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A different approach for almost sequence spaces defined by a generalized weighted means

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ABSTRACT

In this study, we introduce $f(G,B)$, $f_0(G,B)$ and $fs(G,B)$ *sequence spaces* which consisting of all the sequences whose *generalized weighted* B*-difference means* are found in f, f_0 and fs spaces utilising *generalized weighted mean* and B*-difference matrices*. The γ-and the β-duals of the spaces $f(G,B)$ and $fs(G,B)$ are determined. At the same time, we have characterized the infinite matrices $(f(G,B):\mu)$ and $(\mu: f(G,B))$, where μ is any given sequence space.

Keywords: Matrix transformations, sequence spaces, matrix domain of a sequence space, dual spaces

Bir genelleştirilmiş ağırlıklı ortalama ile tanımlanan hemen hemen yakınsak dizi uzayları için bir farklı yaklaşım

ÖZ

Bu çalışmada, B-fark matrisi ile genelleştirilmiş ağırlıklı ortalama metodu yardımıyla inşa edilen $f(G,B)$, $f_0(G,B)$ ve $fs(G,B)$ dizi uzayları tanımlandı. Bu uzaylar, genelleştirilmiş ağırlıklı B-fark ortalamaları sırasıyla f, f_0 ve fs uzaylarında olan dizilerin uzayıdır. $f(G,B)$ ve $fs(G,B)$ uzaylarının γ- ve β-dualleri elde edildi. Ayrıca, μ herhangi bir dizi uzayı olmak üzere $(f(G,B):\mu)$ ve $(\mu: f(G,B))$ sonsuz matrisleri karakterize edildi.

Anahtar Kelimeler: Matris dönüşümleri, dizi uzayları, bir dizi uzayının matris alanı, dual uzaylar

1. INTRODUCTION

Let's start with the definition of *sequence space,* which is the basic concept of summability theory. As usual, the symbol w denotes the space of all real valued sequences. A *sequence* space is known as any *vector subspace* of w. By l_∞, c, c_0, $l_p(1 \le p < \infty)$, bs and cs, we demonstrate the sets of all *bounded, convergent, null sequences,* $p-$ *absolutely convergent series* , *bounded series and convergent series*, respectively. At the same, we are going to use representation that $e = (1,1,\ldots,1,\ldots)$ and $e^{(n)}$ is the sequence space in which only non-zero terms is 1 in the n-th place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\ldots\}$.
Let ϑ and η be arbitrary *sequence spaces* and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. we can defines a matrix transformation as follows. If $Ay = \{A_n(y)\}$, the A $-$transform of y, is in η for each $y = (y_k) \in \vartheta$, we call A as a matrix transformation from ϑ into η and denote the class of all such matrices by (ϑ, η). If a matrix A is an element of this class, then the series $A_n(y)$ is convergence for each $n \in \mathbb{N}$ and $y \in \vartheta$, where

$$A_n(y) = \sum_k a_{nk} y_k, \quad \text{for each } n \in \mathbb{N}$$

and $A_n = (a_{nk})_{k \in \mathbb{N}}$ is the sequence of elements in the n-th row of A. For sake of briefness, henceforward, the summation without limits runs from 0 to ∞.
A matrix E is called triangle if main diagonal's elements aren't zero and elements on the top of the main diagonal are zero. For triangle matrices E, F and a sequence y, the equality $E(Fy) = (EF)y$ holds. Further, a triangle matrix W uniquely has an inverse $W^{-1} = Z$, also a triangle matrix. The

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equality $y = W(Zy) = Z(Wy)$ yields for talked about matrices.

If there exists a single sequence (c_n) of scalars satisfied the following equation, then the sequence (c_n) is known a *Schauder basis* (or shortly *basis*) for a normed sequence space μ, where mentioned above equation is, for every $y \in \mu$,

$$\lim_{n\to\infty}\left\| y - \sum_{k=0}^{n} \alpha_k c_k \right\| = 0.$$

The series $\sum_k \alpha_k b_k$ which has the sum y is called the enlargement of y according to (c_n), and written as $y = \sum_k \alpha_k c_k$. *Schauder basis* and *algebraic basis* coincide for finite sequence spaces. Let us present the definition of some *triangle limitation matrices* which are required in text.

