathcal{R}\eta(p,q,\gamma;\xi)$ for $k = 1, 2$. Then the convolution $F_1 * F_2$ belongs to $R_\eta(p,q,\gamma;\xi)$.*

Proof. Let $F_1(z) = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{U}_{\mathfrak{m}} z^{\mathfrak{m}}$ and $F_2(z) = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{V}_{\mathfrak{m}} z^{\mathfrak{m}}$ and

$$F(z) = (F_1 * F_2)(z) = z^\eta + \sum_{\mathfrak{m}=\eta+1}^{\infty} \mathfrak{U}_{\mathfrak{m}} \mathfrak{V}_{\mathfrak{m}} z^{\mathfrak{m}}.$$

Considering

$$\frac{F^{(\xi)}(z)}{z^{\eta-\xi}} = \frac{F_1^{(\xi)}(z)}{z^{\eta-\xi}} * \frac{F_2(z)}{z^p},$$
$$\frac{F^{(\xi+1)}(z)}{z^{\eta-\xi-1}} = \frac{F_1^{(\xi+1)}(z)}{z^{\eta-\xi-1}} * \frac{F_2(z)}{z^p},$$
$$\frac{F^{(\xi+2)}(z)}{z^{\eta-\xi-2}} = \frac{F_1^{(\xi+2)}(z)}{z^{\eta-\xi-2}} * \frac{F_2(z)}{z^p},$$

we obtain

$$\begin{aligned} &\frac{1}{\sigma-\gamma}\left(p\frac{F^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{F^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{F^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right) \\ &= \frac{1}{\sigma-\gamma}\left(p\frac{F_1^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{F_1^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{F_1^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right) * \frac{F_2(z)}{z^\eta}. \end{aligned} \tag{4.1}$$

Since $F_1 \in \mathcal{R}_\eta(p,q,\gamma;\xi)$,

$$\mathrm{Re}\left\{p\frac{F_1^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{F_1^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{F_1^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right\} > 0 \quad (z \in \mathbb{E})$$

Furthermore, by applying Lemma 4.4, we deduce that $\mathrm{Re}\left\{\frac{F_2(z)}{z^\eta}\right\} > \frac{1}{2}$ within the domain $\mathbb{E}$. Utilizing Lemma 4.3 on the expression in (4.1) yields $\mathrm{Re}\left(p\frac{F^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{F^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{F^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right) > 0$ in $\mathbb{E}$. Thus, $F = F_1 * F_2 \in \mathcal{R}_\eta(p,q,\gamma;\xi)$.

□

Utilizing Lemma 4.5, we demonstrate the closed nature of the set $\mathcal{RH}_\eta(p,q,\gamma;\xi)$ when subjected to convolutions among its constituents. The convolution operations are executed following the methodologies and techniques introduced by Dorff [6].

Theorem 4.6. *Suppose $\mathfrak{f}_k \in \mathcal{RH}\eta(p,q,\gamma;\xi)$ for $k=1,2$. Then the convolution $\mathfrak{f}_1 * \mathfrak{f}_2$ is also in $\mathcal{RH}_\eta(p,q,\gamma;\xi)$.*

Proof. Suppose $\mathfrak{f}_k = \mathfrak{u}_k + \overline{\mathfrak{v}_k} \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$ $(k=1,2)$ and $\mathfrak{f}_1 * \mathfrak{f}_2 = \mathfrak{u}_1 * \mathfrak{u}_2 + \overline{\mathfrak{v}_1 * \mathfrak{v}_2}$. To prove that $\mathfrak{f}_1 * \mathfrak{f}_2 \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$, we need to demonstrate that $\mathfrak{F}_\delta = \mathfrak{u}_1 * \mathfrak{u}_2 + \delta(\mathfrak{v}_1 * \mathfrak{v}_2) \in \mathcal{R}_\eta(p,q,\gamma;\xi)$ for each δ $(|\delta|=1)$. According to Lemma 4.5, one can conclude that the set $\mathcal{R}_\eta(p,q,\gamma;\xi)$ remains closed under convolution. Hence, $\mathfrak{u}_k + \delta\mathfrak{v}_k \in \mathcal{R}_\eta(p,q,\gamma;\xi)$ for $k=1,2$. Both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ can be expressed as

