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RESEARCH ARTICLE

Homogeneous Geodesics of 4-dimensional Solvable Lie Groups

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

We study homogeneous geodesics in 4-dimensional solvable Lie groups Sol_0^4, Sol_1^4, $\mathrm{Sol}_{m,n}^4$ and Nil_4.

Keywords: *Homogeneous geodesic, solvable Lie group, 4-dimensional geometry.*
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1. Introduction

Totally geodesic submanifolds constitute the most fundamental class of submanifolds in Riemannian geometry. Professor Bang-yen Chen and professor Tadashi Nagano developed the so-called (M_+, M_-)-method for the study of totally geodesic submanifolds of Riemannian symmetric spaces [7, 9, 11, 12]. Klein studied totally geodesic submanifolds in complex quadrics, quaternionic 2-plane Grassmannians as well as exceptional Riemannian symmetric spaces of rank 2 [41, 42, 43, 44].

Chen and Nagano proved that a simply connected, irreducible Riemannian symmetric space admits a totally geodesic hypersurface if and only if it is of constant curvature. Tojo [73, 74] proved that a naturally reductive homogeneous space of dimension $n = 3, 4, 5$ or a normal homogeneous space admits a totally geodesic hypersurface if and only if it is of constant curvature. Tsukada [77] generalized Tojo's result on naturally reductive homogeneous spaces to arbitrary dimension $n > 2$. Nikolayevsky [62] showed that a simply connected homogeneous Riemannian space M which admits a totally geodesic hypersurface is isometric to either

- the Riemannian product $M = M_1(c) \times M_2$ of a space $M_1(c)$ of constant curvature c and a homogeneous Riemannian space M_2,
- the warped product $\mathbb{E}^{m_1} \times_f M_2$ of the Euclidean space $\mathbb{E}^{m_1}$ and a homogeneous Riemannian space $M_2 = G_2/H_2$. The warping function f is given by $f(gH_2) = \chi(g)$, where $\chi : G \to (\mathbb{R}, +)$ is a nontrivial Lie group homomorphism satisfying $\chi(H_2) = 1$,
- the twisted product $\mathbb{E}^1 \times_f M_2$ of the the Euclidean line $\mathbb{E}^1$ and homogeneous Riemannian space M_2.

Let us turn our attention to 1-dimensional totally geodesic submanifolds, *i.e.*, geodesics, in homogeneous Riemannian spaces. In homogeneous Riemannian spaces, we may restrict our attention to geodesics starting at the origin. Riemannian symmetric spaces, more generally naturally reductive homogeneous spaces have a particularly nice property (*geodesic orbit property*) that those geodesics are *homogeneous*. More precisely every geodesic starting at the origin of a naturally reductive homogeneous space is the orbit of the origin under the action of the one-parameter subgroup of the largest group of isometries. Kowalski and Vanhecke [51] introduced the notion of Riemannian g. o. space in 1983 (see also Kostant [46] and Vinberg [82]).

A reductive homogeneous Riemannian space $M = G/K$ is said to be a *Riemannian g. o. space* if it satisfies the geodesic orbit property. Kowalski and Vanhecke classified Riemannian g. o. spaces of dimension up to 6. In particular, Riemannian g. o. spaces of dimension $n \leq 4$ are naturally reductive [51]. Kajzer [40] proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic. Kowalski and Szenthe extended this result to all homogeneous Riemannian manifolds [49].

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There are three model spaces in 2-dimensional geometry:

Model space	Isotropy	Property
$\mathbb{S}^2$	SO_2	constant positive curvature
$\mathbb{E}^2$	SO_2	flat
$\mathbb{H}^2$	SO_2	constant negative curvature

There are eight model spaces in 3-dimensional geometry. The list of 3-dimensional model spaces was obtained by Thurston [72]:

Model space	Isotropy	Property
$\mathbb{S}^3, \mathbb{E}^3, \mathbb{H}^3$	SO_3	Riemannian space form
$\mathbb{S}^2 \times \mathbb{E}^1, \mathbb{H}^2 \times \mathbb{E}^1$	SO_2	Riemannian symmetric
$\mathrm{Nil}_3, \widetilde{\mathrm{SL}}_2\mathbb{R}$	SO_2	naturally reductive
Sol_3	trivial	Riemannian 4-symmetric

Except Sol_3, all the 3-dimensional model spaces are Riemannian g. o. spaces. The homogeneous geodesics of Sol_3 were determined by Marinosci [59].

According to Filipkiewicz [27], there are 19 kinds of model spaces in 4-dimensional geometry. Recently submanifold geometry of 4-dimensional geometry has received attention of differential geometers. See a survey [58]. Among the list of Filipkiewicz, there are 14 naturally reductive homogeneous spaces. Thus other spaces; Sol^4_0, Sol^4_1, $\mathrm{Sol}^4_{m,n}$ (including $\mathrm{Sol}_3 \times \mathbb{R}$), Nil_4 and F^4 are not Riemannian g. o. spaces. In this article we classify homogeneous geodesics in these model spaces. Note that homogeneous geodesics in F^4 are classified in [48] and our previous work [26], we study homogeneous geodesics in Sol^4_0, Sol^4_1, $\mathrm{Sol}^4_{m,n}$ (including $\mathrm{Sol}_3 \times \mathbb{R}$), and Nil_4 in this article.

Next, professor Chen made great progress on submanifold geometry of Kähler manifolds, especially in complex space forms. A unit speed curve $\gamma(s)$ in a Riemannian manifold (M, g) is said to be a *Riemannian circle* if there exits a positive constant k and a unit vector field E_2 along $\gamma(s)$ satisfying

$$\nabla_{\dot\gamma}\dot\gamma = k\,E_2, \quad \nabla_{\dot\gamma}E_2 = -k\,\dot\gamma.$$

In other words, γ is a Frenet curve of osculating order 2 with positive constant first curvature $\kappa_1 = k$. Note that geodesics are regarded as a a Frenet curve of osculating order 1. When the ambient space (M, g) is an almost Hermitian manifold, then the *complex torsion* τ_{12} of a Riemannian circle $\gamma(s)$ is defined by

$$\tau_{12}(s) = g(\dot\gamma(s), JE_2).$$

A Riemannian circle in an almost Hermitian manifold is said to be *holomorphic* if its complex torsion is constant.

Here we pick up Chen's research on circles in homogeneous Riemannian spaces, especially complex space forms [8, 10].

All the geodesics of a naturally reductive homogeneous space are homogeneous. However, Riemannian circles of a naturally reductive homogeneous space are not necessarily homogeneous. Mashimo and Tojo proved that every circle of a homogeneous Riemannian space M is homogeneous if and only if M is either a Euclidean space or a Riemannian symmetric space of rank one.

Chen [8] proved that a finite type isometric immersion $f : M \to \mathbb{E}^n$ of a compact irreducible homogeneous Riemannian space M into Euclidean n-space carries every homogeneous curve in M to a curve of finite type in $\mathbb{E}^n$.

In [10], Chen and Maeda studied circles in the complex projective n-space $\mathbb{C}P_n(4)$ through the first standard imbedding:

$$\mathbb{C}P_n(4) \hookrightarrow \mathbb{S}^{n(n+2)-1}(2(n+1)/n) \subset \mathbb{E}^{n(n+2)}.$$

For instance, the image of a circle in $\mathbb{C}P_n(4)$ with complex torsion τ_{12} under the first standard imbedding is of 1-type, 2-type or 3-type in $\mathbb{E}^{n(n+2)}$ according as $\tau_{12} = \pm 1$, $\tau_{12} = 0$ or $\tau_{12} \neq \pm 1, 0$.

The study of holomorphic circles in a Kähler manifold with complex torsion ± 1 has another motivation. To explain this, here we recall the notion of J-trajectory as well as that of Kähler magnetic trajectory [4].

Let (M, g, J) be an almost Hermitian manifold. Then a regular curve $\gamma(t)$ in M is said to be a *J-trajectory* if it satisfies

$$\nabla_{\dot\gamma}\dot\gamma = qJ\dot\gamma.$$

Here q is a constant called the *charge*. When (M, g, J) is an almost Kähler manifold, then a J-trajectory is called a *Kähler magnetic trajectory* since it is a magnetic trajectory with respect to the Kähler magnetic field $g(J, \cdot)$.

Let us consider a unit speed Kähler magnetic trajectory $\gamma(s)$ in a Kähler manifold (M, g, J), then $\gamma(s)$ is a Riemannian circle of constant first curvature $|q|$ and complex torsion ± 1. Indeed, we can take $E_2 = \varepsilon J\dot{\gamma}$, where $\varepsilon = \pm 1$ and $\tau = -\varepsilon$.

The model spaces Sol_0^4 and Sol_1^4 admit a compatible complex structure. The resulting homogeneous Hermitian surfaces are globally conformal Kähler. The globally conformal Kähler surfaces Sol_0^4 and Sol_1^4 are universal coverings of Inoue surfaces (see [68, 75]). It should be mentioned that Chen and Piccinni [13] studied foliations of locally conformal Kähler manifolds (LCK manifolds).

On the other hand, $\mathrm{Sol}_{m,n}^4$ does not admit a compatible complex structure. The model space Nil_4 does not admit a compatible complex structure, but has compatible symplectic structure. As a result, Nil_4 admits a compatible almost Kähler structure.

The second purpose of this article is to determine homogeneous J-trajectories in the model spaces Sol_0^4, Sol_1^4 and Nil_4.

2. Riemannian geodesic orbit spaces

2.1. Homogeneous geodesics

Let $M = G/K$ be a homogeneous Riemannian space. A curve $\gamma(s)$ starting at the origin $o \in M$ is said to be *homogeneous* with respect to the coset space representation G/K if it is represented as

$$\gamma(s) = \exp_{\mathfrak{g}}(sX) \cdot o$$

for some $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. When $\gamma(s)$ is a geodesic, then the vector X is called a *geodesic vector*.

Definition 2.1. A homogeneous Riemannian space $M = G/K$ is called a *space with homogeneous geodesics* or a *Riemannian g.o. space* if every geodesic $\gamma(s)$ of M is an orbit of a one-parameter subgroup of the *largest* connected group of isometries.

As is well known, every homogeneous Riemannian space $M = G/K$ admits a *Lie subspace* $\mathfrak{m}$, that is, a linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ satisfying

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{m}. \tag{2.1}$$

Here $\mathfrak{k}$ is the Lie algebra of the isotropy subgroup K (called the *isotropy algebra*).

The decomposition (2.1) is called a *reductive decomposition* of $\mathfrak{g}$. For a vector $X \in \mathfrak{g}$, we denote by $X_{\mathfrak{k}}$ and $X_{\mathfrak{m}}$, the $\mathfrak{k}$-component and $\mathfrak{m}$-component of X, respectiveley, *i.e.,*

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \quad X_{\mathfrak{k}} \in \mathfrak{k}, \quad X_{\mathfrak{m}} \in \mathfrak{m}.$$

A homogeneous Riemannian space $M = G/K$ with a fixed Lie subspace $\mathfrak{m}$ is called a *reductive homogeneous Riemannian space*. Hereafter we only consider reductive homogeneous Riemannian spaces. Denote by $\pi : G \to G/K$, the projection. Take the differential map $\pi_{*\mathsf{e}}$ at the identity $\mathsf{e} \in G$. Then $\pi_{*\mathsf{e}}|_{\mathfrak{m}} : \mathfrak{m} \to T_oM$ is a linear isomorphism. We identify T_oM with $\mathfrak{m}$ and regard it as a linear subspace of $\mathfrak{g}$ through the inverse mapping of $\pi_{*\mathsf{e}}|_{\mathfrak{m}}$.

Example 2.1 (The space of inner products)**.** Let us denote by $\widetilde{\mathcal{M}}(\mathbb{R}^n)$ the set of all inner products on $\mathbb{R}^n$. Next, let

$$\mathrm{Sym}_n^+\mathbb{R} = \{F \in \mathrm{GL}_n\mathbb{R} \mid \det F > 0\}.$$

be the set of all positive definite symmetric matrices of degree n. As is well known, $\widetilde{\mathcal{M}}(\mathbb{R}^n)$ is identified with $\mathrm{Sym}_n^+\mathbb{R}$. The identification is given by

$$\mathrm{Sym}_n^+\mathbb{R} \ni F \longmapsto \mathsf{F} := \mathsf{F}_0(F, \cdot) \in \widetilde{\mathcal{M}}(\mathbb{R}^n),$$

where F_0 is the Euclidean inner product of $\mathbb{R}^n$. The inner product F is defined by

$$\mathsf{F}(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{F}_0(F\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

The general linear group $\mathrm{GL}_n\mathbb{R}$ acts on $\widetilde{\mathcal{M}}(\mathbb{R}^n)$ via the action

$$\mathrm{GL}_n\mathbb{R} \times \widetilde{\mathcal{M}}(\mathbb{R}^n) \to \widetilde{\mathcal{M}}(\mathbb{R}^n); \quad (A \cdot \mathsf{F})(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{F}(A^{-1}\boldsymbol{x}, A^{-1}\boldsymbol{y}).$$

The isotropy subgroup at the Euclidean inner product F_0 is O_n. Hence we get

$$\widetilde{\mathcal{M}}(\mathbb{R}^n) = \mathrm{Sym}_n^+\mathbb{R} = \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n.$$

The tangent space $T_{\mathsf{F}_0}\widetilde{\mathcal{M}}(\mathbb{R})$ is identified with the linear space $\mathfrak{m} = \mathrm{Sym}_n\mathbb{R}$ of symmetric matrices. The isotropy algebra is $\mathfrak{o}_n$. Hence we get a reductive decomposition $\mathfrak{gl}_n\mathbb{R} = \mathfrak{o}_n \oplus \mathfrak{m}$. Thus $\widetilde{\mathcal{M}}(\mathbb{R}) = \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$ is a reductive homogeneous space. Moreover we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. Note that $\exp : \mathrm{Sym}_n\mathbb{R} \to \mathrm{Sym}_n^+\mathbb{R}$ is surjective. The inner product

$$\langle X, Y\rangle = \mathrm{tr}(XY), \quad X, Y \in \mathfrak{m}$$

is $\mathrm{Ad}(\mathrm{O}_n)$-invariant. The reductive homogeneous Riemannian space $\mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$ is a Riemannian symmetric space. Hence $\mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$ is a Riemannian g. o. space.

Next, we introduce a tensor $\mathsf{U}_\mathfrak{m} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$2\langle \mathsf{U}_\mathfrak{m}(X, Y), Z\rangle = -\langle X, [Y, Z]_\mathfrak{m}\rangle + \langle Y, [Z, X]_\mathfrak{m}\rangle, \quad X, Y, Z \in \mathfrak{m}.$$

A homogeneous Riemannian space is said to be *naturally reductive* if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ with vanishing $\mathsf{U}_\mathfrak{m}$. As we will explain later, naturally reductive homogeneous spaces are Riemannian g. o. spaces (see Corollary 2.1).