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For $u \in U$, let $\frac{1}{u} = \left(\frac{1}{u_k}\right)$. Let $u, v \in U$ and define the matrix $G(u,v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k, & (k < n), \\ u_n v_n, & (k = n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$, where u_n is only attached to n and v_k bounds up with only k. The matrix $G(u,v)$ described above, is entitled as *generalized weighted mean or factorable matrix.* Another matrix $B(r,s) = \{b_{nk}(r,s)\}$ known as *generalized difference matrix* is defined as below:

$$b_{nk}(r,s) = \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ 0, & (0 \le k < n-1 \; or \; k > n), \end{cases}$$

where r and s are non-zero *real numbers.* The matrix $B(r,s)$ can be degraded to the *difference matrix* $\Delta^{(1)}$ in case of $r = 1,\ s = -1$. Therefore, the obtained conclusions concerned with *domain of the matrix* $B(r,s)$ are the generalization of the consequences corresponding of the matrix domain of $\Delta^{(1)}$, where $\Delta^{(1)} = (\delta_{nk})$ is described as

$$\delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \le k \le n), \\ 0, & (0 \le k < n-1 \;\; or \;\; k > n). \end{cases}$$

The matrix $S = (s_{nk})$ is defined as

$$s_{nk} = \begin{cases} 1, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}.$$

The domain of an infinite matrix K on a *sequence space* μ is a sequence space denoted by μ_K and this space is recognized by the set

$$\mu_K = \{y = (y_k) \in w : Ky \in \mu\}. \qquad (1)$$

Generally, the new *sequence space* μ_K is the enlargement or the shrinkage of the original space μ, in some cases it can be sighted that those spaces overlap. Also, If μ is one of the *sequence space* of *bounded, convergent* and *null sequence* spaces, then inclusion relationship $\mu_S \subset \mu$ strictly holds. Further it can be acquired easily that the inclusion relationship $\mu \subset \mu_{\Delta^{(1)}}$ yields for $\mu \in \{l_\infty, c, c_0, l_p\}$.

Combined with *a linear topology a sequence space μ* is denominated a K *−space*, if for each $i \in \mathbb{N}$, coordinate maps $p_i : \mu \to \mathbb{C}$, described by $p_i(y) = y_i$ are continuous, where $\mathbb{C}$ is the *complex numbers field.* A K −space which is a *complete linear metric space* is entitled an *FK space*. An FK −space whose topology is normable is called a BK − *space* [1] which comprises ϕ, the set of all finitely nonzero sequences.

Let us assume that K is a *triangle matrix*, in that case, we can obviously say that the *sequence spaces* μ_K and μ are *linearly isomorphic*, i.e., $\mu_K \cong \mu$ and if μ is a BK *−space*, then μ_K is also a BK −space with the norm given by $\|y\|_{\mu_K} = \|Ky\|_\mu$, for all $x \in \mu_K$. As well as above mentioned *sequence spaces* l_∞, c, c_0 and *almost convergent sequence space* f are BK *−spaces* with the ordinary supnorm described by

$$\|y\|_\infty = \sup_{k\in\mathbb{N}} |y_k|.$$

Also l_p are BK −spaces with the ordinary norm defined by

$$\|y\|_p = \left(\sum_k |y_k|^p\right)^{\frac{1}{p}}, \qquad (1 \le p < \infty).$$

A continuous linear functional ψ on l_∞ is said a *Banach limit*, if

i) For every $y = (y_k)$, $\psi(y) \ge 0$,

ii) $\psi(y_{\rho(k)}) = \psi(y_k)$, where ρ is shift operator which is described onto w with $\rho(k) = k + 1$,

iii) $\psi(e) = 1$, where $e = (1,1,1, \dots)$.

A sequence $y = (y_k) \in l_\infty$ is entitled to be *almost convergent* to generalized limit l, if all *Banach Limits* of y are l [2] and denoted by $f - \lim y = l$. In an other saying, $f - \lim y_k = l$ iff uniformly in n

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} y_{k+n} = l.$$

We indicate the sets of all *almost convergent sequences* by f and series by fs and define as follow:

$$f = \left\{ y = (y_k) \in w : \lim_{m\to\infty} s_{mn}(y) = l \right\},$$

where l exists uniformly in n and

$$s_{mn}(y) = \frac{1}{m+1} \sum_{k=0}^{m} y_{k+n},$$

and

$$fs = \Big\{ y = (y_k) \in w : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \frac{y_j}{m+1} = l, uniformly\ in\ n \Big\}.$$