$$\mathfrak{F}_1 = (\mathfrak{u}_1 - \mathfrak{v}_1) * (\mathfrak{u}_2 - \delta\mathfrak{v}_2) \quad \text{and} \quad \mathfrak{F}_2 = (\mathfrak{u}_1 + \mathfrak{v}_1) * (\mathfrak{u}_2 + \delta\mathfrak{v}_2),$$

belong to $\mathcal{R}_\eta(p,q,\gamma;\xi)$. Since $\mathcal{R}_\eta(p,q,\gamma;\xi)$ is closed under convex combinations, the function

$$\mathfrak{F}_\delta = \frac{1}{2}(\mathfrak{F}_1 + \mathfrak{F}_2) = \mathfrak{u}_1 * \mathfrak{v}_2 + \delta(\mathfrak{v}_1 * \mathfrak{v}_2)$$

belongs to $\mathcal{R}_\eta(p,q,\gamma;\xi)$. Hence, $\mathcal{RH}_\eta(p,q,\gamma;\xi)$ is closed under convolution. □

The Hadamard product of an analytic function ψ in $\mathbb{E}$. and a harmonic function $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ was defined by Goodloe [9] as follows:

$$\mathfrak{f} \tilde{*} \psi = \mathfrak{u} * \psi + \overline{\mathfrak{v} * \psi},$$

where ψ is an analytic function and $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ is a harmonic function in $\mathbb{E}$.

Theorem 4.7. *Assume $\mathfrak{f} \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$ and $\psi \in \mathcal{A}_\eta$ with the condition $Re\left(\frac{\psi(z)}{z^\eta}\right) > \frac{1}{2}$ for $z \in \mathbb{E}$. Then, the convolution $\mathfrak{f}\tilde{*}\psi$ belongs to $\mathcal{RH}_\eta(p,q,\gamma;\xi)$.*

Proof. Assume $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{RH}_\eta(p,q,\gamma;\xi)$, then $\mathfrak{F}_\delta = \mathfrak{u} + \delta\mathfrak{v} \in \mathcal{R}_\eta(p,q,\gamma;\xi)$ for each δ $(|\delta|=1)$. By Theorem 2.1, to show that $\mathfrak{f}\tilde{*}\psi \in \mathcal{RH}_\xi(p,q,\gamma;\xi)$, we need to demonstrate that $\mathfrak{G} = \mathfrak{u} * \psi + \delta(\mathfrak{v} * \psi) \in \mathcal{R}_\eta(p,q,\gamma;\xi)$ for each δ $(|\delta|=1)$. Express $\mathfrak{G}$ as $\mathfrak{G} = \mathfrak{F}_\delta * \psi$, and

$$\begin{aligned}
&\frac{1}{\sigma-\gamma}\left(p\frac{\mathfrak{G}^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{G}^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{G}^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right) \\
&= \frac{1}{\sigma-\gamma}\left(p\frac{\mathfrak{F}_\delta^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{F}_\delta^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{F}_\delta^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right) * \frac{\psi(z)}{z^\eta}.
\end{aligned}$$

Given $\text{Re}\left(\frac{\psi(z)}{z^\eta}\right) > \frac{1}{2}$ and $\text{Re}\left\{p\frac{\mathfrak{F}_\delta^{(\xi)}(z)}{z^{\eta-\xi}} + q\frac{\mathfrak{F}_\delta^{(\xi+1)}(z)}{z^{\eta-\xi-1}} + \left(\frac{q-p}{2}\right)\frac{\mathfrak{F}_\delta^{(\xi+2)}(z)}{z^{\eta-\xi-2}} - \gamma\right\} > 0$ in $\mathbb{E}$, Lemma 4.3 establishes that $\mathfrak{G} \in \mathcal{R}_\eta(p,q,\gamma;\xi)$. □

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