Let G be a compact semi-simple Lie group, then the Killing form B is negative definite on $\mathfrak{g}$. Thus for any positive constant c, $-c\mathsf{B}$ induces a bi-invariant Riemannian metric on G. A homogeneous Riemannian space $M = G/K$ with compact semi-simple G is said to be *normal* if its G-invariant Riemannian metric is derived from a bi-invariant Riemannian metric of G. It is well known that every normal homogeneous space is naturally reductive. Moreover (irreducible) Riemannian symmetric spaces are naturally reductive.

2.2. *The homogeneous geodesic equation*

Let $M = G/K$ be a reductive homogeneous Riemannian space with Lie subspace $\mathfrak{m}$. Take vectors X, $Z \in \mathfrak{g}$ and set $\phi_t = \exp(tX)$ and $\psi_s = \exp(sZ)$. The Killing vector field $X^\sharp$ derived from X is defined by

$$X^\sharp_p = \left.\frac{d}{dt}\right|_{t=0} \exp_\mathfrak{g}(tX) \cdot p, \quad p \in M.$$

At any point $x \in M$, we have [52, p. 193]:

$$Z^\sharp_{\phi_t(p)} = \phi_{t*(\phi_t^{-1}\circ\psi_s\circ\phi_t)(p)} \left(Z - t[X, Z] + o(t^2)\right)^\sharp_p, \quad X^\sharp_{\psi_s(p)} = \psi_{s*(\psi_s^{-1}\circ\phi_t\circ\psi_s)(p)} \left(X - t[Z, X] + o(t^2)\right)^\sharp_p. \tag{2.2}$$

From the Koszul formula we have

$$2g(\nabla_{X^\sharp}X^\sharp, Z^\sharp) = 2X^\sharp g(X^\sharp, Z^\sharp) - Z^\sharp g(X^\sharp, X^\sharp) + 2g([Z^\sharp, X^\sharp], X^\sharp).$$

From (2.2), we get

$$X^\sharp_p\, g(X^\sharp, Z^\sharp) = g_p(X^\sharp, [X^\sharp, Z^\sharp]), \quad Z^\sharp_p\, g(X^\sharp, X^\sharp) = 2g_p(X^\sharp, [Z^\sharp, Z^\sharp]).$$

Hence, we deduce that

$$g_p(\nabla_{X^\sharp}X^\sharp, Z^\sharp) = -g_p(X^\sharp, [X, Z]^\sharp) = -\langle X_\mathfrak{m}, [X, Z]_\mathfrak{m}\rangle = -\langle [X, Z]_\mathfrak{m}, X_\mathfrak{m}\rangle. \tag{2.3}$$

This equation implies the following useful criterion ([52, Proposition 2.1], [5, Theorem 5.2], see also [46, 82]). Here we give a proof for completeness and later use.

Proposition 2.1. *Let $M = G/K$ be a reductive homogeneous Riemannian space equipped with a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Take a vector $X = X_\mathfrak{k} + X_\mathfrak{m} \in \mathfrak{g}$. Then*

$$\gamma(s) = \exp_\mathfrak{g}(sX) \cdot o$$

is a geodesic if and only if one of the following conditions are fulfilled:

1. $[X_{\mathfrak{k}}, X_{\mathfrak{m}}] + \mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}}) = 0$.
2. $\langle [X_{\mathfrak{k}}, X_{\mathfrak{m}}], Z\rangle = \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z]_{\mathfrak{m}}\rangle$ *for any* $Z \in \mathfrak{m}$.
3. $\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0$ *for any* $Z \in \mathfrak{m}$.

Proof. The equation (2.3) implies that the geodesic equation is equivalent to (3).
The tensor $\mathsf{U}_{\mathfrak{m}}$ satisfies

$$\langle \mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}}), Z_{\mathfrak{m}}\rangle = -\langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z_{\mathfrak{m}}]_{\mathfrak{m}}\rangle.$$

Next, we get

$$\begin{aligned}\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle =& \langle X_{\mathfrak{m}}, [X_{\mathfrak{k}} + X_{\mathfrak{m}}, Z_{\mathfrak{k}} + Z_{\mathfrak{m}}]_{\mathfrak{m}}\rangle = \langle X_{\mathfrak{m}}, [X_{\mathfrak{k}}, Z_{\mathfrak{m}}] + [X_{\mathfrak{m}}, Z_{\mathfrak{k}}] + [X_{\mathfrak{m}}, Z_{\mathfrak{m}}]_{\mathfrak{m}}\rangle \\ =& \langle X_{\mathfrak{m}}, [X_{\mathfrak{k}}, Z_{\mathfrak{m}}] + [X_{\mathfrak{m}}, Z_{\mathfrak{k}}]\rangle + \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z_{\mathfrak{m}}]_{\mathfrak{m}}\rangle \\ =& \langle X_{\mathfrak{m}}, [X_{\mathfrak{k}}, Z_{\mathfrak{m}}] + [X_{\mathfrak{m}}, Z_{\mathfrak{k}}]\rangle - \langle \mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}}), Z_{\mathfrak{m}}\rangle.\end{aligned}$$

The G-invariance of the metric g implies the $\mathrm{Ad}(K)$-invariance of the inner product $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$, we have

$$\langle X_{\mathfrak{m}}, [X_{\mathfrak{k}}, Z_{\mathfrak{m}}] + [X_{\mathfrak{m}}, Z_{\mathfrak{k}}]\rangle = \langle X_{\mathfrak{m}}, [X_{\mathfrak{k}}, Z_{\mathfrak{m}}]\rangle + \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z_{\mathfrak{k}}]\rangle = \langle [X_{\mathfrak{m}}, X_{\mathfrak{k}}], Z_{\mathfrak{m}}\rangle.$$

Hence we get

$$g_p(\nabla_{X^\sharp} X^\sharp, Z^\sharp) = -\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = \langle [X_{\mathfrak{k}}, X_{\mathfrak{m}}] + \mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}}), Z_{\mathfrak{m}}\rangle.$$

This equation implies that $\gamma(s)$ is a geodesic if and only if X satisfies (1).
Finally, for any $Z \in \mathfrak{m}$, we have

$$-\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = \langle [X_{\mathfrak{k}}, X_{\mathfrak{m}}]\rangle - \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z]_{\mathfrak{m}}\rangle.$$

Thus we show the equivalence of the geodesic equation and (2). ☐

Corollary 2.1. *Let $M = G/K$ be a naturally reductive homogeneous space with naturally reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then any geodesic γ starting at the origin o with initial tangent vector $X \in \mathfrak{m}$ is expressed as $\gamma(s) = \exp(sX)\cdot o$.*

Kowalski and Szenthe [49] proved the following fundamental fact.

Theorem 2.1. *Every homogeneous Riemannian space has at least one homogeneous geodesic passing through a given point.*

For more information on Riemannian g. o. spaces, we refer to [5, 29].

2.3. *Naturally reductive homogeneous metrics on non-compact Lie groups*

Here we exhibit typical examples of naturally reductive homogeneous spaces.
Let G be a connected non-compact semi-simple Lie group with Lie algebra $\mathfrak{g}$, then there exists an involutive automorphism θ of $\mathfrak{g}$ satisfying the condition that the symmetric bilinear form

$$\mathsf{B}_\theta(X, Y) := -\mathsf{B}(X, \theta Y), \quad X, Y \in \mathfrak{g}$$

is positive definite. Such an involutive automorphism is unique up to G-conjugation and is called the *Cartan involution*. Since the eigenvalues of θ are 1 and -1, one obtains an eigenspace decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}.$$

This decomposition is called the *Cartan decomposition*. One can see that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The eigenspace $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. Let K be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. We consider the action of the product Lie group $G \times K$ on G by

$$(G \times K) \times G \to G; \quad (a, k)b = abk^{-1}.$$

This action is transitive. The isotropy subgroup at the identity e is

$$\Delta K = \{(k, k) \mid k \in K\}$$

with Lie algebra
$$\Delta\mathfrak{k} = \{(V, V) \mid V \in \mathfrak{k}\}.$$
The Lie group G is expressed by $G = (G \times K)/\Delta K$ as a reductive homogeneous space with reductive decomposition
$$\mathfrak{g} \times \mathfrak{k} = \Delta\mathfrak{k} \oplus \mathfrak{m},$$
where the Lie subspace $\mathfrak{m}$ is given by
$$\mathfrak{m} = \{(Y + W, -W) \mid Y \in \mathfrak{p},\ W \in \mathfrak{k}\}.$$
Every $(X, V) = (X_{\mathfrak{k}} + X_{\mathfrak{p}}, V) \in \mathfrak{g} \times \mathfrak{k}$ is decomposed as
$$(X, V) = \left(\frac{1}{2}(X_{\mathfrak{k}} + V), \frac{1}{2}(X_{\mathfrak{k}} + V)\right) + \left(X_{\mathfrak{p}} + \frac{1}{2}(X_{\mathfrak{k}} + V) - V, -\frac{1}{2}(X_{\mathfrak{k}} + V) + V\right).$$
Thus the $\Delta\mathfrak{k}$-part and $\mathfrak{m}$-part of (X, V) are
$$(X, V)_{\Delta\mathfrak{k}} = \left(\frac{1}{2}(X_{\mathfrak{k}} + V), \frac{1}{2}(X_{\mathfrak{k}} + V)\right),$$
$$(X, V)_{\mathfrak{m}} = \left(X_{\mathfrak{p}} + \frac{1}{2}(X_{\mathfrak{k}} + V) - V, -\frac{1}{2}(X_{\mathfrak{k}} + V) + V\right).$$
Proposition 2.2. *Let G be a non-compact semi-simple Lie group with maximal compact subgroup K and Cartan involution θ. Represent G as a reductive homogeneous space $G = (G \times K)/\Delta K$ with Lie subspace $\mathfrak{m} = \{(Y + W, -W) \mid Y \in \mathfrak{p}, W \in \mathfrak{k}\}$. With respect to the $(G \times K)$-invariant Riemannian metric g induced from $\mathsf{B}_\theta \times \mathsf{B}_\theta$, every geodesic $\gamma(s)$ starting at the origin $o = \mathsf{e} \in G$ with initial velocity $X = X_{\mathfrak{k}} + X_{\mathfrak{m}} \in \mathfrak{g}$ is represented by*
$$\gamma(s) = \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{p}})\} \cdot \exp_K\{2s(X_{\mathfrak{k}})\}. \tag{2.4}$$
It should be remarked that
$$\exp_{G\times K}(sX) = \exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{p}})\} \cdot \exp_K\{2s(X_{\mathfrak{k}})\}$$
holds. Hence all the geodesics starting at the origin of $(G \times K)/\Delta K$ are homogeneous. Indeed, $(G \times K)/\Delta K$ is naturally reductive (and hence it is a Riemannian g. o. space).

More generally the following result is known ([14, 28, 32, 85]):

Theorem 2.2. *Let G be a non-compact semi-simple Lie group with maximal compact subgroup K and Cartan involution θ. Introduce an inner product*
$$\langle X, Y\rangle^{(c)} := -c\mathsf{B}(X_{\mathfrak{k}}, X_{\mathfrak{k}}) + \mathsf{B}(X_{\mathfrak{p}}, X_{\mathfrak{p}})$$
on $\mathfrak{g}$. Here $c > 0$ is a constant. Let us regard G as a reductive homogeneous Riemannian space $(G \times K)/\Delta K$ with Lie subspace
$$\mathfrak{m} = \{(-cX_{\mathfrak{k}} + X_{\mathfrak{p}}, -(1 + c)X_{\mathfrak{k}}) \mid X_{\mathfrak{k}} \in \mathfrak{k},\ X_{\mathfrak{p}} \in \mathfrak{p}\} \tag{2.5}$$
and the $(G \times K)$-invariant Riemannian metric $g^{(c)}$ induced from $\langle\cdot,\cdot\rangle^{(c)}$. Then $G = (G \times K)/\Delta K$ is naturally reductive and every geodesic starting at the origin with $o = \mathsf{e} \in G$ with initial velocity $X = X_{\mathfrak{k}} + X_{\mathfrak{m}} \in \mathfrak{g}$ is represented by
$$\exp_{G\times K}(sX) = \exp_G\{s(-cX_{\mathfrak{k}} + X_{\mathfrak{p}})\} \cdot \exp_K\{s(1 + c)(X_{\mathfrak{k}})\}. \tag{2.6}$$

2.4. Naturally reductive homogeneous metrics on compact Lie groups

Let G be a compact semi-simple Lie group. Take a non-compact real form G' of the complexification of $G^{\mathbb{C}}$ and set $K = G \cap G'$. The Lie algebra $\mathfrak{g}'$ of G' has a Cartan involution θ and admits the corresponding Cartan decomposition $\mathfrak{g}' = \mathfrak{k} \oplus \mathfrak{p}'$. Then we have the decomposition
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} = \sqrt{-1}\mathfrak{p}'.$$
Then we can introduce an inner product
$$\langle X, Y\rangle^{(c)} := c\mathsf{B}(X_{\mathfrak{k}}, X_{\mathfrak{k}}) - \mathsf{B}(X_{\mathfrak{p}}, X_{\mathfrak{p}}),$$
where $c > 0$ is a constant. The Riemannian metric $g^{(c)}$ induced from $\langle\cdot,\cdot\rangle^{(c)}$ is invariant under the action of $G \times K$ on G. Thus we obtain a reductive homogeneous Riemannian space $((G \times K)/\Delta K, g^{(c)})$ with Lie subspace (2.5). The resulting homogeneous Riemannian space is naturally reductive. Every geodesic starting at the origin with $o = \mathsf{e} \in G$ with initial velocity $X = X_{\mathfrak{k}} + X_{\mathfrak{m}} \in \mathfrak{g}$ is represented by (2.6).

2.5. Low dimensional naturally reductive homogeneous spaces

Kowalski and Vanhecke [52] proved that every n-dimensional Riemannian g. o. space of dimension $n < 5$ is naturally reductive.

The 3-dimensional naturally reductive homogeneous spaces are classified by Tricerri and Vanhecke.

Theorem 2.3 ([76]). *A 3-dimensional simply connected naturally reductive homogeneous space is a Riemannian symmetric space or one of the following spaces*:

- *The Heisenberg group* $\mathrm{Nil}_3 = (\mathrm{Nil}_3 \rtimes \mathrm{SO}_2)/\mathrm{SO}_2$.
- *The Berger 3-sphere* $(\mathrm{SU}_2 \times \mathrm{U}_1)/\mathrm{U}_1$.
- *The universal covering* $\widetilde{\mathrm{SL}}_2\mathbb{R}$ *of the homogeneous space* $\mathrm{SL}_2\mathbb{R} = (\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}_2)/\mathrm{SO}_2$.