As known that the containments $c \subset f \subset l_\infty$ are precisely acquired. Owing to these containments, norms $\|.\|_f$ and $\|.\|_\infty$ of the spaces f and l_∞ are equivalent. Therefore the sets f and f_0 are *BK- spaces* having the following norm

$$\|y\|_f = \sup_{m,n} |s_{mn}(y)|.$$

For a sequence $y=(y_k)$, we demonstrate the *difference sequence space* by $\Delta y=(y_k-y_{k-1})$. Kızmaz first presented the *difference sequence spaces* as follows:

$$\mu(\Delta)=\{y=(y_k)\in w:\Delta y=(y_k-y_{k+1})\in\mu\}.$$

It was proved by Kızmaz [3] that $\mu(\Delta)$ is a Banach space with the norm

$$\|y\|_\Delta=|y_1|+\|\Delta y\|_\infty;\quad y=(y_k)\in\mu(\Delta)$$

and the containment relation $\mu\subset\mu(\Delta)$ strictly holds. The author at the same time investigated the $\alpha-,\beta-,\gamma-$ *duals of the difference spaces* and determined the classes $(\mu(\Delta):\nu)$ and $(\nu:\mu(\Delta))$ of infinite matrices, here $\mu,\nu\in\{l_\infty,c\}$. When we look according to summability theory perspective, we can see that to define new *Banach spaces* by the matrix domain of triangle and investigate their algebraical, geometrical and topological properties is well-known. Therefore, many authors were interested in this subject and by using some known matrices, many studies were done.

In literature, it was investigated domain of following matrices on the *almost convergent* and *null almost convergent sequence spaces* in the sources mentioned: *the generalized weighted mean G* in [4], the double band matrix $B(r,s)$ in [5], the *Riesz matrix* in [6], *Cesaro matrix* of order 1 in [13], the matrix B in [7] can be seen. Further, using *generalized difference Fibonacci matrix*, Candan and Kayaduman defined $\hat{c}^{f(r,s)}$ space [24]. Furthermore, it can be looked at those works about this topic nearly: [9], [10], [11], [25], [26], [27], [28], [29], [30], [31] [32] [33] [34] [35], [36].

Recently, A. Karaisa and F. Özger [12] the spaces $f(u,v,\Delta)$, $f_0(u,v,\Delta)$ and $fs(u,v,\Delta)$ defined and studied. By taking inspiration from this work, we decided to study this subject of this paper. By using *generalized weighted mean* and $B-$*difference matrices*, we familiarize $f(G,B)$, $f_0(G,B)$ and $fs(G,B)$ *sequence spaces* consisting of all sequences whose *generalized weighted* $B-$*difference means* are in the f, f_0 and fs spaces .

We assume throughout this paper $u=(u_k)$ and $v=(v_k)\in U$ (as above talk about) and $r,s\in\mathbb{R}-\{0\}$, further, we shall write for briefness that $R=R(G,B)=G(u,v).B(r,s)$, where

$$R(G,B)=\{r_{nk}\}=\begin{cases}u_nv_kr+u_nv_{k+1}s, & k<n,\\ u_nv_nr, & k=n,\\ 0, & k>n.\end{cases}$$

In following definitions, let $y=(y_k)$ be the $R(G,B)-$transform of a sequence $x=(x_k)$. Then $y_0=ru_0v_0x_0,$ and for $k\geq 1$

$$y_k=u_k\left(\textstyle\sum_{i=0}^{k-1}(rv_i+sv_{i+1})x_i+rv_kx_k\right), \tag{2}$$

and for each $j,k\in\mathbb{N}$ we shall write for briefness

$$\tilde{\nabla}_{jk}=(-1)^{j-k}\left(\frac{s^{j-k}}{r^{j-k+1}v_k}+\frac{s^{j-k-1}}{r^{j-k}v_{k+1}}\right) \tag{3}$$

and if $y=(y_k)=R(G,B)(x)\in f$, it means that $\exists l\in\mathbb{C}$ such that uniformly in n,

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}u_{k+n}\left(\textstyle\sum_{i=0}^{k+n-1}(rv_i+sv_{i+1})x_i+rv_{k+n}x_{k+n}\right)=l, \tag{4}$$

Now, let us define the *sequence space* $f(G,B)$

$$f(G,B)=\{x=(x_k)\in w:R(G,B)(x)\in f\}.$$

Similarly, we can define $f_0(G,B)$ and $fs(G,B)$ spaces as

$$f_0(G,B)=\{x=(x_k)\in w:R(G,B)(x)\in f_0\},$$

if $y=(y_k)\in f_0$, we know that in (4), $\alpha=0.$ Further,

$$fs(G,B)=\{x=(x_k)\in w:R(G,B)(x)\in fs\},$$

i.e. $y=(y_k)=R(G,B)(x)\in fs$, then $\exists l\in\mathbb{C}\ni$, uniformly in n,

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k+n}\left[u_j\left(\textstyle\sum_{i=0}^{j-1}(rv_i+sv_{i+1})x_i+rv_jx_j\right)\right]=l.$$

We can redefine the spaces $fs(G,B)$, $f(G,B)$ and $f_0(G,B)$ by the notation of (1),

$$f_0(G,B)=(f_0)_{R(G,B)},\ f(G,B)=(f)_{R(G,B)},\ fs(G,B)=(fs)_{R(G,B)}.$$

In this paper, we investigate some topological properties, *beta-* and *gamma- duals* of these spaces and study to acquire some matrix characterizations between these spaces and standard spaces.

