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NOTES ON THE GEOMETRY OF COTANGENT BUNDLE AND UNIT COTANGENT SPHERE BUNDLE

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Abstract. Let $(\mathtt{N}, \mathfrak{g})$ be a Riemannian manifold, by using the musical isomorphisms ♪ and ♮ induced by $\mathfrak{g}$, we built a bridge between the geometry of the tangent bundle $\mathtt{TN}$ (resp. the unit tangent sphere bundle $\mathtt{T}_1\mathtt{N}$) equipped with the Sasaki metric $\mathfrak{g}_S$ (resp. the induced Sasaki metric $\bar{\mathfrak{g}}_S$) and that of the cotangent bundle $\mathtt{T}^*\mathtt{N}$ (resp. the unit cotangent sphere bundle $\mathtt{T}^*_1\mathtt{N}$) endowed with the Sasaki metric $\mathfrak{g}_{\widetilde{S}}$ (resp. the induced Sasaki metric $\tilde{\mathfrak{g}}_{\widetilde{S}}$). Moreover, we prove that $\mathtt{T}^*_1\mathtt{N}$ carries a contact metric structure and study some of its properties.

1. Introduction

The geometry of tangent bundles of differentiable manifolds is of particular interest in different areas of mathematics and physics. The research in this domain began in 1958, with a very fundamental paper by Sasaki [16]. He constructed a Riemannian metric $\mathfrak{g}_S$ (called the Sasaki metric) on the tangent bundle $\mathtt{TN}$ of a Riemannian manifold $(\mathtt{N}, \mathfrak{g})$, which depends on the metric $\mathfrak{g}$. Since then, the geometry of $(\mathtt{TN}, \mathfrak{g}_S)$ or the (unit) tangent sphere bundle $\mathtt{T}_1\mathtt{N}$ endowed with the induced Sasaki metric $\bar{\mathfrak{g}}_S$ has acquired extensive literature; see, for instance, [4, 5, 9, 11, 12] and the survey [18].

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On the other hand, the geometry of cotangent bundles of differentiable manifolds developed in parallel with that of tangent bundles, as can be seen in [18]. In classical mechanics, the cotangent bundle can be viewed as the phase space. Akbulut, Özdemir and Salimov [1] defined the Sasaki metric analogue $\mathfrak{g}_{\widetilde{S}}$ on the cotangent bundle $\mathtt{T}^*\mathtt{N}$ of $(\mathtt{N}, \mathfrak{g})$ and studied some of its properties. Afterwards, Salimov and Agca [15] gave some of its curvature properties. In [10], the authors proved that the musical isomorphisms ♪ and ♮ induced by Riemannian metric $\mathfrak{g}$ are isometries between $(\mathtt{TN}, \mathfrak{g}_S)$ and $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}})$.

In this paper, we shed more light on the geometry of cotangent bundle (resp. unit cotangent sphere bundle). Firstly, by using ♪ and ♮ we studied the relationship between the geometry of $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}})$ and that of $(\mathtt{TN}, \mathfrak{g}_S)$ and vice versa, and this improved the results of [15]. Secondly, after we defined the unit cotangent sphere bundle $\mathtt{T}_1^*\mathtt{N}$ of $(\mathtt{N}, \mathfrak{g})$ and endowed it by the induced Sasaki metric $\tilde{\mathfrak{g}}_{\widetilde{S}}$, we showed that ♪ and ♮ induced by Riemannian metric $\mathfrak{g}$ are isometries between $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$ and $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$, which allowed us to deduce the geometric properties of $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ from those of $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$. Finally, like the unit tangent sphere bundle of $(\mathtt{N}, \mathfrak{g})$, we showed that the unit cotangent sphere bundle of $(\mathtt{N}, \mathfrak{g})$ carries a contact metric structure and studied some of its properties.