In this list, the naturally reductive homogeneous structures on the Berger 3-sphere and $\mathrm{SL}_2\mathbb{R}$ are those exhibited in Section 2.4 and Section 2.3, respectively.

All the 3-dimensional model spaces except the model space Sol_3 are Riemannian g. o. spaces. Here we give the list of homogeneous geodesics in Sol_3 (see Marinosci [59]). The model space Sol_3 is the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ equipped with metric

$$g = e^{-2z}dx^2 + e^{2z}dy^2 + dz^2.$$

The model space Sol_3 is identified with the linear Lie group

$$\left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra of Sol_3 is given by

$$\mathfrak{sol}_3 = \left\{ \begin{pmatrix} w & 0 & u \\ 0 & -w & v \\ 0 & 0 & 0 \end{pmatrix} \middle| \; u, v, w \in \mathbb{R} \right\}.$$

The metric g is left invariant. We can take a left invariant orthonormal frame field:

$$e_1 = \frac{1}{\sqrt{2}}\left(-e^z \frac{\partial}{\partial x} + e^{-z}\frac{\partial}{\partial y}\right), \quad e_2 = \frac{1}{\sqrt{2}}\left(e^z \frac{\partial}{\partial x} + e^{-z}\frac{\partial}{\partial y}\right), \quad e_3 = \frac{\partial}{\partial z}.$$

This frame field satisfies the commutation relations.

$$[e_1, e_2] = 0, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = -e_2.$$

The full isometry group of Sol_3 is $\mathrm{Sol}_3 \rtimes \mathrm{D}_4$. The action of the dihedral group D_4 with 8 elements on Sol_3 is described as:

$$(x, y, z) \longmapsto (y, -x, -z), \quad (x, y, z) \longmapsto (-x, y, z),$$
$$(x, y, z) \longmapsto (-x, -y, z), \quad (x, y, z) \longmapsto (-y, x, -z), \quad (x, y, z) \longmapsto (y, x, -z),$$
$$(x, y, z) \longmapsto (y, x, z), \quad (x, y, z) \longmapsto (x, -y, z).$$

Hence, the action of $\mathrm{Sol}_3 \rtimes \mathrm{D}_4$ is described as

$$(x, y, z) \longmapsto (\pm e^c x + a, \pm e^{-c} y + b, z + c)$$

or

$$(x, y, z) \longmapsto (\pm e^c y + a, \pm e^{-c} x + b, z + c).$$

The identity component of the full isometry group is Sol_3. Thus we regard Sol_3 as a reductive homogeneous space $\mathrm{Sol}_3/\{\mathsf{e}\}$.

Proposition 2.3 ([59]). *Any unit speed homogeneous geodesic starting at the origin of* Sol_3 *has the form*:

$$\exp(se_1), \quad \exp(se_2), \; \textit{or} \; \exp(se_3).$$

Proof. The symmetric tensor $\mathsf{U} = \mathsf{U}_{\mathfrak{sol}_3}$ is computed as

$$\mathsf{U}(e_1, e_2) = -e_3, \quad \mathsf{U}(e_1, e_3) = \frac{1}{2}e_2, \quad \mathsf{U}(e_2, e_3) = \frac{1}{2}e_1.$$

For a unit vector $X = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \mathfrak{sol}_3$, $\mathsf{U}(X, X) = 0$ holds if and only if

$$X = \pm e_1, \quad \pm e_2, \text{ or } \pm e_3.$$

□

The 4-dimensional naturally reductive homogeneous spaces are classified by Kowalski and Vanhecke.

Theorem 2.4 ([51]). *A 4-dimensional simply connected naturally reductive homogeneous space is a Riemannian symmetric space or the direct product $N \times \mathbb{R}$, where N is a 3-dimensional simply connected naturally reductive homogeneous space.*

All the simply connected naturally reductive homogeneous spaces (with respect to the coset space representation of the largest isometry group) are model spaces of 4-dimensional geometry.

Theorem 2.5. *All the 4-dimensional simply connected naturally reductive homogeneous spaces are one of the following model spaces*:

Model space	*Isotropy*	*Property*
$\mathbb{E}^4$, $\mathbb{S}^4$, $\mathbb{H}^4$	SO_4	*Riemannian space form*
$\mathbb{C}P_2$, $\mathbb{C}H_2$	U_2	*Complex space form*
$\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{H}^3 \times \mathbb{R}$	SO_4	*Riemannian symmetric*
$\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{S}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$	$\mathrm{SO}_2 \times \mathrm{SO}_2$	*Riemannian symmetric*
$\mathrm{Nil}_3 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$,	SO_2	*naturally reductive*

and the product manifold $\{(\mathrm{SU}_2 \times \mathrm{U}_1)/\mathrm{U}_1\} \times \mathbb{E}^1$ of the Berger 3-sphere $(\mathrm{SU}_2 \times \mathrm{U}_1)/\mathrm{U}_1$ equipped with naturally reductive metric and Euclidean line $\mathbb{E}^1$.

This list provides all the simply connected 4-dimensional Riemannian g. o. spaces.

2.6. *Two-step homogeneous geodesics*

As a generalization of homogeneous geodesics, the notion of 2-step homogeneous geodesic was introduced (see [3]).

Definition 2.2. A geodesic $\gamma(s)$ starting at the origin o of a reductive homogeneous Riemannian space $M = G/K$ is said to be a 2-*step homogeneous geodesic* if it has the form

$$\gamma(s) = \{\exp_G(sX)\exp_G(sY)\} \cdot o$$

for some $X, Y \in \mathfrak{g}$.

Let $M = G/K$ be a reductive homogeneous Riemannian space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let us assume that the Lie subspace $\mathfrak{m}$ admits a splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ satisfying

$$[\mathfrak{k}, \mathfrak{m}_1] \subset \mathfrak{m}_1, \quad [\mathfrak{k}, \mathfrak{m}_2] \subset \mathfrak{m}_2, \quad [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1 \tag{2.7}$$

and there exits a nonzero constant c such that

$$\langle [X, Y]_{\mathfrak{m}_2}, Z\rangle - c\,\langle X, [Z, Y]\rangle = 0, \quad X, Y \in \mathfrak{m}_1,\ Z \in \mathfrak{m}_2. \tag{2.8}$$

Under these assumptions, Dohira proved the following result.

Proposition 2.4. *Every geodesic $\gamma(s)$ staring at the origin o with initial vector $X = X_1 + X_2 \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is represented as*

$$\gamma(s) = \{\exp_{\mathfrak{g}}\{s(X_1 - cX_2)\}\exp_{\mathfrak{g}}\{s(1 + c)X_2\}\} \cdot o$$

Let us recall the naturally reductive homogeneous space $(G \times K)/\Delta K$ exhibited in Section 2.3. The Lie subspace $\mathfrak{m}$ is decomposed as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$;

$$\mathfrak{m}_1 = \{(Y, 0) \mid Y \in \mathfrak{p}\}, \quad \mathfrak{m}_2 = \{(W, -W) \mid W \in \mathfrak{k}\}.$$

One can see that the splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ satisfies (2.7). Next,

$$\begin{aligned}(\mathsf{B}_\theta \times \mathsf{B}_\theta)\Big|_{\mathfrak{m}\times\mathfrak{m}}((X+V, -V), (Y+W, -W)) =& \mathsf{B}_\theta(X+V, Y+W) + \mathsf{B}_\theta(V, W)\\ =& 2\mathsf{B}(V, W) - \mathsf{B}(X, Y).\end{aligned}$$

This shows that the $(G \times K)$-invariant Riemannian metric induced from $(\mathsf{B}_\theta \times \mathsf{B}_\theta)|_{\mathfrak{g}\times\mathfrak{k}}$ satisfies (2.8) with $c = 1$. Hence we can apply Dohira's result (Proposition 2.4) to $(G \times K)/\Delta K$. Then we retrieve Proposition 2.2. Although Dohira [19] does not mention, the 2-step geodesic is rewritten as

$$\exp_G\{s(-X_{\mathfrak{k}} + X_{\mathfrak{p}})\} \cdot \exp_K\{2s(X_{\mathfrak{k}})\} = \exp_{G\times K}(sX).$$

Namely those geodesics are homogeneous ones.

3. Complex structures

3.1. Almost Käher and GCK-manifolds

Let (M, g, J) be an almost Hermitian manifold. If its almost complex structure J is integrable, then (M, g, J) is said to be a *Hermitian manifold*.

The *fundamental* 2*-form* of (M, g, J) is a non-degenerate 2-form defined by

$$\Omega(X, Y) = g(X, JY).$$

An almost Hermitian manifold (M, g, J) is said to be an *almost Kähler manifold* if its fundamental 2-form is closed. Note that the fundamental 2-form of an almost Kähler manifold is symplectic. A Hermitian manifold with closed fundamental 2-form is called a *Kähler manifold*. An almost Kähler manifold is said to be *strict* if its almost complex structure is non-integrable.

An almost Hermitian manifold (M, g, J) is said to be a *locally conformal Kähler manifold* (LCK-manifold, in short) if there exists an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ together with a family of smooth functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that $(U_\alpha, e^{-\sigma_\alpha} g|_{U_\alpha}, J|_{U_\alpha})$ is Kähler for all $\alpha \in \Lambda$. In case $U_\alpha = M$, a locally conformal Kähler manifold M is called a *globally conformal Kähler manifold* (GCK-manifold, in short). On an LCK-manifold M, $\omega = d\sigma_\alpha$ is globally defined and called the *Lee form*. The Lee form is characterized by the equation $d\Omega = \omega \wedge \Omega$. The vector field B metrically dual to ω is called the *Lee field*. On the other hand $A := JB$ is is called the *anti-Lee field*. An LCK manifold is called a *Vaisman manifold* if B is parallel. On a Vaisman manifold M, the distribution spanned by A and B is integrable. The foliation determined by this distribution is called the *canonical foliation*.

Chen and Piccinni [13] studied the following three kinds of foliations on arbitrary (non-Käher) LCK manifolds:

- The foliation $\mathcal{F}$ defined by the Pfaff equation $\omega = 0$.
- The foliation $\mathcal{F}^\perp$ generated by B.
- The foliation $\mathcal{D}^\perp$.

The third foliation $\mathcal{D}^\perp$ is defined in the following manner:

Let N be a leaf of $\mathcal{F}$. Then at any point p of N, we denote by $\mathcal{D}_{N;p}$ the maximal J-invariant linear subspace of T_pN. Then we obtain a distribution $\mathcal{D}_N$ on N by the correspondence $p \longmapsto \mathcal{D}_{N;p}$. Next by taking the orthogonal complement $(\mathcal{D}^N_p)^\perp$ of $\mathcal{D}^N_p$, we obtain an integrable distribution $\mathcal{D}^\perp_N$.

3.2. Hermitian model spaces

The 4-dimensional model spaces which admit compatible complex structure are classified as the following list due to Wall [83, 84]:

Complex space form	Hermitian symmetric	Kähler	Globally conformal Kähler
$\mathbb{C}P^2, \mathbb{C}H^2, \mathbb{E}^4$	$\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{E}^2, \mathbb{S}^2 \times \mathbb{H}^2$ $\mathbb{E}^2 \times \mathbb{H}^2, \mathbb{H}^2 \times \mathbb{H}^2$	F^4	$\mathbb{S}^3 \times \mathbb{E}^1, \mathrm{Nil}_3 \times \mathbb{E}^1, \widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$ $\mathrm{Sol}^4_0, \mathrm{Sol}^4_1$

In this list the fundamental 2-forms of GCK-model spaces are not symplectic. On the other hand, as we will see later the model space Nil_4 admits compatible strictly almost Kähler structures.

Ovando [65] studied invariant complex structures on solvable Lie groups. She classified invariant complex structures and invariant symplectic structures on 4-dimensional Lie groups [66]. As a result, Ovando obtained the classification of left invariant Kähler structures on 4-dimensional Lie groups. Snow [70] also classified invariant complex structures on 4-dimensional solvable Lie groups.

Model space	Notation in [2]	Notation in [6]	Notation in [56]	Notation in [70]
$\mathrm{Sol}^4_{m,n}$	$\mathfrak{r}_{4,\mu,-1-\mu}$	$\mathfrak{g}_{4.5}^{-1,-1-\beta}$	$U1[1,1,1]$	
Sol^4_0	$\mathfrak{r}_{4,-\frac{1}{2},-\frac{1}{2}}$	$\mathfrak{g}_{4.5}^{-\frac{1}{2},-\frac{1}{2}}$	$U1[1,1,1], \quad \lambda=\mu=1$	
Sol^4_1	$\mathfrak{d}_4$	$\mathfrak{g}_{4.8}^{-1} \oplus \mathfrak{g}_1$	$U3I0$	H1
Nil_4	$\mathfrak{n}_4$	$\mathfrak{g}_{4.1}$	$U1[3]$	S4

Remark 3.1. Şukiloviç [71] gave a classification of left invariant metrics on 4-dimensional solvable Lie groups in terms of curvatures.

3.3. *Curve theory in almost Hermitian manifolds*

Definition 3.1. If γ is a curve in a Riemannian manifold M, parametrized by arc length s, we say that γ is a *Frenet curve of osculating order* r if there exist orthonormal vector fields E_1, $E_2, \cdots, E_r$ along γ such that

$$\begin{aligned} &\dot{\gamma} = E_1,\ \nabla^g_{\dot{\gamma}}E_1 = \kappa_1 E_2,\ \nabla^g_{\dot{\gamma}e}E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \cdots, \\ &\nabla^g_{\dot{\gamma}}E_{r-1} = -\kappa_{r-2}E_{r-2} + \kappa_{r-1}E_r,\ \nabla^g_{\dot{\gamma}}E_r = -\kappa_{r-1}E_{r-1}, \end{aligned} \tag{3.1}$$

where $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are positive C^∞ functions of s. The function κ_j is called the j-th *curvature* of γ.

A *geodesic* is regarded as a Frenet curve of osculating order 1. A *Riemannian circle* (also called a *geodesic circle*) is defined as a Frenet curve of osculating order 2 with *constant* κ_1. Note that Riemannian circles are not necessarily closed.

A *helix* of order r is a Frenet curve of osculating order r, such that all the curvatures $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are constant.

For Frenet curves in almost Hermitian manifolds, we recall the following notion:

Definition 3.2. Let $\gamma(s)$ be a Frenet curve of osculating order $r > 0$ in an almost Hermitian manifold (M, J, g). The *complex torsions* τ_{ij} $(1 \leq i < j \leq r)$ are smooth functions along γ defined by $\tau_{ij} = g(E_i, JE_j)$. A helix of order r in (M, J, g) is said to be a *holomorphic helix* of order r if all complex torsions are constant. In particular holomorphic helices of order 2 are called *holomorphic circles*.

3.4. *J-trajectories*

Let (M, g, J) be an almost Hermitian manifold. Then a regular curve $\gamma(s)$ is said to be a J-*trajectory* of *charge* q if it satisfies

$$\nabla_{\dot{\gamma}}\dot{\gamma} = qJ\dot{\gamma},$$

where q is a constant. On can see that every J-trajectory has constant speed. A J-trajectory is said to be *normal* if it is unit speed.

