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SEMI-SLANT LIGHTLIKE SUBMERSIONS

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Abstract. In this paper, we intend to study semi-slant lightlike submersions from indefinite Kaehler manifolds onto lightlike manifolds. After giving definitions and basic properties, we obtain conditions for a lightlike submersion to be a semi-slant lightlike submersion. We indicate some relevant examples. Finally, we investigate the geometric properties of foliations that appeared with a semi-slant lightlike submersion.

1. Introduction

Studying differentiable maps defined between manifolds are one of the methods used to compare geometric structures. One of these maps is submersion, in which the rank of the transformation is equal to the dimension of the target manifold. Moreover, if this map is isometric, it is called Riemannian submersion.

Riemannian submersions were first defined by O' Neill and Gray independently of each other [15], [7]. This definition was extended to manifolds with different differentiable structures. After some important developments in complex and contact geometry, the Riemannian submersions have become interesting. The differential geometry of manifolds with special structures have been examined by using different kind of Riemannian submersions [1, 6, 8–10, 17, 23, 24, 26].

On the other hand, a major shortcoming of the semi-Riemannian manifold is that there are no suitable types of functions from one manifold to the next to satisfy its geometrical properties.This flaw was fixed by O' Neill in 1983 [16]. As the generalizations of Riemannian submersions, O' Neill introduced the notion of semi-Riemannian submersions. A well known fact is that for a defined Riemannian submersion between two Riemannian manifolds, the fibers are always Riemannian but the fibers of semi-Riemannian manifolds on a semi-Riemann submersions may not be semi-Riemannian manifold.

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The another importance of such maps is their applications in mathematics [24], in theoretical physics (supergravity and superstring theories [13,14],Yang-Mills theory [3,27] and Kaluza-Klein theory [4,11]) and robotic theory [2].

On the other hand, although there have been many publications on the geometry of Riemannian submersions, there have been very few on semi-Riemannian submersions and lightlike submersions. First, Şahin investigated submersions between lightlike manifolds and semi-Riemannian manifolds [21,22]. Here, he obtained O'Neill's tensors defined on Riemannian submersions for lightlike submersions and showed the differences between the two maps for these tensors. Moreover, he studied lightlike harmonic map.

In [5], Duggal investigated harmonic maps between two semi-Riemannian manifolds. He showed that these maps behave differently. Moreover, he obtained that harmonic maps between two semi-Riemannian manifolds must be subject to some restricted classes of semi-Riemannian manifolds. Thus, harmonic maps from a semi-Riemannian manifold into a lightlike manifold were studied only when the target manifold is a Riemannian hypersurface of a lightlike manifold.

In [25], Şahin and Gündüzalp investigated lightlike submersions from a semi-Riemannian manifold onto lightlike manifold. After this definition, different structure in Riemannian submersion theory began to be examined for lightlike submersion as well. Firstly, Sachdeva et all. introduced slant lightlike submersions [19]. Later, Prasad et all. studied slant lightlike submersion for indefinite nearly Kaehler manifold [18]. They established the existence theorems for slant lightlike submersions and investigated geometry of foliations. Kaushal et all. introduced pointwise slant lightlike submersions [12]. Shukla et all. studied screen slant lightlike submersions [20].

Under the motivations and the light of these studies, we defined semi-slant lightlike submersion from indefinite Kaehler manifold onto lightlike manifold. We aim is to present some general properties of this type of submersions and after that to obtain major results on the geometry of them. In Section 2 we review some the standard facts on semi-Riemann submersions and lightlike submersions. After giving the definition of semi-slant lightlike submersions from indefinite Kaehler manifold into lightlike manifold in Section 3 we indicate related examples. In section 4 we study of minimality, integrability and totally geodesic conditions of distributions.

2. Preliminaries

In this section, we introduce lightlike submersions. We define lightlike submersions and O'Neill's tensors for lightlike submersions.

Let (M, g_M) and (B, g_B) be a semi-Riemannian manifold and an r-lightlike manifold, respectively. Therefore, we have a submersion $\psi : M \to B$. Moreover, $\psi^{-1}(q)$ is a submanifold of M, where $dim\psi^{-1} = dimM - dimB$. Then, for $q \in B$, $\psi^{-1}(q)$ is said to be fiber.

Thus, the kernel of ψ_* at the point p is defined by

$$\ker\psi_* = \{X \in T_pM : \psi_*(X) = 0\}.$$

On the other hand, we denote

$$(\ker\psi_*)^\perp = \{Y \in T_pM : g_M(X, Y) = 0, \forall X \in \ker\psi_*\}.$$

Since T_pM is a semi-Riemannian manifold, $(\ker\psi_*)^\perp$ cannot be a supplement to $\ker\psi_*$.

Assume $\triangle = \ker\psi_* \cap (\ker\psi_*)^\perp \neq \{0\}$. Therefore, we have different four cases of submersions:

Case1: Then consider $0 < dim\triangle < \min\{\dim(\ker\psi_*), \dim(\ker\psi_*)^\perp\}$.

Thus $\triangle$ is the radical subspace of T_pM.