2. Some Results on the Geometry of Cotangent Bundle

First, we start with a brief review on the geometry of tangent and cotangent bundles following [7, 11, 18]. Let $(\mathtt{N}, \mathfrak{g})$ be an n-dimensional Riemannian manifold, ∇ its Levi-Civita connection and $(\mathtt{TN}, \pi, \mathtt{N})$ (resp. $(\mathtt{T}^*\mathtt{N}, \widetilde{\pi}, \mathtt{N})$) be its tangent bundle (resp. its cotangent bundle). The tangent space $\mathtt{T}_\mathfrak{p}\mathtt{TN}$ (resp. $\mathtt{T}_\mathfrak{q}\mathtt{T}^*\mathtt{N}$) at a point $\mathfrak{p} = (p, \mathtt{v})$ in $\mathtt{TN}$ (resp. at a point $\mathfrak{q} = (p, \mathfrak{v})$ in $\mathtt{T}^*\mathtt{N}$) splits into the direct sum of the vertical subspace $\mathcal{V}_\mathfrak{p} = \ker(d\pi\mid_\mathfrak{p})$ (resp. $\widetilde{\mathcal{V}}_\mathfrak{q} = \ker(d\widetilde{\pi}\mid_\mathfrak{q})$) and the horizontal subspace $\mathcal{H}_\mathfrak{p}$ (resp. $\widetilde{\mathcal{H}}_\mathfrak{q}$), with respect to ∇: $\mathtt{T}_\mathfrak{p}\mathtt{TN} = \mathcal{H}_\mathfrak{p} \oplus \mathcal{V}_\mathfrak{p}$ (resp. $\mathtt{T}_\mathfrak{q}\mathtt{T}^*\mathtt{N} = \widetilde{\mathcal{H}}_\mathfrak{q} \oplus \widetilde{\mathcal{V}}_\mathfrak{q}$).

The Sasaki metric $\mathfrak{g}_S$ on $\mathtt{TN}$ is defined for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ by

$$\mathfrak{g}_S(\Upsilon_1^h, \Upsilon_2^h) = \mathfrak{g}_S(\Upsilon_1^v, \Upsilon_2^v) = \mathfrak{g}(\Upsilon_1, \Upsilon_2) \circ \pi,\ \mathfrak{g}_S(\Upsilon_1^v, \Upsilon_2^h) = 0,$$

where Υ_1^h and Υ_1^v are the horizontal and the vertical lifts of Υ_1 respectively. It is well known from [7] that $(\mathtt{TN}, \mathfrak{g}_S, J)$ is an almost Hermitian manifold, where the structure J is defined by

$$\begin{cases} J(\Upsilon_1^h) = \Upsilon_1^v\ , \\ J(\Upsilon_1^v) = -\Upsilon_1^h\ , \end{cases} \tag{1}$$

for any $\Upsilon_1 \in \Gamma(\mathtt{TN})$. Furthermore, J defines an almost Kählerian structure. It is a Kählerian manifold if and only if $(\mathtt{N}, \mathfrak{g})$ is flat.

On the other hand, the Sasaki metric $\mathfrak{g}_{\widetilde{S}}$ on $\mathtt{T}^*\mathtt{N}$ is defined for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ and any $\omega, \theta \in \Gamma(\mathtt{T}^*\mathtt{N})$ by:

$$\mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \theta^{\widetilde{v}}) = \mathfrak{g}^{-1}(\omega, \theta) \circ \widetilde{\pi},\ \mathfrak{g}_{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}) = \mathfrak{g}(\Upsilon_1, \Upsilon_2) \circ \widetilde{\pi},\ \mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \Upsilon_2^{\widetilde{h}}) = 0,$$

where $\Upsilon_1^{\widetilde{h}}$ and $\omega^{\widetilde{v}}$ are the horizontal lift of Υ_1 and the vertical lift of ω respectively. Here, $\mathfrak{g}^{-1}(\omega, \theta) = \mathfrak{g}(\natural\omega, \natural\theta)$, in which $\natural : \omega \mapsto \natural(\omega)$, such that $\natural(\omega)$ is the vector field on $\mathtt{N}$ defined by $\mathfrak{g}(\natural(\omega), \Upsilon_2) = \omega(\Upsilon_2)$. Note that the musical isomorphisms $\natural$ and $♪ : \Upsilon_1 \mapsto ♪(\Upsilon_1) = \mathfrak{g}(\Upsilon_1, .)$, define a bundle isomorphism between the tangent and the cotangent bundle of $\mathtt{N}$; moreover, $♪$ and $\natural$ are isometries between $(\mathtt{TN}, \mathfrak{g}_S)$ and $(\mathtt{T^*N}, \mathfrak{g}_{\widetilde{S}})$ [10]. Further we have

$$♪_*(\Upsilon_1^v) = (♪\Upsilon_1)^{\widetilde{v}},\ ♪_*(\Upsilon_1^h) = \Upsilon_1^{\widetilde{h}}, \tag{2}$$

and

$$\natural_*(\omega^{\widetilde{v}}) = (\natural\omega)^v,\ \natural_*(\Upsilon_1^{\widetilde{h}}) = \Upsilon_1^h. \tag{3}$$