2. SOME TOPOLOGICAL PROPERTIES OF THESE SPACES

Theorem 1: *i) The sequence space $f(G,B)$ is normed space with*

$$\|x\|_{f(G,B)}=\sup_{m,n}\left|\frac{1}{m+1}\sum_{k=0}^{m}u_{k+n}\left(\textstyle\sum_{i=0}^{k+n-1}(rv_i+sv_{i+1})x_i+rv_{k+n}x_{k+n}\right)\right|,$$

ii) The *sequence space* $fs(G,B)$ is *normed space with*

$$\|x\|_{fs(G,B)}=\sup_{m,n}\left|\frac{1}{m+1}\sum_{k=0}^{m}\left(\textstyle\sum_{j=0}^{k+n}u_j\left(\sum_{i=0}^{j-1}(rv_i+sv_{i+1})x_i+rv_jx_j\right)\right)\right|.$$

Theorem 2: *The sets $f(G,B)$, $f_0(G,B)$ and $fs(G,B)$ are linearly isomorphic to the sets f,f_0 and fs respectively, i.e., $f(G,B)\cong f$, $f_0(G,B)\cong f_0$, $fs(G,B)\cong fs$.*

Proof: Firstly, let us attest that $f(G,B)\cong f$. For this purpose, we have to show that there exists a linear bijection among the spaces $f(G,B)$ and f. Let us take into account the transformation T described by the relation of (1) from $f(G,B)$ to f with $x\to y=Tx=R(G,B)x\in f$, for $x\in f(G,B)$. The linearity of T is clear. Moreover, it is obvious that $x=0$ when $Tx=0$, thus T is injective.

Let us assume $y=(y_k)\in f$ and describe $x=(x_k)$ by

$$x_k=\sum_{j=0}^{k-1}\frac{1}{u_j}\tilde{\nabla}_{kj}y_j+\frac{1}{ru_kv_k}y_k,\quad(k\in\mathbb{N}).$$

Then, we have

$$u_k\left(\sum_{j=0}^{k-1}(rv_j+sv_{j+1})x_j+rv_kx_k\right)$$

$$=u_k\sum_{j=0}^{k-1}(rv_j+sv_{j+1})\left[\sum_{i=0}^{j-1}\frac{1}{u_i}\widetilde{\nabla}_{ji}y_i+\frac{1}{ru_jv_j}y_j\right]$$

$$+u_krv_k\left(\sum_{j=0}^{k-1}\frac{1}{u_j}\widetilde{\nabla}_{kj}y_j+\frac{1}{ru_kv_k}y_k\right)$$

$$=\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})\sum_{i=0}^{j-1}\frac{1}{u_i}\widetilde{\nabla}_{ji}y_i$$

$$+\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})\frac{1}{ru_jv_j}y_j+u_krv_k\left(\sum_{j=0}^{k-1}\frac{1}{u_j}\widetilde{\nabla}_{kj}y_j\right)+y_k$$

$$=\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})\sum_{i=0}^{j-1}\frac{1}{u_i}\widetilde{\nabla}_{ji}y_i$$

$$+\left[\sum_{j=0}^{k-1}(rv_j+sv_{j+1})\frac{1}{ru_jv_j}+rv_k\sum_{j=0}^{k-1}\frac{1}{u_j}\widetilde{\nabla}_{kj}\right]u_ky_j+y_k$$

$$=y_k$$

for all $k\in\mathbb{N}$, which leads us to the truth that

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}u_{k+n}\left(\sum_{i=0}^{k+n-1}(rv_i+sv_{i+1})x_i+rv_{k+n}x_{k+n}\right)$$

$$=\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}y_{k+n}\quad (uniformly\ in\ n)$$

$$=f-lim y_k.$$

It means that $x=(x_k)\in f(G,B)$. Hereby, we reach the truth that T is *surjective*. So, T is *a linear bijection*, and it means that the spaces $f(G,B)$ and f are *linearly isomorphic*, as desired. The fact $f_0(G,B)\cong f_0$ can be analogously attested. Due to the well known fact that *the matrix domain* λ_A of *the normed sequence space* denoted by λ, has got a *base* iff λ has got a *base,* whenever a matrix $A=(a_{nk})$ is a triangle [14] ($Remark$ 2.4) and since the space f has no *Schauder basis*, we have;