On the other hand, $\ker\psi_*$ is a reel lightlike vector space.Then, there is supplementary non degenerate sub-space to $\triangle$. Let $S(\ker\psi_*)$ be a supplementary non degenerate sub space to $\triangle$ in $\ker\psi_*$. Thus we given by

$$\ker\psi_* = \triangle \perp S(\ker\psi_*).$$

By a similar method, we see that

$$(\ker\psi_*)^\perp = \triangle \perp S(\ker\psi_*)^\perp,$$

where $S(\ker\psi_*)^\perp$ is a supplementary sub-space of $\triangle$ in $(\ker\psi_*)^\perp$. However $S(\ker\psi_*)$ is non-degenerate in T_pM, we have

$$T_pM = S(\ker\psi_*) \perp S(\ker\psi_*)^\perp$$

where $S(\ker\psi_*)^\perp$ is the supplementary sub-space of $S(\ker\psi_*)$ in T_pM. On the other hand $S(\ker\psi_*)$ and $S(\ker\psi_*)^\perp$ are non degenerate, we deduce,

$$(S(\ker\psi_*))^\perp = S(\ker\psi_*)^\perp \perp (S(\ker\psi_*)^\perp)^\perp.$$

In that case, for all $\alpha, \beta \in \{1, ..., t\}$ and $i, j \in \{1, ..., r\}$, we get

$$g_M(\xi_i, \xi_j) = g_M(N_i, N_j) = 0, \quad g_M(\xi_i, N_j) = \delta_{ij}$$

$$g_M(W_\alpha, \xi_j) = g_M(W_\alpha, N_j) = 0, \quad g_M(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta},$$

where $\{\xi_i\}$ is base of $\triangle$, $\{N_i\}$ are null vector fields of $(S(\ker\psi_*)^\perp)^\perp$, $\{W_\alpha\}$ are bases of $S(\ker\psi_*)^\perp$. We can construct the set of vector fields $\{N_i\}$ for $ltr(\ker\psi_*)$, therefore, we arrive

$$tr(\ker\psi_*) = ltr(\ker\psi_*) \perp S(\ker\psi_*)^\perp.$$

We emphasize that $\ker\psi_*$ and $ltr(\ker\psi_*)$ are not orthogonal. Therefore, we show that $\mathcal{H} = tr(\ker\psi_*)$ the horizontal space and $\mathcal{V} = \ker\psi_*$ the vertical space of T_pM as is usual in the theory of Riemannian submersions. Hence we have,

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p.$$

We note that $\mathcal{H}$ and $\mathcal{V}$ are not orthogonal.

Now, we can give the definition of a lightlike submersion.

Definition 1. *[25], Let $\psi : (M, g_M) \to (B, g_B)$ be a submersion, where M and B are a semi-Riemannian manifold and an r-lightlike manifold, respectively. Therefore, ψ is said to be an r-lightlike submersion if,*

(i) $\dim \triangle = \dim\{\ker \psi_* \cap (\ker \psi_*)^{\perp}\} = r, 0 < r < \min\{\dim(\ker \psi_*), \dim(\ker \psi_*)^{\perp}\}$.

(ii) $g_M(X, Y) = g_B(\psi_* X, \psi_* Y)$ *for all* $X, Y \in \Gamma(\mathcal{H})$.

Case2: $\dim \triangle = \dim \ker \psi_* < \dim(\ker \psi_*)^{\perp}$.
Therefore, $\mathcal{H} = S(\ker \psi_*)^{\perp} \perp ltr(\ker \psi_*)$ and $\mathcal{V} = \triangle$. Then, ψ is said to be an isotropic submersion.

Case3: $\dim \triangle = \dim(\ker \psi_*)^{\perp} < \dim \ker \psi_*$.
Therefore $\mathcal{H} = ltr(\ker \psi_*)$ and $\mathcal{V} = S(\ker \psi_*) \perp \triangle$. Then, ψ is said to be a co-isotropic submersion.

Case4: $\dim \triangle = \dim(\ker \psi_*)^{\perp} = \dim \ker \psi_*$.
Therefore $\mathcal{H} = ltr(\ker \psi_*)$ and $\mathcal{V} = \triangle$. Then, ψ is said to be a totally lightlike submersion.

Now, we follow the lemma that we will use in the definition of semi-slant lightlike submersion.

Lemma 1. *[19], Let $\psi : (M, J, g_M) \to (B, g_B)$ be a r-lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Let $J\triangle$ be a distribution on M such that $\triangle \cap J\triangle = 0$. Then any distribution complementary to $J\triangle \oplus J(ltr(\ker \psi_*))$ in $S(\ker \psi_*)$ is Riemannian.*

On the other hand, O'Neill was defined tensors $\mathcal{T}$ and $\mathcal{A}$ for Riemannian submersions [15]. Şahin and Gündüzalp are characterized tensors $\mathcal{T}$ and $\mathcal{A}$ for lightlike submersions as follows:

$$\mathcal{T}_E F = h\overset{M}{\nabla}_{vE} vF + v\overset{M}{\nabla}_{vE} hF \tag{1}$$

and

$$\mathcal{A}_E F = h\overset{M}{\nabla}_{hE} hF + v\overset{M}{\nabla}_{hE} vF, \tag{2}$$

for all $E, F \in \Gamma(TM)$, where h and v are the horizontal and vertical projections. Therefore from (1) and (2), we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + v\overset{M}{\nabla}_U V \tag{3}$$

$$\overset{M}{\nabla}_U X = \mathcal{T}_U X + h\overset{M}{\nabla}_U X \tag{4}$$

$$\overset{M}{\nabla}_X U = v\overset{M}{\nabla}_X U + \mathcal{A}_X U \tag{5}$$

$$\overset{M}{\nabla}_X Y = \mathcal{A}_X Y + h\overset{M}{\nabla}_X Y, \tag{6}$$

for all $U, V \in \Gamma(\ker \psi_*)$ and $X, Y \in \Gamma(tr(\ker \psi_*))$, [25].

Now, let's remember the definition of indefinite Kaehler manifold. A 2m-dimensional differentiable manifold $M = (M, J, g_M)$ is said to be indefinite Kaehler manifold if there exist a semi-Riemannian metric g_M and a complex structure J,

$$J^2 = -I, \quad g_M(JE, JF) = g_M(E, F) \tag{7}$$

and

$$(\nabla_E J)F = 0, \tag{8}$$

for any $E, F \in \Gamma(TM)$.

3. Semi-Slant Lightlike Submersions

Firstly, let's define the semi-slant lightlike submersions and give examples.