In case $q = 0$, J-trajectories are nothing but geodesics. Moreover when (M, g, J) is an almost Kähler manifold, then J-trajectories are *Kähler magnetic trajectories* with respect to the Kähler magnetic field $-\Omega$. Thus the notion of J-trajectory is a slight extension of Kähler magnetic trajectory on arbitrary almost Hermitian manifolds.

The second origin of the notion of J-trajectory is the geometry of holomorphically planar curves. Let (M, J, D) be an almost complex manifold equipped with an almost complex connection D, *i.e.*, $DJ = 0$. Then a smooth curve $\gamma(t)$ in M is said to be a *holomorphically planar curve* (*h-planar curve* in short) if it remains, under parallel translation along the curve, in the distribution generated by the vectors $\dot\gamma$ and $J\dot\gamma$. Namely γ satisfies

$$D_{\dot\gamma}\dot\gamma = a(t)\dot\gamma(t) + b(t)J\dot\gamma(t)$$

for some functions $a(t)$ and $b(t)$ defined along γ. This notion was introduced by Otsuki and Tashiro [64] in 1954. When (M, J, D) is a Kähler manifold and D is the Levi-Civita connection ∇, obviously J-trajectories (Kähler magnetic trajectories) are holomorphically planar.

When M is a Kähler manifold, then Kähler magnetic trajectories have particular properties. Indeed every Kähler magnetic trajectory of charge $q \neq 0$ is a Riemannian circle of curvature $|q|$.

Now let $M = G/K$ be a homogeneous Riemannian space equipped with a G-invariant orthogonal almost complex structure J.

By the proof of Proposition 2.1, one can confirm that a homogeneous curve $\gamma(s) = \exp_{\mathfrak{g}}(sX)\cdot o$ is a J-trajectory if and only if X satisfies

$$[X_{\mathfrak{k}}, X_{\mathfrak{m}}] + \mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}}) = qJX_{\mathfrak{m}}. \tag{3.2}$$

Now let $\gamma(s)$ be a normal J-trajectory of charge q in an almost Hermitian manifold $M = (M, J, g)$. First we observe that the first curvature κ_1 is constant $|q|$ by comparing the J-trajectory equation and the Frenet formula (3.1). The Frenet formula implies that the first normal vector field E_2 is given by $E_2 = \pm J\dot\gamma$. Let $\varepsilon = q/|q|$, then we have $E_2 = \varepsilon\, J\dot\gamma$ and $\kappa_1 = \varepsilon q > 0$.

If a Frenet curve γ in an almost Hermitian manifold (M, J, g) is a J-trajectory, then

$$\tau_{12} = g(E_1, JE_2) = -\varepsilon.$$

If M is a Kähler manifold, then every J-trajectory is a holomorphic circle.

4. The space of left invariant metrics on Lie groups

Let $\widetilde{\mathcal{M}}(\mathfrak{g})$ be the set of all inner products on the Lie algebra $\mathfrak{g}$ of a Lie group G. Then the group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra automorphisms acts on $\widetilde{\mathcal{M}}(\mathfrak{g})$ by

$$(a\cdot \mathsf{F})(X, Y) = \mathsf{F}(a^{-1}X, a^{-1}Y), \quad a \in \mathrm{Aut}(\mathfrak{g}),\ \mathsf{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}). \tag{4.1}$$

Proposition 4.1. *Let g and g' be left invariant Riemannian metrics on a simply connected Lie group G. Denote by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$, the inner products on $\mathfrak{g}$ induced from g and g', respectively. Then,*

1. *If a Lie group automorphism $\alpha \in \mathrm{Aut}(G)$ is an isometry from (G, g) to (G, g'), then its differential $\alpha_{*\mathsf{e}} : (\mathfrak{g}, \langle\cdot,\cdot\rangle) \to (\mathfrak{g}, \langle\cdot,\cdot\rangle')$ is an isometric Lie algebra isomorphism. Namely, $\alpha_{*\mathsf{e}}$ is a Lie algebra automorphism and satisfies*

$$\langle\alpha_{*\mathsf{e}}X, \alpha_{*\mathsf{e}}Y\rangle' = \langle X, Y\rangle$$

for all $X, Y \in \mathfrak{g}$.

2. *If $a \in \mathrm{Aut}(\mathfrak{g})$ is a linear isometry from $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ to $(\mathfrak{g}, \langle\cdot,\cdot\rangle')$, then there exits a isometry $\alpha : (G, g) \to (G, g')$ such that $\alpha_{*\mathsf{e}} = a$.*

If two inner products lie in the same $\mathrm{Aut}(\mathfrak{g})$-orbit, then they induce isometric left invariant metrics on the corresponding simply connected Lie group G.

This proposition says that to classify the left invariant metrics on a simply connected Lie group G up to automorphism, it suffices to classify the inner products on $\mathfrak{g}$ up to automorphism. It should be remarked that the classification up to automorphism is *finer* in general than that up to isometry, since (G, g) and (G, g') will be isometric if and only if they are isometric by an isometry fixing the identity. However, such an isometry need not be an automorphism of G, in general. Alekseevskii [1] proved that if the Lie algebra $\mathfrak{g}$ has only real roots, two left invariant metrics on G are isometric if and only if they are isometric by an automorphism of G (see also Gordon and Wilson [30, Corollary 5.3]).

When $\mathfrak{g}$ is nilpotent, Wilson [86] proved the following fact (see also [53, Proposition 1.3]).

Proposition 4.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with corresponding connected Lie group G. If $\alpha : (G, g) \to (G, g')$ is an isometry fixing the identity, then α is a Lie group automorphism.*

Now let us discuss about the moduli space of left invariant metrics.

Take two inner products F and $\mathsf{F}' \in \widetilde{\mathcal{M}}(\mathfrak{g})$, we introduce an equivalence relation $\cong$ by

$$\mathsf{F} \cong \mathsf{F}' \Longleftrightarrow \exists a \in \mathrm{Aut}(\mathfrak{g});\ \mathsf{F}' = a \cdot \mathsf{F}.$$

The quotient set $\mathcal{M}(\mathfrak{g})/\cong$ is denoted by

$$\mathcal{M}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathcal{M}}(\mathfrak{g})$$

and referred as to the *moduli space of left invariant metrics*. In case, $\dim G = n$, by taking a basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathfrak{g}$, we identify $\mathfrak{g}$ with $\mathbb{R}^n$ via $\{e_1, e_2, \ldots, e_n\}$. Then, we have an identification

$$\widetilde{\mathcal{M}}(\mathfrak{g}) \cong \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n.$$

Denote by $\{\vartheta^1, \vartheta^2, \ldots, \vartheta^n\}$ the dual basis of $\{e_1, e_2, \ldots, e_n\}$. Then the inner product F corresponding to $F = (F_{ij}) \in \mathrm{Sym}_n^+\mathbb{R}$ is expressed as

$$\mathsf{F} = \sum_{i,j=1}^{n} F_{ij}\vartheta^i \otimes \vartheta^j.$$

The automorphism group $\mathrm{Aut}(\mathfrak{g})$ is regarded as a subgroup of $\mathrm{GL}_n\mathbb{R}$. The automorphism group $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathrm{Sym}_n\mathbb{R}$ by

$$\mathrm{Aut}(\mathfrak{g}) \times \mathrm{Sym}_n\mathbb{R} \to \mathrm{Sym}_n\mathbb{R}; \quad a \cdot F = {}^t(a^{-1})Fa^{-1}.$$

Then

$$((a \cdot F)\boldsymbol{x}|\boldsymbol{y}) = (a \cdot \mathsf{F})(\boldsymbol{x}, \boldsymbol{y}).$$

Thus the action of $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathrm{Sym}_n\mathbb{R}$ is equivariant to the action of $\mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathcal{M}}(\mathfrak{g})$.

Here we recall the classification procedure of left invariant metrics. Any invertible matrix $A \in \mathrm{GL}_n\mathbb{R}$ defines a left invariant metric on G by declaring the column vectors to be an orthonormal basis for $\mathfrak{g}$. Two matrices A_1 and $A_2 \in \mathrm{GL}_n\mathbb{R}$ define the same metric if and only if $A_1 = A_2U$ for some $U \in \mathrm{O}_n$. This retrieves the identification $\widetilde{\mathcal{M}} = \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$ Hence we obtain the double coset space representation:

$$\mathcal{M}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})\backslash\mathrm{GL}_n\mathbb{R}/\mathrm{O}_n.$$

To carry out the classification, first we need to calculate the automorphism group $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}_n\mathbb{R}$ relative to the prescribed basis $\{e_1, e_2, \ldots, e_n\}$. Next we look for simpler form of the representatives in $\mathcal{M}$. As is well known, $\mathrm{GL}_n\mathbb{R}$ has a Lie group splitting $\mathrm{GL}_n\mathbb{R} = \mathrm{T}_n^-\mathbb{R} \cdot \mathrm{O}_n$, called the *polar decomposition* (or *Gram-Schmidt decomposition*). Here $\mathrm{T}_n^-\mathbb{R}$ is the subgroup of lower triangular matrices of positive diagonal entries. For a matrix $A \in \mathrm{GL}_n\mathbb{R}$, we decompose it as $A = TU$ according to the polar decomposition. Then, we may use the lower triangular part T as a representative of $[A] \in \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$. Take a lower triangular matrix T with positive diagonal entries as a representative of a coset $[T] \in \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$. We look for some $A \in \mathrm{Aut}(\mathfrak{g})$ such that $A \cdot T$ has a simple form.

Here we give the procedure for determining $\mathcal{M}(\mathfrak{nil}_3)$ of the Heisenberg algebra.

Example 4.1 (The moduli space of Heisenberg group [31, 45, 53, 69])**.** The Heisenberg algebra is a 3-dimensional 2-step nilpotent Lie algebra generated by the commutation relation $[e_1, e_2] = e_3$. The Heisenberg algebra is realized as

$$\left\{ \left(\begin{array}{ccc} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{array} \right) \middle|\ u, v, w \in \mathbb{R} \right\}.$$

The corresponding simply connected Lie group (*Heisenberg group*) is realized as the linear Lie group

$$\left\{ \left(\begin{array}{ccc} 1 & x & z + (xy)/2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle|\ x, y, z \in \mathbb{R} \right\}.$$

The basis $\{e_1, e_2, e_3\}$ is extended to left invariant vector fields:

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}$$

with dual coframe field

$$\vartheta^1 = dx, \quad \vartheta^2 = dy, \quad \vartheta^3 = dz + \frac{1}{2}(ydx - xdy).$$

The Heisenberg group equipped with the left invariant metric

$$g = (\vartheta^1)^2 + (\vartheta^2)^2 + (\vartheta^3)^2 = dx^2 + dy^2 + \{dz + \frac{1}{2}(ydx - xdy)\}^2$$

is denoted by Nil_3. The Lie algebra of Nil_3 is denoted by $\mathfrak{nil}_3$. The automorphism group of the Heisenberg algebra is given by

$$\mathrm{Aut}(\mathfrak{nil}_3) = \left\{ \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{11}c_{22} - c_{12}c_{21} \end{pmatrix} \middle| \, c_{11}, c_{22} \neq 0, \; c_{11}c_{22} - c_{12}c_{21} \neq 0 \right\}$$

relative to $\{e_1, e_2, e_3\}$. Take $A \in \mathrm{GL}_3\mathbb{R}$ and decomposed it as $A = TU$ according to the polar decomposition. Here T is a *lower triangular* matrix of positive diagonal entries. Then, the coset $[A] \in \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$ is rewritten as $[A] = [TU] = [T]$. By using $T = (t_{ij})$, we set

$$C = \frac{1}{t_{11}t_{22}} \begin{pmatrix} t_{22} & 0 & 0 \\ -t_{21} & t_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathfrak{nil}_3).$$

Then we have

$$CT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t_{33}/(t_{11}t_{22}) \end{pmatrix}$$

Put $\lambda := t_{33}/(t_{11}t_{22}) > 0$, then we have

$$\mathcal{M}(\mathfrak{nil}_3) = \left\{ \mathrm{Aut}(\mathfrak{nil}_3) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \middle| \, \lambda > 0 \right\}.$$

Thus left invariant metrics on the Heisenberg group are isometric to

$$dx^2 + dy^2 + \lambda^{-2}\{dz + \frac{1}{2}(ydx - xdy)\}^2$$

for some $\lambda > 0$. Note that Ha and Lee [31] chose the representative

$$\left\{ \mathrm{Aut}(\mathfrak{nil}_3) \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, \lambda > 0 \right\}.$$

Let us study the equivalence relation (*homothetic*)

$$\mathsf{F} \sim \mathsf{F}' \iff \exists a \in \mathrm{Aut}(\mathfrak{g}) \text{ and } c > 0; \; \mathsf{F}' = c\, a \cdot \mathsf{F}.$$

The quotient set $\mathcal{M}(\mathfrak{g})/\!\sim$ is denoted by $\mathcal{PM}(\mathfrak{g})$ ($\mathfrak{PM}$ in the notation of [45]).

Next we set

$$\mathbb{R}^{\times} \cdot \mathrm{Aut}(\mathfrak{g}) := \{c\,\mathrm{Id} \circ a \mid a \in \mathrm{Aut}(\mathfrak{g})\}.$$

Then the quotient space $\mathcal{PM}(\mathfrak{g})$ is rewritten as

$$\mathcal{PM}(\mathfrak{g}) := \mathbb{R}^{\times} \cdot \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathcal{M}} = \mathbb{R}^{\times} \cdot \mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n.$$

and referred as to the *scale invariant moduli space of left invariant metrics*.

For a matrix $a \in \mathrm{GL}_n\mathbb{R}$, its double coset $[\![a]\!]$ is

$$[\![a]\!] = \mathbb{R}^{\times} \cdot \mathrm{Aut}(\mathfrak{g}) \cdot a \cdot \mathrm{O}_n.$$

Take the origin $\mathsf{F}_0 \in \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$, then the correspondence

$$\mathbb{R}^\times \cdot \mathrm{Aut}(\mathfrak{g})\backslash \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n \to \mathcal{PM}(\mathfrak{g}) = \mathbb{R}^\times \cdot \mathrm{Aut}(\mathfrak{g})\backslash \widetilde{\mathcal{M}}(\mathfrak{g}); \quad [\![a]\!] \longmapsto [a \cdot \mathsf{F}_0]$$

is bijective.

A subset $\mathcal{U}$ of $\mathrm{GL}_n\mathbb{R}$ is said to be a *system of representatives* of $\mathcal{PM}(\mathfrak{g})$ if

$$\mathcal{PM} = \{[a \cdot \mathsf{F}_0] \mid A \in \mathcal{U}\}$$

holds (see [33, p. 177]).