Corollary 1: *The space $f(G,B)$ has no Schauder basis.*

3.THE $\alpha-,\beta-,\gamma-$DUALS OF THESE SPACES

The $\alpha-,\beta-,\gamma-$duals of the *sequence space* X are defined by

$$X^{\alpha}=\{a=(a_k)\in w: ax=(a_kx_k)\in l_1,\ \forall\, x=(x_k)\in X\},$$

$$X^{\beta}=\{a=(a_k)\in w: ax=(a_kx_k)\in cs,\ \forall\, x=(x_k)\in X\},$$

and

$$X^{\gamma}=\{a=(a_k)\in w: ax=(a_kx_k)\in bs,\ \forall\, x=(x_k)\in X\},$$

here cs and bs are defined to be *sequence spaces* of all convergent and bounded series, respectively.

Lemma 1: [15] *So as to the matrix A appertains to the matrix class from f to l_∞ is necessary and sufficient condition*

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty$$

is satisfied.

Lemma 2: [15] *So as to the matrix A appertains to the matrix class from f to c is necessary and sufficient conditions*

i) $sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty$,
ii) $lim_{n\to\infty}a_{nk}=\alpha_k$, *for each* $k\in\mathbb{N}$,
iii) $lim_{n\to\infty}\sum_k a_{nk}=\alpha$,
iv) $lim_{n\to\infty}\sum_k|\Delta(a_{nk}-\alpha_k)|=0$,

are satisfied.

Theorem 3: *The $\gamma-$dual of the space $f(G,B)$ is the intersection of the sets*

$$b_1=\left\{a=(a_k)\in w: \sup_n\sum_{k=1}^{n-1}\left|\frac{a_k}{u_krv_k}+\frac{\widetilde{\nabla}_{jk}}{u_k}\sum_{j=k+1}^{n-1}a_j\right|<\infty\right\},$$

$$b_2=\left\{a=(a_k)\in w: \sup_n\left|\frac{a_n}{ru_nv_n}\right|<\infty\right\}.$$

Proof: For an optional sequence $a=(a_k)\in w$ and take into consideration the following equality

$$\sum_{k=0}^{n}a_kx_k=\sum_{k=0}^{n}a_k\left(\sum_{j=0}^{k-1}\frac{1}{u_j}\widetilde{\nabla}_{kj}y_j+\frac{1}{ru_kv_k}y_k\right)$$

$$=\left[\sum_{k=0}^{n-1}\frac{a_k}{ru_kv_k}+\frac{1}{u_k}\sum_{j=k+1}^{n-1}\widetilde{\nabla}_{jk}a_j\right]y_k+\frac{a_n}{ru_nv_n}y_n \quad (5)$$

$$=(Ey)_n$$

where the general term e_{nk} of the matrix E is determined as follows:

$$\begin{cases}\sum_{k=0}^{n-1}\frac{a_k}{ru_kv_k}+\frac{1}{u_k}\sum_{j=k+1}^{n-1}\widetilde{\nabla}_{jk}a_j, & 0\le k\le n-1,\\ \frac{a_n}{ru_nv_n}, & k=n,\\ 0, & k>n,\end{cases}\quad (6)$$

for all $k,n\in\mathbb{N}$. Thus, we deduce from [5] that $a_kx_k\in bs$ whenever $x=(x_k)\in f(G,B)$ necessary and sufficient condition $Ey\in l_\infty$ whenever $y=(y_k)\in f$, where $E=(e_{nk})$ is described in (6). That's why with assistance of Lemma 1, $f(G,B)^{\gamma}=b_1\cap b_2$.

Theorem 4: *The $\beta-$dual of the space $f(G,B)$ is the intersection of the sets*

$$b_3 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} e_{nk} \quad exists \right\},$$

$$b_4 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k e_{nk} \quad exists \right\},$$

$$b_5 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k \Delta[e_{nk} - \alpha_k] < \infty \right\},$$

where $\alpha_k = lim_{n\to\infty} e_{nk}$. *Then* $\{f(G,B)\}^\beta = \cap_{k=1}^5 b_k$.