Definition 2. *Let (M, J, g_M) and (B, g_B) be an indefinite Kaehler manifold and r-lightlike manifold, respectively. Let $\psi : (M, J, g_M) \to (B, g_B)$ be an r-lightlike submersion. Therefore, ψ is called a semi-slant lightlike submersion if there exist on M two non-degenere orthogonal distributions D_1 and D_2 such that*

(i) $J\triangle$ is a distribution in $\ker\psi_*$ such that $\triangle \cap J\triangle = 0$;

(ii)

$$S(\ker\psi_*) = (J\triangle \oplus J(ltr(\ker\psi_*))\perp D_1 \perp D_2;$$

(iii) D_1 is an invariant distribution, under J, that is $JD_1 = D_1$;

(iv) D_2 is slant distribution with angle $\theta(X)$, such that for all $x \in M$ and $X \in (D_2)_x$.

Moreover, the angle θ is saidto be the semi-slant angle of the lightlike submersion. In particular, if $D_1 = 0$, therefore M is a slant lightlike submersion.

Hence we get,

$$\begin{aligned} TM &= \mathcal{V} \oplus \mathcal{H} \\ &= \{\triangle \perp (J\triangle \oplus J(ltrker\psi_*)) \perp D_1 \perp D_2\} \oplus \{\psi(D_2) \perp \mu \perp ltr(ker\psi_*)\}, \end{aligned}$$

where μ is the orthogonal sub-bundle complementary to $\psi(D_2)$ in $S(ker\psi_*)$.

Example 1. *Every slant lightlike submersion from indefinite Kaehler manifold onto r-lightlike manifold is semi-slant lightlike manifold with $D_1 = 0$.*

Example 2. *Let $(\mathbb{R}^{12}_{0,2,10}, g_1, J)$ and $(\mathbb{R}^{7}_{1,0,6}, g_2)$ be an indefinite Kaehler manifold and lightlike manifold, $g_1 = -(dx_1)^2 - (dx_2)^2 + \sum_{i=3}^{12}(dx_i)^2$ is semi-Riemannian metric and $g_2 = \sum_{j=2}^{7}(dy_j)^2$ is a degenerate metric, where x_i, $i = 1, ...12$ and $y_j, j = 1, ...7$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^7$ respectively. If we set $J(x_1, x_2, ..., x_{11}, x_{12}) = (-x_2, x_1, ..., -x_{12}, x_{11})$ then $J^2 = -I$ and J is complex structure on $\mathbb{R}^{12}$. We define the following map*

$$\psi : \mathbb{R}^{12} \to \mathbb{R}^7$$

$$(x_1, ..., x_{12}) \to (x_1 + x_4, x_2, x_3, \frac{x_5 + x_7}{\sqrt{2}}, \frac{x_6 + x_8}{\sqrt{2}}, \sin\alpha x_9 - \cos\alpha x_{11}, x_{12}).$$

On the other hand, kernel of ψ_* *is*

$$\begin{aligned} \ker\psi_* &= Sp\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3}, V_4 = \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}, \\ V_5 &= \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}, V_6 = -\cos\alpha\frac{\partial}{\partial x_9} - \sin\alpha\frac{\partial}{\partial x_{11}}, V_7 = \frac{\partial}{\partial x_{10}}\}. \end{aligned}$$

Then, we arrive

$$\begin{aligned} (\ker\psi_*)^\perp &= Sp\{Z_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, Z_2 = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}, Z_3 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8} \\ Z_4 &= \sin\alpha\frac{\partial}{\partial x_9} - \cos\alpha\frac{\partial}{\partial x_{11}}, Z_5 = \frac{\partial}{\partial x_{12}}\}. \end{aligned}$$

On the other hand, we have $\ker\psi_* \cap (\ker\psi_*)^\perp = Sp\{V_1\}$. *Moreover, we get* $ltr(\ker\psi_*) = Sp\{N = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4})\}$. *Then the horizontal and vertical spaces are given by*

$$\mathcal{H} = \{N, Z_2, Z_3, Z_4, Z_5\}, \mathcal{V} = Sp\{V_1, V_2, V_3, V_4, V_5, V_6, V_7\},$$

Also by direct computations we obtain, $g_1(N,N) = g_2(\psi_*N, \psi_*N)$, *and* $g_1(Z_i, Z_i) = g_2(\psi_*Z_i, \psi_*Z_i)$ *for all* $i = 2, ..., 5$. *Hence* ψ *is a 1-lightlike submersion. On the other hand, we have* $JV_4 = -V_5, JV_5 = V_4$. *Thus it follows that* $D_1 = Sp\{V_4, V_5\}$ *and* $D_2 = Sp\{V_6, V_7\}$ *are a invariant and slant distribution with slant angle* $\theta = \alpha$, *respectively. Moreover* $JV_1 = V_2 + V_3$, $JN = \frac{1}{2}(-V_2 + V_3)$ *such that* $J\triangle$ *and* $J(ltr(\ker\psi_*))$ *are distributions on* $\mathbb{R}^{12}_2$. *Thus* ψ *is a semi slant lightlike submersion.*

Example 3. *Let* $(\mathbb{R}^{12}_{0,2,10}, g_1, J)$ *and* $(\mathbb{R}^6_{2,0,4}, g_2)$ *be an indefinite Kaehler manifold and lightlike manifold,* $g_1 = -(dx_1)^2 - (dx_2)^2 + \sum_{i=3}^{12}(dx_i)^2$ *is semi-Riemannian metric and* $g_2 = \sum_{j=3}^{6}(dy_j)^2$ *is a degenerate metric, where* x_i, $i = 1, ...12$ *and* $y_j, j = 1, ...6$ *are the canonical coordinates on* $\mathbb{R}^{12}$ *and* $\mathbb{R}^6$ *respectively. If we set* $J(x_1, x_2, ..., x_{11}, x_{12}) = (-x_2, x_1, ..., -x_{12}, x_{11})$ *then* $J^2 = -I$ *and* J *is complex structure on* $\mathbb{R}^{12}$. *We define the following map*

$$\psi : \mathbb{R}^{12} \to \mathbb{R}^7$$

$$(x_1, ..., x_{12}) \to (x_1 + x_5, x_2 + x_6, \frac{x_3 - x_7}{\sqrt{2}}, \frac{x_4 - x_8}{\sqrt{2}}, \frac{x_9 - x_{12}}{\sqrt{2}}, x_{11}).$$