Lemma 4.1. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra. Then $\mathcal{U} \subset \mathrm{GL}_n\mathbb{R}$ is a system of representatives of $\mathcal{PM}(\mathfrak{g})$ if and only if for each $A \in \mathrm{GL}_n\mathbb{R}$, there exists a matrix $P \in \mathcal{U}$ such that $P \in [\![A]\!]$.*

For more information on moduli spaces of left invariant metrics, see [34, 45, 54].

5. Homogeneous geodesics of Sol_0^4

5.1. *The model space* Sol_0^4

The underlying manifold of the model space Sol_0^4 is the Cartesian 4-space $\mathbb{R}^4(x, y, z, t)$ with group operation:

$$(x_1, y_1, z_1, t_1) \cdot (x_2, y_2, z_2, t_2) = (x_1 + e^{t_1}x_2, y_1 + e^{t_1}y_2, z_1 + e^{-2t_1}z_2, t_1 + t_2).$$

The inverse element of (x, y, z, t) is given by $(x, y, z, t)^{-1} = (-e^{-t}x, -e^{-t}y, -e^{2t}z, -t)$.

The underlying manifold M of Sol_0^4 is realized as the following linear Lie group

$$M = \left\{ (x, y, z, t) := \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & e^{-2t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ x, y, z, t \in \mathbb{R} \right\}.$$

Note that M is isomorphic to the solvable Lie group $G_6(1)$ in [27, p. 98]. The Lie group M has no lattices [55].

The Lie algebra $\mathfrak{m}$ of M is given explicitly by

$$\mathfrak{m} = \left\{ \begin{pmatrix} s & 0 & 0 & u \\ 0 & s & 0 & v \\ 0 & 0 & -2s & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ u, v, w, s \in \mathbb{R} \right\}$$

and is spanned by the basis $\{e_1, e_2, e_3, e_4\}$ given by

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The left invariant vector fields determined by e_1, e_2, e_3 and e_4 are

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^t \frac{\partial}{\partial y}, \quad e_3 = e^{-2t} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

These vector fields satisfy the commutation relations:

$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0, \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3. \tag{5.1}$$

These relations imply that $\mathfrak{m}$ is solvable.

5.2.

The automorphism group of $\mathfrak{m}$ is computed by Van Thuong:

$$\mathrm{Aut}(\mathfrak{m}) \cong \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4\mathbb{R} \;\middle|\; a_{33} \neq 0, \quad a_{11}a_{22} - a_{12}a_{21} \neq 0 \right\}.$$

By virtue of this result, he obtained the following fact ([80, Theorem 4.1]):

Lemma 5.1. *Every left invariant Riemannian metric on M is isometric to the one defined by an orthonormal basis $\{e_1, e_2, b_{13}e_1 + e_3, b_{44}e_4\}$, where $b_{44} > 0$ and $b_{13} \geq 0$. The metric has the expression:*

$$e^{-2t}(dx^2 + dy^2) + e^{4t}(1 - 2b_{13} + b_{13}^2)dz^2 - 2b_{13}e^t dxdz + \frac{1}{b_{44}^2}dt^2.$$

We concentrate on the left invariant Riemannian metric g determined by the condition $\{e_1, e_2, e_3, e_4\}$ is orthonormal and given by

$$g = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2.$$

The homogenous Riemannian space (M, g) is denoted by Sol_0^4. The metric g has particular property. See the remark below.

Moreover (M, g) admits a pair of orthogonal complex structures $\{J_+, J_-\}$;

$$J_+e_1 = e_2, \quad J_+e_2 = -e_1, \quad J_+e_3 = e_4, \quad J_+e_4 = -e_3,$$

$$J_-e_1 = e_2, \quad J_-e_2 = -e_1, \quad J_-e_3 = -e_4, \quad J_-e_4 = e_3,$$

The Kähler forms

$$\Omega_\pm(X, Y) = g(X, J_\pm Y)$$

satisfies

$$d\Omega_\pm = \omega_\pm \wedge \Omega_\pm, \quad \omega_\pm = 2dt.$$

Hence $(\mathrm{Sol}_0^4, g, J_+)$ and $(\mathrm{Sol}_0^4, g, J_-)$ are globally conformal Kähler with common Lee form

$$\omega := \omega_+ = \omega_- = 2dt,$$

common Lee field

$$B := B_+ = B_- = 2e_4$$

and the anti-Lee fields

$$A_+ = -2e_3, \quad A_- = 2e_3.$$

Remark 5.1. In our previous work [21] we chose the complex structure $J := -J_-$ which is compatible to the geometric structure (see [84]). The resulting Hermitian surface (Sol_0^4, g, J) is a globally conformal Kähler surface with Lee form $-2dt$. The Lee field is $-2e_4$. The anti-Lee field is $2e_3$. Moreover the Hermitian metric coincides with Tricerri metric [75]. The globally conformal Kähler surface (Sol_0^4, g, J) is the universal covering of the Inoue surface of type S^0 [38].

5.3. Levi-Civita connection

The Levi-Civita connection ∇ is described as

$$\begin{array}{llll} \nabla_{e_1}e_1 = e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = -e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -2e_4, & \nabla_{e_3}e_4 = 2e_3, \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0. \end{array}$$

Hence, we get

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = 2e_1, \quad R(e_1, e_4)e_4 = -e_1,$$

$$R(e_2, e_3)e_3 = 2e_2, \quad R(e_2, e_4)e_4 = -e_2, \quad R(e_3, e_4)e_4 = -4e_3.$$

Introducing an endomorphism field P by

$$PX = g(X, e_4)e_4 = \frac{1}{4}\omega_{\pm}(X)B_{\pm}.$$

Then the Riemannian curvature is expressed by the following formula due to D'haene [15]:

$$\begin{aligned} R(X,Y)Z = \ & 2(X \wedge Y)Z - 3((PX \wedge Y)Z + (X \wedge PY)Z) \\ & - \frac{1}{2}(\, g(J_+Y, Z)J_+X - g(Z, J_+X)J_+Y + 2g(X, J_+Y)J_+Z\,) \\ & - \frac{1}{2}(\, g(J_-Y, Z)J_-X - g(Z, J_-X)J_-Y + 2g(X, J_-Y)JZ\,), \end{aligned}$$

where

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$

5.4. Reductive decomposition

The full isometry group $\mathrm{Iso}(\mathrm{Sol}_0^4)$ of Sol_0^4 is given by $\mathrm{Sol}_0^4 \rtimes (\mathrm{O}_2 \times \mathbb{Z}/2\mathbb{Z})$ and has countably infinite distinct lattices [55].

The identity component G of the full isometry group $\mathrm{Iso}(\mathrm{Sol}_0^4)$ is

$$G = \left\{ \left(\begin{array}{cccc} e^t\cos\theta & -e^t\sin\theta & 0 & x \\ e^t\sin\theta & e^t\cos\theta & 0 & y \\ 0 & 0 & e^{-2t} & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle|\ x, y, z, \in \mathbb{R},\ e^{i\theta} \in \mathbb{S}^1 \right\} \cong \mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2).$$

The isotropy subgroup at the origin $o = (0, 0, 0, 0)$ is

$$K = \left\{ \left(\begin{array}{cccc} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \middle|\ e^{i\theta} \in \mathbb{S}^1 \right\} \cong \mathrm{SO}(2).$$

The Lie algebra of G is given by

$$\mathfrak{g} = \left\{ \left(\begin{array}{cccc} u_4 & -u_5 & 0 & u_1 \\ u_5 & u_4 & 0 & u_2 \\ 0 & 0 & -2u_4 & u_3 \\ 0 & 0 & 0 & 1 \end{array} \right) \middle|\ u_1, u_2, u_3, u_4, u_5 \in \mathbb{R}, \right\}.$$

Obviously $\mathfrak{m} \subset \mathfrak{g}$ and $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is spanned by e_1, e_2, e_3, e_4 and

$$e_5 = \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The commutation relations of $\mathfrak{g}$ are described as

$$\begin{aligned} &[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0, \\ &[e_4, e_1] = e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3, \\ &[e_5, e_1] = e_2, \quad [e_5, e_2] = -e_1. \end{aligned}$$

The isotropy algebra $\mathfrak{k}$ is spanned by e_5. The tangent space of Sol_0^4 at the origin is identified with the Lie subalgebra

$$\mathfrak{m} = \left\{ \left(\begin{array}{cccc} u_4 & 0 & 0 & u_1 \\ 0 & u_4 & 0 & u_2 \\ 0 & 0 & -2u_4 & u_3 \\ 0 & 0 & 0 & 1 \end{array} \right) \middle|\ u_1, u_2, u_3, u_4 \in \mathbb{R}, \right\} \cong \mathfrak{sol}_0^4.$$

One can confirm that G/K is a reductive homogeneous Riemannian space.
Then $\mathsf{U}_{\mathfrak{m}}$ is computed as

$$\mathsf{U}_{\mathfrak{m}}(e_1,e_1)=e_4,\quad \mathsf{U}_{\mathfrak{m}}(e_1,e_2)=0,\quad \mathsf{U}_{\mathfrak{m}}(e_1,e_3)=0,\quad \mathsf{U}_{\mathfrak{m}}(e_1,e_4)=-\frac{1}{2}e_1,$$

$$\mathsf{U}_{\mathfrak{m}}(e_2,e_2)=e_4,\quad \mathsf{U}_{\mathfrak{m}}(e_2,e_3)=0,\quad \mathsf{U}_{\mathfrak{m}}(e_2,e_4)=-\frac{1}{2}e_2,$$

$$\mathsf{U}_{\mathfrak{m}}(e_3,e_3)=-2e_4,\quad \mathsf{U}_{\mathfrak{m}}(e_3,e_4)=e_3,\quad \mathsf{U}_{\mathfrak{m}}(e_4,e_4)=0.$$

Remark 5.2 ($\mathrm{Ad}(\mathrm{SO}(2))$-invariant metrics). D'haene [15] proved that any inner product invariant under $\mathrm{Ad}(\mathrm{SO}(2))$-action is

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \mu & \lambda \end{pmatrix}, \quad \lambda > |\mu|.$$

up to automorphisms. The metric has the expression:

$$\lambda(e^{-2t}(dx^2+dy^2)+e^{4t}dz^2+dt^2)+2\mu e^{2t}\,dzdt.$$

5.5. Homogeneous geodesics

The unit speed homogeneous geodesics in Sol_0^4 are classified as follows:

Proposition 5.1. *The only unit speed homogeneous geodesics of* Sol_0^4 *starting at the origin are*:

$$\gamma_1(s)=\exp_{\mathfrak{m}}(sX)\cdot o,\quad X=X^1e_1+X^2e_2\pm\frac{1}{\sqrt{3}}e_3,\quad (X^1)^2+(X^2)^2=\frac{2}{3}\quad \textit{or}$$

$$\gamma_2(s)=\exp_{\mathfrak{m}}(se_4)\cdot o=(0,0,0,0,s).$$

The homogeneous geodesics $\gamma_1(s)$ and $\gamma_2(s)$ are mutually orthogonal.

Proof. Let us investigate the criterion $[X_{\mathfrak{m}},X_{\mathfrak{k}}]=\mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}},X_{\mathfrak{m}})$. Take a tangent vector

$$X=X^1e_1+X^2e_2+X^3e_3+X^4e_4+X^5e_5\in\mathfrak{g},$$

then we obtain

$$[X_{\mathfrak{m}},X_{\mathfrak{k}}]=[X^1e_1+X^2e_2+X^3e_3+X^4e_4,X^5e_5]=X^2X^5e_1-X^1X^5e_2$$

and

$$\mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}},X_{\mathfrak{m}})=-(X^1X^4)e_1-(X^2X^4)e_2+2X^3X^4e_3+\{(X^1)^2+(X^2)^2-2(X^3)^2\}e_4.$$

Thus we obtain the system:

$$-X^1X^4-X^2X^5=0,\quad X^1X^5-X^2X^4=0,\quad X^3X^4=0,\quad (X^1)^2+(X^2)^2-2(X^3)^2=0.$$

From this system, the vector X has the form

$$X=X^1e_1+X^2e_2\pm\frac{\sqrt{(X^1)^2+(X^2)^2}}{\sqrt{2}}e_3\neq 0, \tag{5.2}$$

or

$$X=X^4e_4+X^5e_5,\quad (X^4)^2+(X^5)^2>0 \tag{5.3}$$

In the former case, we may assume that X is a unit vector. Then X is rewritten as:

$$X=X^1e_1+X^2e_2\pm\frac{1}{\sqrt{3}}e_3,\quad (X^1)^2+(X^2)^2=\frac{2}{3}.$$

In the latter case, since $[e_4,e_5]=0$, we have

$$\exp_{\mathfrak{g}}(s(X^4e_4+X^5e_5))\cdot o=\exp_{\mathfrak{m}}(s(X^4e_4))\cdot\{\exp(s(X^5e_5))_{\mathfrak{k}}\cdot o\}=\exp(s(X^4e_4))_{\mathfrak{m}}\cdot o.$$

Thus, under the arc length parametrization, the geodesic is parametrized as
$\exp(se_4)_{\mathfrak{m}}\cdot o=(0,0,0,s)$. □

This result means that Sol_0^4 is far from naturally reductive homogeneous spaces.

5.6. Homogeneous J-trajectories

Let us study homogeneous J-trajectories $\gamma(s) = \exp(sX) \cdot o$ of Sol_0^4 with respect to the complex structure $J = -J_-$. For the tangent vector $X = X^1 e_1 + X^2 e_2 X^3 e_3 + X^4 e_4 + X^5 e_5 \in \mathfrak{g}$, $\gamma(s) = \exp_{\mathfrak{g}}(sX) \cdot o$ is a J-trajectory with charge $q \neq 0$ if and only if

$$-X^1X^4 - X^2X^5 = qX^2, \quad -X^2X^4 + X^1X^5 = -qX^1, \quad 2X^3X^4 = -qX^4, \quad (X^1)^2 + (X^2)^2 - 2(X^3)^2 = qX^3.$$

When $X^5 = 0$, we deduce that X has the form.

$$X = -\frac{q}{2}e_3 + X^4 e_4.$$

This result retrieves [21, Theorem 2]. If we choose the arc length parameter s, then $q^2 + 4(X^4)^2 = 4$. Moreover $\exp_{\mathfrak{g}}(sX) \cdot o$ has positive constant first curvature $|q|$ and vanishing second curvature.