Proof: Let us take any sequence $a \in w$. By (5), $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(G,B)$ necessary and sufficient condition $Ey \in c$ whenever $y = (y_k) \in f$, where $E = (e_{nk})$ is designated in (6). We reproduce the consequence by Lemma 2 that $\{f(G,B)\}^\beta = \cap_{k=1}^5 b_k$.

Theorem 5: *The* $\gamma -$*dual of the space* $fs(G,B)$ *is the intersection of the sets,*

$$b_6 = \left\{ a = (a_k) \in w : \sup_n \sum_k |\Delta e_{nk}| < \infty \right\},$$

$$b_7 = \left\{ a = (a_k) \in w : \lim_{k\to\infty} e_{nk} = 0 \right\}.$$

In another saying, we get $\{fs(G,B)\}^\gamma = b_6 \cap b_7$.

Proof: This might be acquired in a similar concept as talk about in the proof of Theorem 3 with Lemma 1 in lieu of Lemma 4 (iii). So, we neglect details.

Theorem 6: *Defined the set*

$$b_8 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k |\Delta^2 e_{nk}| < \infty \right\}.$$

Then, $\quad \{fs(G,B)\}^\beta = b_3 \cap b_6 \cap b_7 \cap b_8$.

Proof: This might be acquired in a similar concept as talk about in the proof of Theorem 4 with Lemma 2 in lieu of Lemma 4 (iv). So, we disregard details.

4. SOME MATRIX TRANSFORMATIONS

For briefness, we write

$$a_{nk} = \sum_{j=0}^{n} a_{jk},$$

$$a(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k},$$

$$\Delta a_{nk} = a_{nk} - a_{n,k+1}.$$

Theorem 7: [16] *Let* η *be an* FK*-space,* U *be a triangle,* P *be its inverse and* μ *be optional subset of* w. *Then we have* $A = (a_{nk}) \in (\eta_U : \nu)$ *necessary and sufficient condition*

$$C^{(n)} = \left(C_{mk}^{(n)}\right) \in (\eta, c) \;\; for\ all\ n \in \mathbb{N}, \tag{7}$$

$$C = (c_{nk}) \in (\eta : \nu), \tag{8}$$

where,

$$C_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} a_{nj} p_{jk}, & 0 \le k \le m, \\ 0, & k > m, \end{cases}$$

and

$$c_{nk} = \sum_{j=k}^{\infty} a_{nj} p_{jk}, \; for\ all\ k, m, n \in \mathbb{N}.$$

Lemma 3: *So as to the matrix A appertains to the matrix class from* f *to* f *is necessary and sufficient conditions:*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty,$$

for each fixed $k \in \mathbb{N}$, $f - \lim a_{nk} = \alpha_k \;\; exist,$

$$f - lim \sum_k a_{nk} = \alpha,$$

and uniformly in n

$$\lim_{m\to\infty} \sum_k |\Delta a(n,k,m) - \alpha_k]| = 0,$$

are satisfied.

For an infinite matrix $A = (a_{nk})$, we shall write for briefness that,

$$d_{mk}^n = \tilde{a}_{nk}(m) = \frac{1}{r u_k v_k} a_{nk} + \frac{1}{u_k} \sum_{j=k+1}^{m} \tilde{\nabla}_{jk} a_{nj}, \;\; (k < m), \tag{9}$$

and

$$d_{nk} = \tilde{a}_{nk} = \frac{1}{r u_k v_k} a_{nk} + \frac{1}{u_k} \sum_{j=k+1}^{\infty} \tilde{\nabla}_{jk} a_{nj}, \tag{10}$$

for all $n, k, m \in \mathbb{N}$,

$$\hat{a}_{nk} = u_n \left(\sum_{i=0}^{n-1} (r v_i + s v_{i+1}) a_{ik} + r v_n a_{nk} \right). \tag{11}$$

Theorem 8: *Let us assume that the entries of the infinite matrices given by* $A = (a_{nk})$ *and* $H = (h_{nk})$ *are related by the following relation*

$$h_{nk} = \tilde{a}_{nk} \tag{12}$$

for all k *and* $n \in \mathbb{N}$, μ *is an arbitrary sequence space. Then,* $A \in (f(G,B) : \mu)$ *necessary and sufficient condition for all* $n \in \mathbb{N}$, $\{a_{nk}\}_{k\in\mathbb{N}} \in f(G,B)^\beta$ *and* $H \in (f : \mu)$.

Proof: We assume that μ is a given *sequence space*. Let us assume that (12) yields among the entries of $A = (a_{nk})$ and $H = (h_{nk})$, and consider the fact that the spaces $f(G,B)$ and f are defined to be linearly isomorphic.