On the other hand, kernel of ψ_* *is*

$$\begin{aligned} \ker\psi_* = \quad & Sp\{V_1 = \tfrac{\partial}{\partial x_1} - \tfrac{\partial}{\partial x_5}, V_2 = \tfrac{\partial}{\partial x_2} - \tfrac{\partial}{\partial x_6}, V_3 = \tfrac{\partial}{\partial x_3} + \tfrac{\partial}{\partial x_7}, V_4 = \tfrac{\partial}{\partial x_4} + \tfrac{\partial}{\partial x_8}, \\ & V_5 = \tfrac{\partial}{\partial x_9} + \tfrac{\partial}{\partial x_{12}}, V_6 = \tfrac{\partial}{\partial x_{10}}\}. \end{aligned}$$

Then, we arrive

$$(\ker\psi_*)^\perp = Sp\{Z_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_5}, Z_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_6}, Z_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_7}$$

$$Z_4 \quad = \quad \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_8}, Z_5 = \frac{\partial}{\partial x_9} - \frac{\partial}{\partial x_{12}}, Z_6 = \frac{\partial}{\partial x_{11}}\}.$$

On the other hand, we have $\ker\psi_* \cap (\ker\psi_*)^\perp = Sp\{V_1, V_2\}$. *Moreover, we get* $ltr(\ker\psi_*) = Sp\{N_1 = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), N_2 = \frac{1}{2}(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6})\}$. *Then the horizontal and vertical spaces are given by*

$$\mathcal{H} = \{N_1, N_2, Z_3, Z_4, Z_5, Z_6\}, \mathcal{V} = Sp\{V_1, V_2, V_3, V_4, V_5, V_6\},$$

Also by direct computations we obtain, $g_1(N_j, N_j) = g_2(\psi_* N_j, \psi_* N_j)$, *and* $g_1(Z_i, Z_i) = g_2(\psi_* Z_i, \psi_* Z_i)$ *for all* $i = 3, ..., 6$. *Hence* ψ *is a 2-lightlike submersion. On the other hand, we have* $JV_3 = -V_4, JV_4 = V_3$. *Thus it follows that* $D_1 = Sp\{V_3, V_4\}$ *and* $D_2 = Sp\{V_5, V_6\}$ *are a invariant and slant distribution with slant angle* $\theta = \frac{\alpha}{4}$, *respectively. Moreover* $JV_1 = V_2 + V_3$, $JN = \frac{1}{2}(-V_2 + V_3)$ *such that* $J\triangle$ *and* $J(ltr(\ker\psi_*))$ *are distributions on* $\mathbb{R}^{12}_2$. *Thus* ψ *is a semi slant lightlike submersion.*

Now, let ψ be a r-lightlike submersion. Therefore for $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$, we get

$$JU = \phi U + wU, \quad JX = BX + CX, \tag{9}$$

where $wU(CX\ )$ and $\phi U(BX)$ are the transversal component and tangential of $JU(JX)$, respectively.

Denote by P_1, P_2, P_3, P_4, P_5 the projections onto the distributions $\triangle$, $J\triangle$, $J(ltr(\ker\psi_*))$, D_1, D_2, respectively.

Thus, for any $U \in \Gamma(\mathcal{V})$, we can write

$$U = P_1U + P_2U + P_3U + P_4U + P_5U.$$

We applying J to last equation, we get

$$JU = JP_1U + JP_2U + JP_3U + JP_4U + \phi P_5U + wP_5U, \tag{10}$$

where $\phi P_5U(resp.\ wP_5U)$ denotes the tangential (resp. transversal) component of JP_5U. Then, we have

$$JP_1U = \phi P_1U \in \Gamma(J\triangle), \quad wP_1U = 0,$$

$$JP_2U = \phi P_2U \in \Gamma(\triangle), \quad wP_2U = 0,$$

$$JP_3U = wP_3U \in \Gamma(ltr(\ker\psi_*)), \quad \phi P_3U = 0,$$

$$JP_4U = \phi P_4U \in \Gamma(D_1),$$

$$\phi P_5U \in \Gamma(D_2), \quad wP_5U \in \Gamma(\psi(D_2)).$$

Therefore, we can write

$$\phi U = \phi P_1U + \phi P_2U + \phi P_5U.$$

Theorem 1. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold, respectively. Therefore ψ is a semi-slant lightlike submersion if and only if*

i) $Jltr(\ker \psi_*)$ is a distribution on M,

ii) for all $U \in \Gamma(\ker \psi_*)$,

$$\phi^2 P_5 U = \lambda P_5 U, \tag{11}$$

where, $\lambda = -\cos^2\theta$ and θdenotes the semi-slant angle of D_2.

Proof. Firstly, let ψ be a semi-slant lightlike submersion. Therefore $J\triangle$ is a distribution on $S(\ker \psi_*)$. Then, using *Lemma 1*, $J(ltr(\ker \psi_*))$ is a distribution on M.