Next we consider the vector X with $X^5 \neq 0$. Then X has one of the following forms:

$$X^1 e_1 + X^2 e_2 + X^3 e_3 - q e_5, \quad X^3 e_3 + X^5 e_5, \quad -\frac{q}{2}e_3 + X^4 e_4 + X^5 e_5.$$

In the second case we have

$$\exp_{\mathfrak{g}}(sX) \cdot o = \exp_{\mathfrak{m}}(s(X^3 e_3)) \exp_{\mathfrak{k}}(s(X^5 e_5)) \cdot o = \exp_{\mathfrak{m}}(s(X^3 e_3)).$$

In the third case

$$\exp_{\mathfrak{g}}(sX) \cdot o = \exp_{\mathfrak{m}}(s(-\tfrac{q}{2}e_3 + X^4 e_4)) \exp_{\mathfrak{k}}(s(X^5 e_5)) \cdot o = \exp_{\mathfrak{m}}(s(-\tfrac{q}{2}e_3 + X^4 e_4)).$$

Hence we obtain the following classification which generalizes [21, Theorem 2].

Theorem 5.1. *The homogeneous J-trajectories $\exp_{\mathfrak{g}}(sX)$ have one of the following form:*

$$\begin{aligned}
&\exp_{\mathfrak{m}}\left(s\left(-\frac{q}{2}e_3 + X^4 e_4\right)\right),\\
&\exp_{\mathfrak{m}}\left(s\left(X^1 e_1 + X^2 e_2 + X^3 e_3 - q e_5\right)\right),\\
&\exp_{\mathfrak{m}}(s(X^3 e_3)).
\end{aligned}$$

We may replace J by J_+ or J_-. The classification of homogeneous J_+-trajectories and J_--trajectories are quite analogus to the above classification, so we omit those here.

Remark 5.3. Some minimal submanifolds in Sol_0^4 are investigated in [22].

6. Homogeneous geodesics of Sol_1^4

6.1. The model space Sol_1^4

According to Wall [83, 84], the underlying manifold of the model space Sol_1^4 is realized as the following closed group:

$$\left\{ (x, y, u, v) = \begin{pmatrix} 1 & v & u \\ 0 & y & x \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, u, v \in \mathbb{R},\ y > 0 \right\}$$

of the affine transformation group $\mathrm{GL}_2\mathbb{C} \ltimes \mathbb{C}^2$ of complex Euclidean plane $\mathbb{C}^2$.

The group operation is given explicitly by

$$(x_1, y_1, u_1, v_1) \cdot (x_2, y_2, u_2, v_2) = (x_1 + y_1 x_2, y_1 y_2, u_1 + u_2 + v_1 x_2, v_1 y_2 + v_2).$$

The inverse element of (x, y, u, v) is

$$(x, y, u, v)^{-1} = (-x/y, 1/y, -u + xv/y, -v/y).$$

The Lie group Sol_1^4 acts on the region

$$\mathbb{C} \times \mathbb{H} = \{(w, z) \in \mathbb{C}^2 \mid \mathrm{Im}\, z > 0\}$$

of $\mathbb{C}^2$ via the affine action [84, p. 124]:

$$\begin{pmatrix} 1 & a_4 & a_3 \\ 0 & a_2 & a_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} w + a_4 z + a_3 \\ a_2 z + a_1 \\ 1 \end{pmatrix}.$$

This action is transitive with trivial isotropy. Hence Sol_1^4 is identified with $\mathbb{C} \times \mathbb{H}$. In fact, the formula

$$\begin{pmatrix} 1 & v & u \\ 0 & y & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} u + \sqrt{-1}v \\ x + \sqrt{-1}y \\ 1 \end{pmatrix}$$

shows that the orbit of $(0, \sqrt{-1}) \in \mathbb{C} \times \mathbb{H}$ coincides with the whole $\mathbb{C} \times \mathbb{H}$.

The nilradical of Sol_1^4 is the Heisenberg group

$$\mathrm{Nil}_3 = \left\{ (x, 1, u, v) = \begin{pmatrix} 1 & v & u \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, u, v \in \mathbb{R} \right\}.$$

One can see that

$$\begin{pmatrix} 1 & v & u \\ 0 & y & x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v/y & u \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that $\mathrm{Sol}_1^4 = \mathrm{Nil}_3 \rtimes \mathbb{R}^+$. As explained in [27, p. 101], the underlying manifold of the model space Sol_1^4 is the connected simply connected solvable Lie group G_8 in the classification [27] by Filipkiewicz.

The center $Z = Z(\mathrm{Sol}_1^4)$ is

$$Z = \left\{ (0, 1, u, 0) = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, u, v \in \mathbb{R} \right\}.$$

The quotient group Sol_1^4/Z is isomorphic to the 3-dimensional solvable Lie group Sol_3.

6.2. *The Lie algebra* $\mathfrak{sol}_1^4$

The Lie algebra $\mathfrak{sol}_1^4$ of Sol_1^4 is given by

$$\left\{ \begin{pmatrix} 0 & t_4 & t_3 \\ 0 & t_2 & t_1 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; t_1, t_2, t_3, t_4 \in \mathbb{R} \right\}.$$

Let us take the basis

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the commutation relations are

$$[e_1, e_2] = -e_1, \quad [e_1, e_4] = -e_3, \quad [e_2, e_4] = -e_4.$$

Remark 6.1. The Lie algebra $\mathfrak{sol}_1^4$ is isomorphic to the following Lie algebras: $\mathfrak{d}_4$ in [2], $\mathfrak{g}_{4.8}^{-1} \oplus \mathfrak{g}_1$ in [6], $U3I0$ in [56] and $\mathfrak{s}_4$ in [33].

The exponential map $\exp : \mathfrak{sol}_1^4 \to \mathrm{Sol}_1^4$ is surjective and given explicitly by

$$\exp\left\{ s \begin{pmatrix} 0 & t_4 & t_3 \\ 0 & t_2 & t_1 \\ 0 & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 1 & t_4(e^{st_2} - 1)/t_2 & t_4 t_1 (e^{st_2} - 1 - t_2 s)/t_2^2 + t_3 s \\ 0 & e^{st_2} & t_1 (e^{st_2} - 1)/t_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.1}$$

The left Maurer-Cartan form of Sol_1^4 is $\vartheta^1\,e_1+\vartheta^2\,e_2+\vartheta^3\,e_3+\vartheta^4\,e_4$, where

$$\vartheta^1=\frac{dx}{y},\quad \vartheta^2=\frac{dy}{y},\quad \vartheta^3=du-\frac{v}{y}\,dx,\quad \vartheta^4=dv-\frac{v}{y}\,dy.$$

This formula shows that the left translated vector fields of e_1, e_2, e_3 and e_4 are given by

$$e_1=y\,\frac{\partial}{\partial x}+v\,\frac{\partial}{\partial u},\quad e_2=y\,\frac{\partial}{\partial y}+v\,\frac{\partial}{\partial v},\quad e_3=\frac{\partial}{\partial u},\quad e_4=\frac{\partial}{\partial v}.$$

Remark 6.2. Tricerri [75] chose $e_3=-\partial_u$ and $e_4=-\partial_v$.

For topological studies on Sol_1^4 and its compact quotients, we refer to [35, 39, 81] and references therein.

6.3. *Tricerri metric*

Let us introduce a left invariant Riemannian metric g so that $\{e_1,e_2,e_3,e_4\}$ is orthonormal with respect to it. Then g is computed as

$$g=\frac{(1+v^2)}{y^2}(dx^2+dy^2)-\frac{2v}{y}(dxdu+dydv)+du^2+dv^2.$$

This metric is nothing but the so-called *Tricerri metric* on $\mathbb{C}\times\mathbb{H}$ [75].

6.4. *The Levi-Civita connection*

The Levi-Civita connection ∇ is described as

$$\begin{array}{llll}\nabla_{e_1}e_1=e_2, & \nabla_{e_1}e_2=-e_1, & \nabla_{e_1}e_3=\frac{1}{2}e_4, & \nabla_{e_1}e_4=-\frac{1}{2}e_3\\ \nabla_{e_2}e_1=0, & \nabla_{e_2}e_2=0, & \nabla_{e_2}e_3=0, & \nabla_{e_2}e_4=0\\ \nabla_{e_3}e_1=\frac{1}{2}e_4, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_3=0, & \nabla_{e_3}e_4=-\frac{1}{2}e_1\\ \nabla_{e_4}e_1=\frac{1}{2}e_3, & \nabla_{e_4}e_2=e_4, & \nabla_{e_4}e_3=-\frac{1}{2}e_1, & \nabla_{e_4}e_4=-e_2.\end{array}$$

The Riemannian curvature is given by

$$R(e_1,e_2)e_2=-e_1,\quad R(e_2,e_3)e_4=-\frac{1}{2}e_1\quad R(e_1,e_3)e_3=\frac{1}{4}e_1,$$

$$R(e_1,e_4)e_4=\frac{1}{4}e_1,\quad R(e_2,e_4)e_4=-e_2,\quad R(e_1,e_4)e_3=-\frac{1}{2}e_2,$$

$$R(e_3,e_4)e_4=\frac{1}{4}e_3,\quad R(e_1,e_4)e_2=-\frac{1}{2}e_3.$$

The sectional curvatures $K_{ij}=K(e_i\wedge e_j)$ of a tangent plane spanned by e_i and e_j are given by

$$K_{12}=K_{24}=-1,\quad K_{13}=K_{14}=K_{34}=\frac{1}{4},\quad K_{23}=0.$$

The scalar curvature is $-\frac{5}{2}$.

The full isometry group $\mathrm{Iso}(\mathrm{Sol}_1^4)$ of Sol_1^4 is $\mathrm{Sol}_1^4\rtimes \mathrm{D}_4$. In particular, the identity component $\mathrm{Iso}_\circ(\mathrm{Sol}_1^4)$ of $\mathrm{Iso}(\mathrm{Sol}_1^4)$ is Sol_1^4. For the crystallographic group of Sol_1^4, see [78].

Thus we represent Sol_1^4 by $\mathrm{Sol}_1^4=\mathrm{Sol}_1^4/\{\mathrm{Id}\}$ as a reductive homogeneous space with trivial isotropy algebra and Lie subspace $\mathfrak{m}=\mathfrak{sol}_1^4$.

The symmetric tensor $\mathsf{U}_{\mathfrak{m}}$ is given by

$$\mathsf{U}_{\mathfrak{m}}(e_1,e_1)=e_2,\quad \mathsf{U}_{\mathfrak{m}}(e_1,e_2)=-\frac{1}{2}e_1,\quad \mathsf{U}_{\mathfrak{m}}(e_1,e_3)=\frac{1}{2}e_4,\quad \mathsf{U}_{\mathfrak{m}}(e_1,e_4)=0,$$

$$\mathsf{U}_{\mathfrak{m}}(e_2,e_2)=0,\quad \mathsf{U}_{\mathfrak{m}}(e_2,e_3)=0,\quad \mathsf{U}_{\mathfrak{m}}(e_2,e_4)=\frac{1}{2}e_4,$$

$$\mathsf{U}_{\mathfrak{m}}(e_3,e_3)=0,\quad \mathsf{U}_{\mathfrak{m}}(e_3,e_4)=-\frac{1}{2}e_1,\quad \mathsf{U}_{\mathfrak{m}}(e_4,e_4)=-e_2.$$

6.5. Homogeneous geodesics

Homogeneous geodesics starting at the origin are classified as follows (This corrects [23, Corollary 6.7]):

Proposition 6.1. *The unit speed homogeneous geodesics starting at the origin are given by*

$$\gamma(s) = \exp\left(s\left(\pm be_1 + ae_2 \mp ae_3 + be_4\right)\right), \quad a^2 + b^2 = \frac{1}{2}, \quad b \neq 0.$$

or

$$\gamma(s) = \exp(s(ae_2 + be_4)), \quad a^2 + b^2 = 1 \tag{6.2}$$

In particular $\exp(se_2) = (0, e^s, 0, 0)$ *and* $\exp(se_4) = (0, 1, 0, s)$ *are homogeneous geodesics.*

Proof. For a vector $X = X^1e_1 + X^2e_2 + X^3e_3 + X^4e_4 \in \mathfrak{sol}_1^4$, $\gamma(s) = \exp(sX)$ is a homogeneous geodesic if and only if $\mathsf{U}_{\mathfrak{m}}(X, X) = 0$. The vector $\mathsf{U}_{\mathfrak{m}}(X, X)$ is computed as

$$\mathsf{U}_{\mathfrak{m}}(X, X) = -(X^1X^2 + X^3X^4)e_1 + ((X^1)^2 - (X^4)^2)e_2 + (X^1X^3 + X^2X^4)e_4.$$

Hence X has the form

$$X = \pm X^4e_1 + X^2e_2 \mp X^2e_3 + X^4e_4, \quad X^4 \neq 0$$

or

$$X = X^2e_2 + X^3e_3.$$

If we assume that $\exp(sX)$ is unit speed, then in the former case, we have

$$X = \pm X^4e_1 + X^2e_2 \mp X^2e_3 + X^4e_4, \quad (X^2)^2 + (X^4)^2 = \frac{1}{2}, \quad X^4 \neq 0.$$

In the latter case

$$X = X^2e_2 + X^3e_3, \quad (X^2)^2 + (X^3)^2 = 1.$$

□

Take a unit vector $X = ae_2 + be_4$, then $Y = -be_2 + ae_4$ is orthogonal to X and both the homogeneous curves $\exp(sX)$ and $\exp(sY)$ are geodesics.

Corollary 6.1. *There exits a pair of mutually orthogonal homogeneous geodesics starting at the origin.*

Homogeneous J-trajectories are classified in our previous work [23, Corollary 6.6]

Proposition 6.2. *The unit speed homogenous J-trajectories of charge q in* Sol_1^4 *are represented as* $\exp(sX)$ *with* $X = -qe_1 + ae_2 + be_3$ *for some constants a and b satisfying* $q^2 + a^2 + b^2 = 1$.

6.6. Problem

In [17], Codazzi hypersurfaces and totally umbilical hypersurfaces in Sol_0^4 are classified. Here we propose the following problem:

Problem 1. *Classify Codazzi hypersurfaces and totally umbilical hypersurfaces in* Sol_1^4.

Some minimal submanifolds in Sol_1^4 are investigated in [24].

7. Homogeneous geodesics of $\mathrm{Sol}_{m,n}^4$

7.1. The model space $\mathrm{Sol}_{m,n}^4$

Take a positive integer m, n, consider the cubic equation:

$$f(\lambda) = \lambda^3 - m\lambda^2 + n\lambda - 1 = 0.$$

We assume that this cubic equation has three distinct positive roots $\{e^\alpha, e^\beta, e^\gamma\}$ so that $\alpha > \beta > \gamma$. Then we have $\alpha + \beta + \gamma = 0$ and

$$m = e^\alpha + e^\beta + e^{-(\alpha+\beta)}, \quad n = e^{\alpha+\beta} + e^{-\alpha} + e^{-\beta}.$$

We introduce a representation

$$T_{m,n} : \mathbb{R}(t) \to \mathrm{GL}_3\mathbb{R}; \quad T_{m,n}(t) = \begin{pmatrix} e^{\alpha t} & 0 & 0 \\ 0 & e^{\beta t} & 0 \\ 0 & 0 & e^{\gamma t} \end{pmatrix}.$$

Then the semi-direct product $\mathbb{R} \times_{T_{m,n}} \mathbb{R}^3(x,y,z)$ is realized as the linear Lie group

$$\left\{ \begin{pmatrix} e^{\alpha t} & 0 & 0 & x \\ 0 & e^{\beta t} & 0 & y \\ 0 & 0 & e^{-(\alpha+\beta)t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x,y,z,t \in \mathbb{R} \right\}.$$

The semi-direct products $\mathbb{R} \times_{T_{m,n}} \mathbb{R}^3(x,y,z)$ and $\mathbb{R} \times_{T_{m',n'}} \mathbb{R}^3(x,y,z)$ are isomorphic each other if and only if two matrices

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -(\alpha+\beta) \end{pmatrix}, \quad \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \beta' & 0 \\ 0 & 0 & -(\alpha'+\beta') \end{pmatrix}$$

are proportional. There are infinitely many isomorphism classes.