We take $A \in (f(G,B) : \mu)$ and any $y = (y_k) \in f$. Thus, $H.R(G,B)$ does exist and $\{a_{nk}\}_{k\in\mathbb{N}} \in \cap_{k=1}^5 b_k$ which yields that $\{h_{nk}\}_{k\in\mathbb{N}} \in l_1$ for each $n \in \mathbb{N}$. Hence, Hy exists and thus for all $n \in \mathbb{N}$

$$\sum_k h_{nk} y_k = \sum_k a_{nk} x_k, \tag{13}$$

we have by (12) that $Hy = Ax$, which leads us to consequence $H \in (f : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in f(G,B)^\beta$ for each $n \in \mathbb{N}$ and $H \in (f : \mu)$ yield, and take any $x = (x_k) \in f(G,B)$. Then, Ax exists. Thus, we acquire from the following equality for each $n \in \mathbb{N}$,

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \left[\sum_{j=0}^{k-1} \frac{1}{u_j} \tilde{\nabla}(k,j) a_{nj} y_j + \frac{1}{r u_k v_k} a_{nk} y_k \right], \tag{14}$$

as $m \to \infty$ that $Ax = Hy$ and this shows that $A \in (f(G,B) : \mu)$.

This completes the proof.

Theorem 9: *$A \in (f(G,B): c)$ necessary and sufficient condition $D^{(n)} = \left(d_{mk}^{(n)}\right) \in (f: c)$ and $D = (d_{nk}) \in (f: c)$.*

Theorem 10: *$A \in (f(G,B): l_\infty)$ necessary and sufficient condition $D^{(n)} = \left(d_{mk}^{(n)}\right) \in (f: c)$ and $D = (d_{nk}) \in (f: l_\infty)$.*

If we change the roles for the spaces $f(G,B)$ and f with μ, we have;

Theorem 11: *Assume that the entries of the infinite matrices $A = (a_{nk})$ and $L = (l_{nk})$ are connected with the relation $l_{nk} = \hat{a}_{nk}$, (11), for all $k, n \in N$ and μ be any given sequence space. Then, $A \in (\mu: f(G,B))$ necessary and sufficient condition $L \in (\mu: f)$.*

Proof: Let $x = (x_k) \in \mu$ and take into account the following equality

$$\{R(G,B)(Ax)\}_n = u_n\left(\sum_{j=0}^{n-1}\left(rv_j + sv_{j+1}\right)(Ax)_j + ru_n v_n (Ax)_n\right)$$

$$= u_n\left(\sum_{j=0}^{n-1}\left(rv_j + sv_{j+1}\right)\sum_j a_{nj}x_j\right) + ru_n v_n \sum_k a_{nk}x_k$$

$$= \sum_k\left(\sum_{j=k}^{n-1} u_k\left(rv_{j-k} + sv_{j-k+1}\right)a_{n,j-k}x_{j-k} + ru_n v_n a_{nk} x_k\right)$$

$$= (Lx)_n,$$

which leads us to consequence that $Ax \in f(G,B)$ necessary and sufficient condition $Lx \in f$.
This step completes the proof.

At this time, we are going to denote the following conditions:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{15}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \quad \text{for each fixed } k \in \mathbb{N}, \tag{16}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha, \tag{17}$$

$$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0, \tag{18}$$

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta(a_{nk})| < \infty, \tag{19}$$

$$\lim_{k\to\infty} a_{nk} = 0, \text{for each fixed } n \in \mathbb{N}, \tag{20}$$

$$\lim_{n\to\infty} \sum_k |\Delta^2 a_{nk}| = \alpha, \tag{21}$$

for each fixed $k \in \mathbb{N}$
$$f - \lim a_{nk} = \alpha_k \text{ exists}, \tag{22}$$
uniformly in n

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| = 0, \tag{23}$$
uniformly in n

$$f - \lim \sum_k a_{nk} = \alpha, \tag{24}$$
uniformly in n

$$\lim_{m\to\infty} \sum_k |\Delta[a(n,k,m) - \alpha_k]| = 0, \tag{25}$$
uniformly in n

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q} \Delta[a(n+i,k) - \alpha_k]\right| = 0, \tag{26}$$

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a(n,k)| < \infty, \tag{27}$$
for each fixed $k \in \mathbb{N}$,
$$f - \lim a(n,k) = \alpha_k \text{exists}, \tag{28}$$
uniformly in n