Further, since ψ is semi-slant lightlike submersion, the slant angle betwen JU and D_2 is constant. Then using (10) and (7), we get

$$\cos\theta_{D_2} = -\frac{g_M(U, (\phi P_5)^2 U)}{\|JU\| \, \|\phi P_5 U\|}.$$

On the other hand, from (7), we obtain

$$\cos\theta_{D_2} = \frac{\|JU\|}{\|\phi P_5 U\|}.$$

By the last two equations, we have

$$\cos\theta_{D_2}^2 = -\frac{g_M(U, (\phi P_5)^2 U)}{\|\phi P_5 U\|^2}.$$

Since the angle θ is constant on D_2, we give

$$\phi^2 P_5 U = \lambda^2 P_5 U,$$

where $\lambda = -\cos^2\theta$.

Conversely, from (i), $J\triangle$ is a distribution on $S(\ker \psi_*)$. Moreover, if *lemma 2* is used, the proof is complete. $\square$

Corollary 1. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore, for all $U, V \in \Gamma(\ker \psi_*)$*

$$g_M(\phi U, \phi V) = \cos^2\theta g_M(U, V), \tag{12}$$

$$g_M(wU, wV) = \sin^2\theta g_M(U, V). \tag{13}$$

4. Minimality, Integrability and Totally Geodesic Foliations

In this section, we investigate minimality,totally geodesic and integrability of distributions.

Theorem 2. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore D_1 is integrable if and only if*

i)$\mathcal{T}_U \phi P_4 V - \mathcal{T}_V \phi P_4 U \notin \Gamma(\psi(D_2))$

ii) $g_M(v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U, BN) = g_M(\mathcal{T}_V \phi P_4 U - \mathcal{T}_U \phi P_4 V, CN)$

iii) $v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U \notin \Gamma(\triangle)$,

where $U, V \in \Gamma(D_1), K \in \Gamma(D_2), W \in \Gamma(Jltr(\ker \psi_*)), N \in \Gamma(ltr(\ker \psi_*))$.

Proof. For all $U, V \in \Gamma(D_1)$, since $[U, V] \in \Gamma(\mathcal{V})$ we arrive $g_M([U, V], X) = 0$, where $X \in \Gamma(\mathcal{H})$. Thus, for all $K \in \Gamma(D_2), W \in \Gamma(Jltr(\ker \psi_*))$ and $N \in \Gamma(ltr(\ker \psi_*))$, we get D_1 is integrable if and only if $g_M([U, V], K) = 0, g_M([U, V], N) = 0$ and $g_M([U, V], W) = 0$. Firsly using (7) and (8), we have

$$\begin{aligned} g_M(\nabla_U V, K) &= -g_M(\nabla_U JV - (\nabla_U J)V, JK) \\ &= -g_M(\nabla_U JV, JK). \end{aligned} \tag{14}$$

Then, from (7), (8) and (10) we get

$$\begin{aligned} g_M([U, V], K) &= -g_M(\nabla_U V, J\phi P_5 K) + g_M(\nabla_U JV, wP_5 K) \\ &\quad +g_M(\nabla_V U, J\phi P_5 K) - g_M(\nabla_V JU, wP_5 K). \end{aligned}$$

Also, using (10), (12), (3) and (4) we have

$$\begin{aligned} g_M([U, V], K) &= \cos^2\theta g_M(\nabla_U V, K) + g_M(\mathcal{T}_U \phi P_4 V, wP_5 K) \\ &\quad -\cos^2\theta g_M(\nabla_V U, K) - g_M(\mathcal{T}_V \phi P_4 U, wP_5 K). \end{aligned}$$

After some calculations, we obtain

$$\sin^2\theta g_M([U, V], K) = g_M(\mathcal{T}_U \phi P_4 V - \mathcal{T}_V \phi P_4 U, wP_5 K)$$

which proves (i).

For $N \in \Gamma(ltr(\ker \psi_*))$, from (10), (14), we obtain

$$g_M([U, V], N) = g_M(\nabla_U \phi P_4 V - \nabla_V \phi P_4 U, JN).$$

Thus, using (3) and (9), we get

$$\begin{aligned} g_M([U, V], N) &= g_M(v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U, BN) \\ &\quad +g_M(\mathcal{T}_U \phi P_4 V - \mathcal{T}_V \phi P_4 U, CN) \end{aligned}$$

which gives (ii).

Finally, $W \in \Gamma(Jltr(\ker \psi_*))$, from (10), (14) and (3), we arrive at

$$g_M([U, V], W) = g_M(v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U, \phi P_2 W)$$

which proves (iii). □

Theorem 3. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold, where g_M is semi-Riemannian metric of index 2r. Therfore, the invariant distribution D_1 is minimal.*

Proof. The distribuiton D_1 is minimal iff $T_V V + T_{JV} JV = 0$, for all $V \in \Gamma(D_1)$. By virtue of (7), (8) and (3), we obrtain

$$g(T_V V + T_{JV} JV, X) = g(\nabla_V JV, JX) - g(\nabla_{JV} V, JX)$$

which gives our assertion. □

Theorem 4. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore D_2 is integrable if and only if*
i) $g_M(v\nabla_K \phi P_5 L - v\nabla_L \phi P_5 K, \phi P_4 U) = -g_M(\mathcal{T}_K wP_5 L - \mathcal{T}_L wP_5 K, \phi P_4 U)$
ii) $g_B(\psi_*(h\nabla_K wP_5 L) - \psi_*(h\nabla_L wP_5 K), \psi_*(CN)) = -g_M(\mathcal{T}_K wP_5 L - \mathcal{T}_L wP_5 K, BN)$
iii) $g_B(\psi_*(h\nabla_K wP_5 L) - \psi_*(h\nabla_L wP_5 K), \psi_*(wP_3 W)) = g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, wP_3 W)$,
where $K, L \in \Gamma(D_2), U \in \Gamma(D_1), W \in \Gamma(Jltr(\ker \psi_*)), N \in \Gamma(ltr(\ker \psi_*))$.