Remark 7.1. We assume that the cubic equation has three distinct positive roots. If we permit the case of two equal roots which occurs when $m^2n^2 + 18mn = 4(m^3+n^3) + 27$. One can see that this condition is equivalent to $\alpha = \beta = 1$. As pointed out by Wall [83], the semi-direct product $\mathbb{R} \times_{T_{m,n}} \mathbb{R}^3(x,y,z)$ with $\alpha = \beta = 1$ coincides with the underlying Lie group of Sol_0^4.

7.2.

The Lie algebra of $\mathbb{R} \times_{T_{m,n}} \mathbb{R}^3(x,y,z)$ is given explicitly by

$$\left\{ \begin{pmatrix} \alpha s & 0 & 0 & u \\ 0 & \beta s & 0 & v \\ 0 & 0 & -(\alpha+\beta)s & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; u,v,w,s \in \mathbb{R} \right\}$$

and is spanned by the basis $\{e_1, e_2, e_3, e_4\}$ given by

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & -(\alpha+\beta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The non-trivial commutation relations are

$$[e_4, e_1] = \alpha\, e_1, \quad [e_4, e_2] = \beta\, e_2, \quad [e_4, e_3] = -(\alpha+\beta)e_3. \tag{7.1}$$

These relations imply that the Lie algebra of $\mathbb{R} \times_{T_{m,n}} \mathbb{R}^3(x,y,z)$ is solvable.

7.3. Levi-Civita connection

We take the left invariant Riemannian metric

$$g = e^{-2\alpha t}dx^2 + e^{-2\beta t}dy^2 + e^{2(\alpha+\beta)t}dz^2 + dt^2.$$

The homogeneous Riemannian space $(\mathbb{R} \times_{T_{m,n}} \mathbb{R}^3(x,y,z), g)$ is denoted by $\mathrm{Sol}_{m,n}^4$. We may regard $\mathrm{Sol}_{m,n}^4 = \mathrm{Sol}_0^4$ when $\alpha = \beta = 1$.

In case $m = n$, we obtain $\beta = 0$ and the metric is

$$g = e^{-2\alpha t}dx^2 + dy^2 + e^{2\alpha t}dz^2 + dt^2.$$

Hence $\mathrm{Sol}^4_{n,n}$ is regarded as $\mathrm{Sol}_3 \times \mathbb{E}^1$. Note that if $\beta = 0$, then

$$m = n = 1 + 2\cosh\alpha > 3.$$

The Levi-Civita connection ∇ is described as

$$\begin{array}{llll}
\nabla_{e_1}e_1 = \alpha e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = -\alpha e_1, \\
\nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = \beta e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = -\beta e_2, \\
\nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -(\alpha+\beta)e_4, & \nabla_{e_3}e_4 = (\alpha+\beta)e_3, \\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0.
\end{array}$$

7.4. Reductive decomposition

When $m \neq n$, the full isometry group of $\mathrm{Sol}^4_{m,n}$ is $\mathrm{Sol}^4_{m,n} \rtimes (\mathbb{Z}/2\mathbb{Z})^3$ (see [55, Theorem 3.5]). When $m = n$, we know that the maximal compact subgroup of $\mathrm{Sol}_3 \times \mathbb{R}$ is $\mathrm{D}_4 \times \mathbb{Z}/2\mathbb{Z}$ ([79, Lemma 4.4]). Moreover the full isometry group of $\mathrm{Sol}^4_{n,n}$ is $(\mathrm{Sol}_3 \times \mathbb{R}) \rtimes (\mathrm{D}_4 \times \mathbb{Z}/2\mathbb{Z})$ ([36, §7.3], [55, Theorem 3.4]).

For the crystallographic group of $\mathrm{Sol}^4_{m,n}$, we refer to Van Thuong's Thesis [78] as well as Yoo's article [87].

Kowalksi and Tricerri proved the following characterization.

Theorem 7.1 ([50]). *Each complete, simply connected and irreducible Riemannian 4-manifold admitting a homogeneous Riemannian structure of type $\mathcal{T}_2$ are the model space F^4 or $\mathrm{Sol}^4_{m,n}$ with $\alpha \neq 0$, $\beta \neq 0$ and $\alpha + \beta \neq 0$.*

The identity component of the full isometry group $\mathrm{Iso}(\mathrm{Sol}^4_{m,n})$ is $\mathrm{Sol}^4_{m,n}$. Thus we regard $\mathrm{Sol}^4_{m,n}$ as a reductive homogeneous Riemannian space $\mathrm{Sol}^4_{m,n}/\{\mathsf{e}\}$ with Lie subspace $\mathfrak{m} = \mathfrak{sol}_{m,n}$. The tensor $\mathsf{U}_\mathfrak{m}$ is computed as

$$\mathsf{U}_\mathfrak{m}(e_1,e_1) = \alpha e_4, \quad \mathsf{U}_\mathfrak{m}(e_1,e_2) = 0, \quad \mathsf{U}_\mathfrak{m}(e_1,e_3) = 0, \quad \mathsf{U}_\mathfrak{m}(e_1,e_4) = -\frac{\alpha}{2}e_1,$$

$$\mathsf{U}_\mathfrak{m}(e_2,e_2) = \beta e_4, \quad \mathsf{U}_\mathfrak{m}(e_2,e_3) = 0, \quad \mathsf{U}_\mathfrak{m}(e_2,e_4) = -\frac{\beta}{2}e_2,$$

$$\mathsf{U}_\mathfrak{m}(e_3,e_3) = -(\alpha+\beta)e_4, \quad \mathsf{U}_\mathfrak{m}(e_3,e_4) = \frac{\alpha+\beta}{2}e_3, \quad \mathsf{U}_\mathfrak{m}(e_4,e_4) = 0.$$

Note that $\mathrm{Nil}_4/\{\mathsf{e}\}$ is a generalized affine symmetric space of infinite order ([47, p. 153]).

7.5. Homogeneous geodesics

The unit speed homogeneous geodesics in $\mathrm{Sol}^4_{m,n}$ are classified as follows:

Proposition 7.1. *The unit speed homogeneous geodesics in $\mathrm{Sol}^4_{m,n}$ starting at the origin e are described as follows*:

1. *When $\alpha \neq 0$, $\beta \neq 0$ and $\alpha + \beta \neq 0$,*

$$\exp(s(X^1e_1 + X^2e_2 + X^3e_3)), \quad (X^1)^2 + (X^2)^2 + (X^3)^2 = 1, \quad \alpha(X^1)^2 + \beta(X^2)^2 - (\alpha+\beta)(X^3)^2 = 0$$

or $\exp(se_4)$.

2. *When $m = n$,*

$$\exp(s(ae_1 \pm \sqrt{1-2a^2}e_2 \pm ae_3)), \quad 1 - 2a^2 \geq 0.$$

Proof. Take a tangent vector

$$X = X^1e_1 + X^2e_2 + X^3e_3 + X^4e_4 \in \mathfrak{sol}_{m,n},$$

we have

$$\mathsf{U}_\mathfrak{m}(X_\mathfrak{m}, X_\mathfrak{m}) = -\alpha(X^1X^4)e_1 - \beta(X^2X^4)e_2 + (\alpha+\beta)X^3X^4e_3 + \{\alpha(X^1)^2 + \beta(X^2)^2 - (\alpha+\beta)(X^3)^2\}e_4.$$

1. Under the assumption $\alpha \neq 0$, $\beta \neq 0$, and $\alpha + \beta \neq 0$, we get the system
$$X^1X^4 = 0, \quad X^2X^4 = 0, \quad X^3X^4 = 0, \quad \alpha(X^1)^2 + \beta(X^2)^2 - (\alpha+\beta)(X^3)^2 = 0.$$
Thus X has the form
$$X = X^1e_1 + X^2e_2 + X^3e_3, \quad \alpha(X^1)^2 + \beta(X^2)^2 - (\alpha+\beta)(X^3)^2 = 0$$
or $X = X^4e_4$.
If $\exp(sX)$ is arc length parametrized, the former case is rewritten as
$$X = X^1e_1 + X^2e_2 + X^3e_3, \quad (X^1)^2 + (X^2)^2 + (X^3)^2 = 1, \quad \alpha(X^1)^2 + \beta(X^2)^2 - (\alpha+\beta)(X^3)^2 = 0.$$
In the latter case, $\exp(sX) = \exp(se_4)$.
2. When $\alpha = 1$ and $\beta = 0$, we get
$$(X^1)^2 - (X^3)^2 = 0$$
If $\exp(sX)$ is arc length parametrized, then
$$X = ae_1 \pm \sqrt{1-2a^2}e_2 \pm ae_3, \quad 1 - 2a^2 \geq 0.$$

□

Remark 7.2. Matsushita [61] considered the following left invariant almost complex structures on $\mathrm{Sol}^4_{m,n}$:
$$J_+e_1 = e_2, \quad J_+e_2 = -e_1, \quad J_+e_3 = e_4, \quad J_+e_4 = -e_3,$$
$$J_-e_1 = e_2, \quad J_-e_2 = -e_1, \quad J_-e_3 = -e_4, \quad J_-e_4 = e_3.$$
He confirmed that both the almost complex structures are non-integrable. In addition he also confirmed that both the almost Hermitian structures (g, J_+) and (g, J_-) are not almost Kähler.

8. Homogeneous geodesics in Nil_4

8.1. *The model space* Nil_4

Let us consider the representation
$$\rho(t) = \exp\left\{ t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$
of $(\mathbb{R}(t), +)$ over $\mathbb{R}^3(x, y, z)$. Then the semi-direct product $\mathbb{R} \times_\rho \mathbb{R}^3$ is the Cartesin 4-space $\mathbb{R}^4(x, y, z, t)$ with multiplication:
$$(x_1, y_1, z_1, t_1)(x_2, y_2, z_2, t_2) = (x_1 + x_2 + t_1y_2 + t_1^2z_2/2, y_1 + y_2 + t_1z_2, z_1 + z_2, t_1 + t_2).$$
The semi-direct product $\mathbb{R} \times_\rho \mathbb{R}^3$ is realized as the linear Lie group
$$\left\{ \begin{pmatrix} 1 & t & t^2/2 & x \\ 0 & 1 & t & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z, t \in \mathbb{R} \right\}$$
with Lie algebra
$$\left\{ \begin{pmatrix} 0 & s & 0 & u \\ 0 & 0 & s & v \\ 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; u, v, w, s \in \mathbb{R} \right\}.$$

The left Maurer-Cartan form of $\mathbb{R} \times_\rho \mathbb{R}^3$ is given by

$$\vartheta^1 e_1 + \vartheta^2 e_2 + \vartheta^3 e_3 + \vartheta^4 e_4,$$

where

$$\vartheta^1 = dx - tdy + \frac{t^2}{2}dz, \quad \vartheta^2 = dy - tdz, \quad \vartheta^3 = dz, \quad \vartheta^4 = dt,$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra is spanned by $\{e_1, e_2, e_3, e_4\}$. The non-trivial commutation relations are

$$[e_4, e_2] = e_1, \quad [e_4, e_3] = e_2.$$

The center is spanned by e_1. By left translation, we obtain left invariant vector fields:

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad e_3 = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

Remark 8.1. D'haene [16] chose

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad e_4 = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The non-trivial commutation relations are

$$[e_1, e_3] = e_2, \quad [e_1, e_4] = e_3.$$

On the other hand, Wall [83] chose the basis so that

$$[e_4, e_1] = e_2, \quad [e_4, e_2] = e_3.$$

Here we mention the following fundamental result.

Proposition 8.1 ([57, 67]). *Let $\mathfrak{n}$ be a 4-dimensional nilpotent Lie algebra. Then $\mathfrak{n}$ is isomorphic to one of the following Lie algebras*:

1. *Abelian Lie algebra $\mathbb{R}^4$.*
2. *The direct sum $\mathfrak{nil}_3 \oplus \mathbb{R}$, where $\mathfrak{nil}_3$ is the 3-dimensional Heisenberg algebra.*
3. *The Lie algebra $\mathfrak{nil}_4$.*

8.2. The space of left invariant metrics

Lauret [53] and Van Thuong [80] studied the space of left invariant metrics on Nil_4. The automorphism group of $\mathfrak{nil}_4$ is described as ([53, p. 151],[80]. See also [33, p. 180]):

$$\mathrm{Aut}(\mathfrak{nil}_4) \cong \left\{ \begin{pmatrix} a_{33}(a_{44})^2 & a_{23}a_{44} & a_{13} & a_{14} \\ 0 & a_{33}a_{44} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \in \mathrm{GL}_4\mathbb{R} \;\middle|\; a_{33}, a_{44} \neq 0 \right\}.$$

The maximal compact subgroup of $\mathrm{Aut}(\mathfrak{nil}_4)$ is ([79, Lemma 4.2]):

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

The left invariant metrics on Nil_4 are classified (up to $\mathrm{Aut}(\mathrm{Nil}_4)$) by Lauret [53] and Van Thuong [80]. The moduli space $\mathcal{M}(\mathfrak{nil}_4)$ has three parameters.

Van Thuong's expression is the following one ([80, Theorem 3.1]):

Proposition 8.2. *Any left invariant metric on* Nil_4 *is determined by the condition that*

$$\{b_{11}e_1, b_{12}e_1 + b_{22}e_2, e_3, e_4\}, \quad b_{11}, b_{22} > 0, \quad b_{12} \geq 0.$$

is orthonormal with respect to it. Hence any left invariant metric is isometric to

$$\{(b_{11})^2 + (b_{12})^2\}(\vartheta^1)^2 + (b_{22})^2(\vartheta^2)^2 + b_{12}b_{22}\{\vartheta^1 \otimes \vartheta^2 + \vartheta^2 \otimes \vartheta^2\} + (\vartheta^3)^2 + (\vartheta^4)^2.$$

for some b_{11}, b_{12} *and* b_{22}.

On the other hand, Lauret's representation [53] is the following one.