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q} \Delta^2[a(n+i,k) - \alpha_k]\right| = 0, \tag{29}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a(n,k)| < \infty, \tag{30}$$

$$\sum_n a_{nk} = \alpha_k, \text{ for each fixed } k \in \mathbb{N} \tag{31}$$

$$\sum_n \sum_k a_{nk} = \alpha, \tag{32}$$

$$\lim_{n\to\infty} \sum_k |\Delta a(n,k) - \alpha_k| = 0. \tag{33}$$

Let $A = (a_{nk})$ be an infinite matrix. In that case, the following expressions yield:

Lemma 4*:* *i) $A = (a_{nk}) \in (\ell_\infty: f)$ necessary and sufficient condition (15), (22) and (23) yield. [17]*
ii) $A = (a_{nk}) \in (f: f)$ necessary and sufficient condition (15), (22), (24), and (25) yield. [17]
iii) $A = (a_{nk}) \in (fs: \ell_\infty)$ necessary and sufficient condition (19) and (20) yield.
iv) $A = (a_{nk}) \in (fs: c)$ necessary and sufficient condition (16), (19) and (21) yield. [18]
v) $A = (a_{nk}) \in (c: f)$ necessary and sufficient condition (15), (22) and (24) yield. [19]
vi) $A = (a_{nk}) \in (bs: f)$ necessary and sufficient condition (19), (20), (22) and (26) yield. [20]
vii) $A = (a_{nk}) \in (fs: f)$ necessary and sufficient condition (20), (22), (25) and (26) yield. [21]
viii) $A = (a_{nk}) \in (cs: f)$ necessary and sufficient condition (19) and (22) yield. [22]
ix) $A = (a_{nk}) \in (bs: fs)$ necessary and sufficient condition (20), (26) and (28) yield. [20]
x) $A = (a_{nk}) \in (fs: fs)$ necessary and sufficient condition (26) and (29) yield. [21]
xi) $A = (a_{nk}) \in (cs: fs)$ necessary and sufficient condition (27) and (28) yield. [22]
xii) $A = (a_{nk}) \in (f: cs)$ necessary and sufficient condition (30) and (33) yield. [23]

Corollary 2: *The following statements hold:*
i) $A = (a_{nk}) \in (f(G,B): l_\infty)$ necessary and sufficient condition $\{a_{nk}\}_{k\in\mathbb{N}} \in f(G,B)^\beta$ for all $n \in \mathbb{N}$ and (15) yields with $\tilde{a}_{nk}$ lieu of a_{nk}.
ii) $A = (a_{nk}) \in (f(G,B): c)$ necessary and sufficient condition $\{a_{nk}\}_{k\in\mathbb{N}} \in f(G,B)^\beta$ for all $n \in \mathbb{N}$ and (15), (16), (18) yield with $\tilde{a}_{nk}$ lieu of a_{nk}.
iii) $A = (a_{nk}) \in (f(G,B): bs)$ necessary and sufficient condition $\{a_{nk}\}_{k\in\mathbb{N}} \in f(G,B)^\beta$ for all $n \in \mathbb{N}$ and (30) yields.
iv) $A = (a_{nk}) \in (f(G,B): cs)$ necessary and sufficient condition $\{a_{nk}\}_{k\in\mathbb{N}} \in f(G,B)^\beta$ for all $n \in \mathbb{N}$ and (30), (33) yield with $\tilde{a}_{nk}$ lieu of a_{nk}.

Corollary 3: *The following statements hold:*

i) $A=(a_{nk})\in(l_\infty : f(G,B))$ *necessary and sufficient condition (15), (22) and (23) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

ii) $A=(a_{nk})\in(f : f(G,B))$ *necessary and sufficient condition (15), (22), (24) and (25) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

iii) $A=(a_{nk})\in(c : f(G,B))$ *necessary and sufficient condition (15), (22) and (24) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

Corollary 4: *The following statements hold:*

i) $A=(a_{nk})\in(bs : f(G,B))$ *necessary and sufficient condition (19), (20), (22) and (26) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

ii) $A=(a_{nk})\in(fs : f(G,B))$ *necessary and sufficient condition (20), (22) and (26) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

iii) $A=(a_{nk})\in(cs : f(G,B))$ *necessary and sufficient condition (19), (22) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

Corollary 5: *The following statements hold:*

i) $A=(a_{nk})\in(bs : fs(G,B))$ *necessary and sufficient condition (20), (26) and (28) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

ii) $A=(a_{nk})\in(fs : fs(G,B))$ *necessary and sufficient condition (26) and (29) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

iii) $A=(a_{nk})\in(cs : fs(G,B))$ *necessary and sufficient condition (27) and (28) yield with* $\hat{a}_{nk}$ *lieu of* a_{nk}.

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