Proof. For all $K, L \in \Gamma(D_2), U \in \Gamma(D_1)$, using (10), (14), (3) and (4), we get

$$g_M(\nabla_K L, U) = g_M(\mathcal{T}_K \phi P_5 L + v\nabla_K \phi P_5 L + h\nabla_K wP_5 L + \mathcal{T}_K wP_5 L, \phi P_4 U).$$

After some calculations, we have

$$\begin{aligned} g_M([K, L], U) &= g_M(\nabla_K \phi P_5 L - \nabla_L \phi P_5 K, \phi P_4 U) \\ &\quad + g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, \phi P_4 U) \end{aligned}$$

which proves (i).

For $N \in \Gamma(ltr(\ker \psi_*))$, from (10), (14) and (12), we arrive at

$$\begin{aligned} g_M([K, L], N) &= \cos^2\theta g_M(\nabla_K L, N) + g_M(\mathcal{T}_K wP_5 L, BN) + g_M(h\nabla_K wP_5 L, CN) \\ &\quad - \cos^2\theta g_M(\nabla_L K, N) - g_M(\mathcal{T}_L wP_5 K, BN) - g_M(h\nabla_L wP_5 K, CN). \end{aligned}$$

Now, using the character of ψ, we obtain

$$\begin{aligned} \sin^2\theta g_M([K, L], N) &= g_M(\mathcal{T}_K wP_5 L - \mathcal{T}_L wP_5 K, BN) \\ &\quad + g_B(\psi_*(h\nabla_K wP_5 L) - \psi_*(h\nabla_L wP_5 K), \psi_*(CN)) \end{aligned}$$

which proves (ii).

For $W \in \Gamma(Jltr(\ker \psi_*))$

$$\begin{aligned} g_M([K, L], W) &= g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, wP_3 W) \\ &\quad + g_M(h\nabla_K wP_5 L - h\nabla_L wP_5 K, wP_3 W). \end{aligned}$$

Then, using the character of ψ, we have

$$\begin{aligned} g_M([K, L], W) &= g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, wP_3 W) \\ &\quad + g_B(\psi_*(h\nabla_K wP_5 L) - \psi_*(h\nabla_L wP_5 K), \psi_*(wP_3 W)) \end{aligned}$$

which proves (iii). □

Theorem 5. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore $\triangle$ is a totally geodesic foliation on M if and only if*

$$g_M(\mathcal{T}_K JP_3 Z + \mathcal{T}_K wP_5 Z, JL) = g_M(\nabla_K JP_2 Z + \nabla_K JP_4 Z + \nabla_K \phi P_5 Z, JL),$$

for any $K, L \in \Gamma(\triangle), Z \in S(\ker \psi_*)$.

Proof. For any $K, L \in \Gamma(\triangle), Z \in S(\ker \psi_*)$, using, (10) in (14), we have

$$\begin{aligned} g_M(\nabla_K L, Z) &= -g_M(\nabla_K JP_2 Z, JL) - g_M(\nabla_K JP_3 Z, JL) - g_M(\nabla_K JP_4 Z, JL) \\ &\quad - g_M(\nabla_K \phi P_5 Z, JL) - g_M(\nabla_K wP_5 Z, JL). \end{aligned}$$

Then by (3) and (4), imply

$$\begin{aligned} g_M(\nabla_K L, Z) &= -g_M(\nabla_K JP_2 Z, JL) - g_M(\mathcal{T}_K JP_3 Z, JL) - g_M(\nabla_K JP_4 Z, JL) \\ &\quad - g_M(\nabla_K \phi P_5 Z, JL) - g_M(\mathcal{T}_K wP_5 Z, JL) \end{aligned}$$

which gives our assertion. □

Theorem 6. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore D_1 is a totally geodesic foliation on M if and only if*

$$g_M(\mathcal{T}_U \phi P_5 Z, JV) = -g_M(\nabla_U \phi P_5 Z, JV)$$

and

$$\nabla_U JN \notin \Gamma(D_1), \ \nabla_U JW \notin \Gamma(D_1),$$

for all $U, V \in \Gamma(D_1), Z \in \Gamma(D_2), W \in \Gamma(Jltr(\ker \psi_*)), N \in \Gamma(ltr(\ker \psi_*))$.

Proof. Invariant distribution D_1 defines a totally geodesic foliation iff $g_M(\nabla_U V, Z) = 0$, $g_M(\nabla_U V, Z) = 0$ and $g_M(\nabla_U V, W) = 0$ for any $U, V \in \Gamma(D_1), Z \in \Gamma(D_2)$, $N \in \Gamma(ltr(\ker \psi_*))$, $W \in \Gamma(Jltr(\ker \psi_*))$.

For $U, V \in \Gamma(D_1), Z \in \Gamma(D_2)$, using (7) and (8), we have

$$g_M(\nabla_U V, Z) = -g_M(\nabla_U JZ, JV). \tag{15}$$

By virtue of (10), (3) and (4) in (15) imply that

$$g_M(\nabla_U V, Z) = -g_M(\mathcal{T}_U \phi P_5 Z, JV) - g_M(\nabla_U \phi P_5 Z, JV).$$

Moreover, for $N \in \Gamma(ltr(\ker \psi_*)), W \in \Gamma(Jltr(\ker \psi_*))$, using (7), (8), (3) and (5), we arrive at

$$\begin{aligned} g_M(\nabla_U V, N) &= -g_M(\nabla_U JN, JV) \\ &= -g_M(v\nabla_U JN, JV) \end{aligned}$$

and $W \in \Gamma(Jltr(\ker \psi_*))$

$$\begin{aligned} g_M(\nabla_U V, W) &= -g_M(\nabla_U JW, JV) \\ &= -g_M(v\nabla_U JW, JV), \end{aligned}$$

which gives our assertion. □

Theorem 7. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore slant distribution D_2 is a totally geodesic foliation on M if and only if*

$$g_M(\mathcal{T}_U JZ, wV) = -g_M(v\nabla_U JZ, \phi V),$$
$$g_M(\mathcal{T}_U JN, wV) = -g_M(v\nabla_U JN, \phi V)$$

and

$$g_M(\mathcal{T}_U JW, \phi V) = -g_B(\psi_*(h\nabla_U JW), \psi_*(wV)),$$

for all $U, V \in \Gamma(D_2), Z \in \Gamma(D_1), W \in \Gamma(Jltr(\ker\psi_*)), N \in \Gamma(ltr(\ker\psi_*))$.