Proposition 8.3. *The moduli space* $\mathcal{M}(\mathfrak{nil}_4)$ *of* Nil_4 *is expressed as*

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix} \cdot \mathsf{F}_0 \;\middle|\; ac - b^2 > 0 \right\}.$$

Here F_0 *is the inner product*

$$\mathsf{F}_0 = (\vartheta^1)^2 + (\vartheta^2)^2 + (\vartheta^3)^2 + (\vartheta^4)^2.$$

Thus every left invariant metric on Nil_4 *is isometric to*

$$g_{a,b,c} := (\vartheta^1)^2 + (\vartheta^2)^2 + \frac{b^2 + c^2}{(ac - b^2)^2}(\vartheta^3)^2 + \frac{a^2 + b^2}{(ac - b^2)^2}(\vartheta^4)^2 - \frac{b(a + c)}{(ac - b^2)^2}\{\vartheta^3 \otimes \vartheta^4 + \vartheta^4 \otimes \vartheta^3\}$$

for some a, b *and* c.

Hashinaga [33, Lemma 3.4] described the moduli space of left invariant metrics up to automorphims and homotheties (scalings):

Proposition 8.4. *The subset*

$$\left\{ \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \lambda > 0,\ \mu \in \mathbb{R} \right\} \subset \mathrm{GL}_4\mathbb{R}$$

is a system of representatives of $\mathcal{PM}(\mathfrak{nil}_4)$. *The left invariant metrics corresponding to the above representatives are expressed as*

$$(\vartheta^1)^2 + (1 + \mu^2)(\vartheta^2)^2 - \mu(\vartheta^1 \otimes \vartheta^2 + \vartheta^2 \otimes \vartheta^1) + \lambda^{-2}(\vartheta^3)^2 + (\vartheta^4)^2$$

for some $\lambda > 0$ *and* $\mu \in \mathbb{R}$.

Remark 8.2. D'haene [16] proposed to study the 4-parameter family $\{g_{\tau_1,\tau_2,\tau_3,\alpha}\}$ of left invariant metrics with orthonormal basis:

$$\left\{ e_1, \frac{e_2}{\sqrt{\tau_2}}, \frac{e_3 - \alpha e_1}{\sqrt{\tau_3 - \alpha^2}}, \frac{e_4}{\sqrt{\tau_1}} \right\}, \quad \tau_1, \tau_2, \tau_3 > 0,\ \alpha \neq \pm\sqrt{\tau_3}.$$

8.3. *Levi-Civita connection*

In this article, we choose a left invariant Riemannian metric

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3 + \vartheta^4 \otimes \vartheta^4 \tag{8.1}$$

which is invariant under $\mathrm{Nil}_4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. Then we obtain

$$d\vartheta^1 = dy \wedge dz - tdz \wedge dt = \vartheta^2 \wedge \vartheta^3 - \vartheta^2 \wedge \vartheta^4, \quad d\vartheta^2 = dx \wedge dt = \vartheta^1 \wedge \vartheta^4, \quad d\vartheta^3 = d\vartheta^4 = 0.$$

By using the first structure equations, we obtain the following table of Levi-Civita connections:

$$\nabla_{e_1} e_2 = \frac{1}{2} e_4, \quad \nabla_{e_1} e_4 = -\frac{1}{2} e_2,$$

$$\nabla_{e_2}e_1 = \frac{1}{2}e_4, \quad \nabla_{e_2}e_3 = \frac{1}{2}e_4, \quad \nabla_{e_2}e_4 = -\frac{1}{2}(e_1+e_3),$$
$$\nabla_{e_3}e_2 = \frac{1}{2}e_4, \quad \nabla_{e_3}e_4 = -\frac{1}{2}e_2,$$
$$\nabla_{e_4}e_1 = -\frac{1}{2}e_2, \quad \nabla_{e_4}e_2 = \frac{1}{2}(e_1-e_3), \quad \nabla_{e_4}e_3 = \frac{1}{2}e_2.$$

The Riemannian curvature R is described as

$$R(e_1,e_2)e_1 = -\frac{1}{4}e_2, \quad R(e_1,e_2)e_2 = \frac{1}{4}(e_1+e_3), \quad R(e_1,e_2)e_3 = -\frac{1}{4}e_2, \quad R(e_1,e_2)e_4 = 0,$$
$$R(e_1,e_3)e_1 = R(e_1,e_3)e_2 = R(e_1,e_3)e_3 = R(e_1,e_3)e_4 = 0,$$
$$R(e_1,e_4)e_1 = -\frac{1}{4}e_4, \quad R(e_1,e_4)e_2 = 0, \quad R(e_1,e_4)e_3 = \frac{1}{4}e_4, \quad R(e_1,e_4)e_4 = \frac{1}{4}(e_1-e_3),$$
$$R(e_2,e_3)e_1 = \frac{1}{4}e_2, \quad R(e_2,e_3)e_2 = -\frac{1}{4}(e_1+e_3), \quad R(e_2,e_3)e_3 = \frac{1}{4}e_2, \quad R(e_2,e_3)e_4 = 0,$$
$$R(e_2,e_4)e_1 = 0, \quad R(e_2,e_4)e_2 = \frac{1}{2}e_4, \quad R(e_2,e_4)e_3 = 0, \quad R(e_2,e_4)e_4 = -\frac{1}{2}e_2,$$
$$R(e_3,e_4)e_1 = \frac{1}{4}e_4, \quad R(e_3,e_4)e_2 = 0, \quad R(e_3,e_4)e_3 = \frac{3}{4}e_4, \quad R(e_3,e_4)e_4 = -\frac{1}{4}e_1 - \frac{3}{4}e_3.$$

The sectional curvatures are given by

$$K_{12} = \frac{1}{4}, \quad K_{13} = 0, \quad K_{14} = \frac{1}{4}, \quad K_{23} = \frac{1}{4}, \quad K_{24} = -\frac{1}{2}, \quad K_{34} = -\frac{3}{4}.$$

The Ricci tensor field is given by

$$\mathrm{Ric} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -1. \end{pmatrix}.$$

8.4. Left invariant symplectic forms

Take a left invariant 2-form

$$\Omega = \Omega_{12}\,\vartheta^1 \wedge \vartheta^2 + \Omega_{13}\,\vartheta^1 \wedge \vartheta^3 + \Omega_{14}\,\vartheta^1 \wedge \vartheta^4 + \Omega_{23}\,\vartheta^2 \wedge \vartheta^3 + \Omega_{24}\,\vartheta^2 \wedge \vartheta^4 + \Omega_{34}\,\vartheta^3 \wedge \vartheta^4,$$

The exterior differentials of the basis of 2-forms are given by

$$d(\vartheta^1 \wedge \vartheta^2) = dx \wedge dz \wedge dt - t\,dy \wedge dz \wedge dt, \quad d(\vartheta^1 \wedge \vartheta^3) = -dy \wedge dz \wedge dt$$

and $d(\vartheta^i \wedge \vartheta^j) = 0$ for other $i,j \in \{1,2,3,4\}$ with $i<j$. Hence the space of all left invariant closed 2-forms is given by

$$\{\Omega = \Omega_{14}\,\vartheta^1 \wedge \vartheta^4 + \Omega_{23}\,\vartheta^2 \wedge \vartheta^3 + \Omega_{24}\,\vartheta^2 \wedge \vartheta^4 + \Omega_{34}\,\vartheta^3 \wedge \vartheta^4 \mid \Omega_{14},\Omega_{23},\Omega_{24},\Omega_{34} \in \mathbb{R}\}.$$

Ovando proved that there exists a 4-parameter family of left invariant symplectic forms on Nil_4.

Proposition 8.5 ([66]). *Left invariant symplectic forms on* Nil_4 *have the form*

$$\Omega = \Omega_{14}\,\vartheta^1 \wedge \vartheta^4 + \Omega_{23}\,\vartheta^2 \wedge \vartheta^3 + \Omega_{24}\,\vartheta^2 \wedge \vartheta^4 + \Omega_{34}\,\vartheta^3 \wedge \vartheta^4, \quad \Omega_{14},\Omega_{23},\Omega_{24},\Omega_{34} \in \mathbb{R}$$

satisfying $\Omega_{14} \neq 0$ *and* $\Omega_{23} \neq 0$.

Proof. A left invariant closed 2-form Ω is non-degenerate if and only if $\Omega_{14}\Omega_{23} \neq 0$. □

By using Ω and the metric g, we obtain a 4-parameter family of left invariant almost complex structures defined by

$$g(X,JY) = \Omega(X,Y), \quad X,Y \in \mathfrak{nil}_4.$$

As Wall proved, there is no left invariant complex structures on Nil_4.

In this article, we only consider the following symplectic forms:

$$\Omega_+ = \vartheta^1 \wedge \vartheta^4 + \vartheta^2 \wedge \vartheta^3, \quad \Omega_- = \vartheta^1 \wedge \vartheta^4 - \vartheta^2 \wedge \vartheta^3.$$

The corresponding almost complex structures are

$$J_+e_1 = e_4, \quad J_+e_2 = -e_3, \quad J_+e_3 = e_2, \quad J_+e_4 = -e_1,$$
$$J_-e_1 = e_4, \quad J_-e_2 = e_3, \quad J_-e_3 = -e_2, \quad J_-e_4 = -e_1.$$

8.5. The reductive decomposition of Nil_4

The full isometry group of (Nil_4, g) is $\mathrm{Nil}_4 \rtimes (\mathbb{Z}/2\mathbb{Z})^3$. The action of $\mathbb{Z}/2\mathbb{Z}$ on Nil_4 is described as:

$$(x, y, z, t) \longmapsto (-x, -y, -z, t), \quad (x, y, z, t) \longmapsto (x, -y, z, -t).$$

The identity component of $\mathrm{Iso}(\mathrm{Nil}_4, g)$ is Nil_4. We regard Nil_4 as a reductive homogeneous Riemannian space $\mathrm{Nil}_4/\{\mathsf{e}\}$ with Lie subspace $\mathfrak{m} = \mathfrak{nil}_4$. Then the tensor $\mathrm{U}_{\mathfrak{m}}$ is given by

$$\mathrm{U}_{\mathfrak{m}}(e_1, e_2) = \frac{1}{2}e_4, \quad \mathrm{U}_{\mathfrak{m}}(e_1, e_4) = -\frac{1}{2}e_2, \quad \mathrm{U}_{\mathfrak{m}}(e_2, e_3) = \frac{1}{2}e_4, \quad \mathrm{U}_{\mathfrak{m}}(e_2, e_4) = -\frac{1}{2}e_3.$$

8.6. Homogeneous geodesics

Homogeneous geodesics starting at the origin are classified as follows:

Theorem 8.1. *The unit speed homogeneous geodesics of* Nil_4 *starting at the origin are given by*

$$\gamma(s) = \exp\left(s\left(ae_3 + be_4\right)\right), \quad a^2 + b^2 = 1,$$

$$\gamma(s) = \exp\left(s\left(ae_1 + be_3\right)\right), \quad a^2 + b^2 = 1,$$

or

$$\gamma(s) = \exp\left(s\left(ae_1 + be_2 - ae_3\right)\right), \quad 2a^2 + b^2 = 1.$$

Proof. For a vector $X = X^1e_1 + X^2e_2 + X^3e_3 + X^4e_4 \in \mathfrak{nil}_4$, $\gamma(s) = \exp(sX)$ is a homogeneous geodesic if and only if $\mathsf{U}_{\mathfrak{m}}(X, X) = 0$. The vector $\mathsf{U}_{\mathfrak{m}}(X, X)$ is computed as

$$\mathsf{U}_{\mathfrak{m}}(X, X) = -X^1X^4e_2 - X^2X^4e_3 + (X^1X^2 + X^2X^3)e_4.$$

Hence X has the form

$$X = X^3e_3 + X^4e_4$$

$$X = X^1e_1 + X^3e_3$$

or

$$X = X^1e_1 + X^2e_2 - X^1e_3.$$

□

8.7. Homogeneous J-trajectories

Let us investigate homogeneous Kähler magnetic trajectories in Nil_4 with respect to $\Omega_{\pm}$.

Theorem 8.2. *The only homogenous Kähler magnetic trajectories in* Nil_4 *with respect to* $\Omega_{\pm}$ *are homogeneous geodesics.*

8.8. Problems

Problem 2. *Determine Kähler magnetic trajectories in* Nil_4.

Problem 3. *Study minimal surfaces in* Nil_4 *invariant under* $J_{\pm}$.

Problem 4. *Study minimal surfaces in* Nil_4 *which are* $\Omega_{\pm}$*-Lagrangian.*

Consider the left invariant distribution $\mathfrak{D}$ spanned by e_1, e_2 and e_3. Then $\mathfrak{D}$ is integrable. The integral hypersurface of $\mathfrak{D}$ through (x_0, y_0, z_0, t_0) is the hypersurface

$$M(1, 2, 3; t_0) = \{(x, y, z, t_0) \in \mathrm{Nil}_4\}.$$

We can take e_4 as a unit normal vetor field to $M(1, 2, 3; t_0)$. Then the shape operator derived from e_4 is given by

$$\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

Hence N is non-totally geodesic minimal hypersurface. One can check that $M(1, 2, 3; t_0)$ is a parallel hypersurface. In [18], the authors claimed that the only Codazzi hypersurfaces are integral hypersurfaces of $\mathfrak{D}$. In particular Nil_4 has no totally geodesic hypersurfaces.

Problem 5. *Classify totally umbilical hypersurfaces in* Nil_4.

Remark 8.3 ($\mathbb{H}^3 \times \mathbb{E}^1$). According to Wall, the model space $\mathbb{H}^3 \times \mathbb{E}^1$ equipped with the product metric does not have compatible complex structure. If we relax the compatibility condition (invariance under $\mathrm{SO}^+_{1,2} \times \mathbb{R}$), we have some options:

- Let us identify $\mathbb{H}^3$ with the solvable part $\mathcal{S}$ of the Iwasawa decomposition $\mathrm{SL}_2\mathbb{C} = \mathcal{S} \cdot \mathrm{SU}_2$ of $\mathrm{SL}_2\mathbb{C}$. Then the Poincaré metric is a left invariant metric on $\mathbb{H}^3 = \mathcal{S}$. There exists a left invariant Kenmotsu structure on $\mathbb{H}^3 = \mathcal{S}$. By extending the Kenmotsu structure to the Riemannian product $\mathbb{H}^3 \times \mathbb{E}^1 = \mathcal{S} \times \mathbb{E}^1$, one obtains a globally conformal Kähler structure. Some minimal submanifolds in $\mathbb{H}^3 \times \mathbb{E}^1$ are investigated in [37, 25].
- Oguro and Sekigawa [63] gave a strictly almost Kähler structure on $\mathbb{H}^3 \times \mathbb{E}^1$. Kähler magnetic trajectories in $\mathbb{H}^3 \times \mathbb{E}^1$ are investigated in [20].

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Competing interests

The authors declare that they have no competing interests.

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