Proof. For all $U, V \in \Gamma(D_2), Z \in \Gamma(D_1)$,using (7) and (8), we give

$$g_M(\nabla_U V, Z) = -g_M(\nabla_U JZ, JV). \tag{16}$$

Now, from (3) and (9), we arrive at

$$g_M(\nabla_U V, Z) = -g_M(\mathcal{T}_U JZ, wV) - g_M(v\nabla_U JZ, \phi V).$$

Moreover, for $W \in \Gamma(Jltr(\ker\psi_*))$ and $N \in \Gamma(ltr(\ker\psi_*))$, using (9), (5) and (4), we have

$$g_M(\nabla_U V, N) = -g_M(\mathcal{T}_U JN, wV) - g_M(v\nabla_U JN, \phi V)$$

and

$$g_M(\nabla_U V, W) = -g_M(\mathcal{T}_U JW, \phi V) - g_M(h\nabla_U JW, wV)$$

which gives our assertion. □

Theorem 8. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore $\mathcal{V}$ is a totally geodesic foliation on M if and only if*

$$g_M(v\nabla_E BN + T_E CN, JF) = -g_M(T_E BN + h\nabla_E CN, JF)$$

and

$$v\nabla_E\phi P_1F + v\nabla_E\phi P_2F + v\nabla_E\phi P_4F + v\nabla_E\phi P_5F + T_E wP_3F + T_E wP_5F \notin \Gamma(D_2),$$

where $E, F \in \Gamma(\mathcal{V}), N \in \Gamma(ltr(\ker\psi_*))$.

Proof. For any $E, F \in \Gamma(\mathcal{H}), N \in \Gamma(ltr(\ker\psi_*))$, using (7), (8) and (9), we have

$$g_M(\nabla_E F, N) = -g_M(\nabla_E BN + \nabla_E CN, F).$$

Then, from (3) and (4), we arrive at

$$g_M(\nabla_E F, N) = -g_M(T_E BN + v\nabla_E BN + T_E CN + h\nabla_E CN, JF).$$

On the other hand, for $K \in \Gamma(D_2)$, using (7), (8) and (10), we get

$$g_M(\nabla_E F, JK) = g_M(J\nabla_E\phi P_1F + J\nabla_E\phi P_2F + J\nabla_E\phi P_3F + J\nabla_E\phi P_4F + J\nabla_E\phi P_5F, JK).$$

By virtue of (3) and (4), we arrive at

$$g_M(\nabla_E F, N) = g_M(w(v\nabla_E \phi P_1 F + v\nabla_E \phi P_2 F + v\nabla_E \phi P_4 F + v\nabla_E \phi P_5 F + T_E w P_3 F + T_E w P_5 F), JK)$$

which completes proof. □

Theorem 9. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore $\mathcal{H}$ is a totally geodesic foliation on M if and only if*

$$g_M(A_X BY + h\nabla_X CY, wP_5 K) = g_M(v\nabla_X BY + A_X CY, \phi P_5 K),$$

$$A_X CY + v\nabla_X BY \notin \Gamma(D_1),$$

and

$$A_X BY + h\nabla_X CY \notin \Gamma(ltr(\ker \psi_*)),$$

where $X, Y \in \Gamma(\mathcal{H}), K \in \Gamma(D_2)$.

Proof. For all $X, Y \in \Gamma(\mathcal{H}), K \in \Gamma(D_2)$, from (7), (8) and (9), we have

$$g_M(\nabla_X Y, K) = -g_M(\nabla_X BY + \nabla_X CY, JK).$$

By virtue of (5) and (6), we get

$$g_M(\nabla_X Y, K) = -g_M(v\nabla_X BY + A_X BY + A_X CY + h\nabla_X CY, \phi P_5 K + wP_5 K).$$

Moreover, for $U \in \Gamma(D_1)$ and for $W \in \Gamma(Jltr(\ker \psi_*))$, by virtue of (5) and (6) we arrive at

$$g_M(\nabla_X Y, K) = -g_M(v\nabla_X BY + A_X CY, K)$$

and

$$g_M(\nabla_X Y, W) = -g_M(A_X BY + h\nabla_X CY, wP_3 W),$$

which gives our assertion. □

Theorem 10. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore M is a locally product manifold of the leaves of $\mathcal{V}$ and $\mathcal{H}$ if and only if*

$$g_M(v\nabla_E BN + T_E CN, JF) = -g_M(T_E BN + h\nabla_E CN, JF),$$

$$v\nabla_E \phi P_1 F + v\nabla_E \phi P_2 F + v\nabla_E \phi P_4 F + v\nabla_E \phi P_5 F + T_E w P_3 F + T_E w P_5 F \notin \Gamma(D_2),$$

and

$$g_M(A_X BY + h\nabla_X CY, wP_5 K) = g_M(v\nabla_X BY + A_X CY, \phi P_5 K),$$

$$A_X CY + v\nabla_X BY \notin \Gamma(D_1),$$

$$A_X BY + h\nabla_X CY \notin \Gamma(ltr(\ker \psi_*)),$$

where $E, F \in \Gamma(\mathcal{V}), N \in \Gamma(ltr(\ker \psi_*)),\ X, Y \in \Gamma(\mathcal{H}), K \in \Gamma(D_2)$.

Conclusion 1. *Submersions, lightlike manifolds and semi-Riemannian manifolds have potential for applications in many fields of physics, engineering and mathematics. In particular it is applicable to the theory of liquid crystals (Harmonic morphisms), theory of spacetimes, theory of relativity. Research in this theory has been increasing in recent years After the defination of submersions from semi-Riemannian manifolds onto lightlike manifolds, slant lightlike submersions were studied. In this paper, the idea of examining semi-slant lightlike submersions are emphasized. We defined and studied semi-slant lightlike submersions from an indefinite Kaehler manifold to an r-lightlike manifold. We introduced geometry of foliatons. The works on this subject will be useful tools for the applications of semi-slant lightlike submersion with various manifolds.*

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References

[1] Akyol, M. A., Gündüzalp, Y., Semi-invariant semi Riemannian submersions, *Commun. Fac. Sci. Univ. Ank. Series A1*, 67(1) (2018), 80-92. https://doi.org/10.1501/Commua1-0000000832

[2] Altafini, C., Redundant robotic chains on Riemannian submersions, *IEEE ,Transactions on Robotics and Automation*, 20(2) (2004), 335-340. https://doi.org/10.1109/TRA.2004.824636

[3] Bourguignon, J. P., Lawson, H. B., Stability and isolation phenomena for Yang-mills fields, *Commun. Math. Phys.*, 79 (1981), 189-230.

[4] Bourguignon, J. P., Lawson, H. B., A mathematician's visit to Kaluza-Klein theory, *Rend. Sem. Mat. Univ. Politec. Torino*, Special Issue, (1989), 143-163.

[5] Duggal, K. L., Harmonic maps, morphisms and globally null manifolds, *Int. J. of Pure and App. Math.*, 6(4) (2003), 421-438.

[6] Falcitelli, M., Ianus, S., Pastore, A. M., Riemannian Submersions and Related Topics, World Scientific, River Edge, NJ, 2004.

[7] Gray, A., Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.*, 16 (1967), 715-737.

[8] Gündüzalp, Y., Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds, *Journal of Function Spaces and Applications*, (2013), Article ID 720623.

[9] Gündüzalp, Y., Slant submersions from almost product Riemannian manifolds, *Turkish J. Math.*, 37 (2013), 863-873. https://doi.org/10.3906/mat-1205-64

[10] Gündüzalp, Y., Semi-slant submersions from almost product Riemannian manifolds, *Demonstratio Mathematica*, 49(3) (2016), 345-356. https://doi.org/10.1515/dema-2016-0029

[11] Ianus, S., Visinescu, M., Kaluza-Klein theory with scalar fields and generalized Hopf manifolds, *Class. Quantum Gravity*, 4 (1987), 1317-1325.

[12] Kaushal, R., Kumar, R., Kumar Nagaich, R., On geometry of pointwise slant lightlike submersions with totally umbilical fibers, *Afrika Mathematica*, 33(1) (2022), 22. https://doi.org/10.1007/s13370-022-00963-4

[13] Ianus, S., Visinescu, M., Space-time compaction and Riemannian submersions, *The Mathematical Heritage of C. F. Gauss,River Edge, World Scientific*, (1991),358-371.

[14] Mustafa, M. T., Applications of harmonic morphisms to gravity, *J. Math. Phys.*, 41 (2000), 6918-6929. https://doi.org/10.1063/1.1290381

[15] O'Neill, B., The fundamental equations of a submersion, *Mich. Math. J.*, 13 (1966), 458-469.

[16] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York-London, 1983.

[17] Park, K. S., Prasad, R., Semi-slant submersions, *Bull. Korean Math. Soc.*, 50(3) (2013), 951-962. https://doi.org/10.4134/BKMS.2012.49.2.329

[18] Prasad, R., Kumar Singh, P., Kumar, S., Slant lightlike submersions from an indefinite nearly Kaehler manidolds into a lightlike manifold, *J. Math. Comput. Sci.*, 8 (2018), 324-349. https://doi.org/10.28919/jmcs/3561

[19] Sachdeva, R., Kumar, R., Bhatia, S. S., Slant lightlike submersions from an indefinite almost Hermitian manifold into a lightlike manifold, *Ukrainian Mathematical Journal*, 68(7) (2016), 225-240. https://doi.org/10.1007/s11253-016-1280-8

[20] Shukla, S. S., Omar, S., Yadav, S. K., Screen slant lightlike submersions, *J. Appl. Math. Informatic*, 40(5-6) (2022), 1073-1082. https://doi.org/10.1134/S1995080222060300

[21] Şahin, B., Screen conformal submersions between lightlike manifolds and semi-Riemannian manifolds and their harmonicity, *International Journal of Geometric Methods in Modern Physics*, 4(6) (2007), 987-1003. https://doi.org/10.1142/S0219887807002405

[22] Şahin, B., On a submersion between Reinhart lightlike manifolds and semi-Riemannian manifolds, *Mediterr. J. Math.*, 5 (2008), 273-284. https://doi.org/10.1007/s00009-008-0149-y

[23] Şahin, B., Slant submersions from almost Hermitian manifolds, *Bull. Math. Soc. Sci. Math. Roumanie*, 1 (2011), 93-105.

[24] Şahin, B., Riemannian Submersions, Riemannian Maps in Hermitian Geometry and Their Applications, Elsevier, Academic Press, 2017.

[25] Şahin, B., Gündüzalp, Y., Submersions from semi-Riemannian manifolds onto lightlike manifolds, *Hacet. J. Math. Stat.*, 39 (2010), 41-53.

[26] Vilcu, G. E., New class of semi-Riemannian submersions, *Romanian Journal of Physics*, 54(9-10) (2009), 815-821.

[27] Watson, B., *G, G'*-Riemannian submersions and nonlinear gauge field equations of general relativity, *Teubner-Texte Math.*, (1983), 324-349.