Taking account of Eq. (2) and Eq. (3) and Lemma 2 in [7], the Lie brackets of vertical and horizontal lifts to $\mathtt{T^*N}$ are given as follows:

Lemma 1.

$$\begin{aligned}
\left[\omega^{\widetilde{v}}, \theta^{\widetilde{v}}\right]_\zeta &= 0,\\
\left[\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{v}}\right]_\zeta &= (\nabla_{\Upsilon_1}\omega)_\zeta^{\widetilde{v}},\\
\left[\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}\right]_\zeta &= [\Upsilon_1, \Upsilon_2]_\zeta^{\widetilde{h}} - (R(\Upsilon_1, \Upsilon_2)\varpi)_\zeta^{\widetilde{v}},
\end{aligned}$$

for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ and any $\omega, \theta \in \Gamma(\mathtt{T^*N})$, where $\zeta \in \mathtt{T^*N}$, ϖ is a 1-form on $\mathtt{N}$ such that $\varpi_{\widetilde{\pi}(\zeta)} = \zeta$ and $R(\Upsilon_1, \Upsilon_2) = [\nabla_{\Upsilon_1}, \nabla_{\Upsilon_2}] - \nabla_{[\Upsilon_1, \Upsilon_2]}$ is the curvature tensor of $\mathtt{N}$.

From Eq. (2) and Eq. (3) and the Levi-Civita connection $\overline{\nabla}$ of $\mathfrak{g}_S$ given by the formulas (8)-(11) in [11], we obtain the following

Lemma 2. *The Levi-Civita connection $\widetilde{\nabla}$ of $\mathfrak{g}_{\widetilde{S}}$ is described completely by*

$$\begin{aligned}
(\widetilde{\nabla}_{\Upsilon_1^{\widetilde{h}}}\Upsilon_2^{\widetilde{h}})_\zeta &= (\nabla_{\Upsilon_1}\Upsilon_2)_\zeta^{\widetilde{h}} - \frac{1}{2}(R(\Upsilon_1, \Upsilon_2)\varpi)_\zeta^{\widetilde{v}},\\
(\widetilde{\nabla}_{\Upsilon_1^{\widetilde{h}}}\theta^{\widetilde{v}})_\zeta &= (\nabla_{\Upsilon_1}\theta)_\zeta^{\widetilde{v}} + \frac{1}{2}(R(\natural(\varpi), \natural(\theta))\Upsilon_1)_\zeta^{\widetilde{h}},\\
(\widetilde{\nabla}_{\omega^{\widetilde{v}}}\Upsilon_2^{\widetilde{h}})_\zeta &= \frac{1}{2}(R(\natural(\varpi), \natural(\omega))\Upsilon_2)_\zeta^{\widetilde{h}},\\
(\widetilde{\nabla}_{\omega^{\widetilde{v}}}\theta^{\widetilde{v}})_\zeta &= 0,
\end{aligned} \tag{4}$$

for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ and any $\omega, \theta \in \Gamma(\mathtt{T^*N})$, where $\zeta \in \mathtt{T^*N}$ and ϖ is a 1-form on $\mathtt{N}$ such that $\varpi_{\widetilde{\pi}(\zeta)} = \zeta$.

Proposition 1. *Let $\widetilde{R}$ be the curvature tensor of $(\mathtt{T^*N}, \mathfrak{g}_{\widetilde{S}})$. Then the following formulae hold*

$$\left\{\widetilde{R}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}})\Upsilon_3^{\widetilde{h}}\right\}_\zeta = \Big\{R(\Upsilon_1, \Upsilon_2)\Upsilon_3 + \frac{1}{4}R(\natural\varpi, R(\Upsilon_3, \Upsilon_2)\natural\varpi)\Upsilon_1$$

$$+\frac{1}{4}R(\natural\varpi, R(\Upsilon_1,\Upsilon_3)\natural\varpi)\Upsilon_2+\frac{1}{2}R(\natural\varpi, R(\Upsilon_1,\Upsilon_2)\natural\varpi)\Upsilon_3\Big\}_\zeta^{\widetilde{h}}$$

$$+\frac{1}{2}\Big\{(\nabla_{\Upsilon_3}R)(\Upsilon_1,\Upsilon_2)\varpi\Big\}_\zeta^{\widetilde{v}},$$

$$\Big\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\Upsilon_2^{\widetilde{h}})\omega^{\widetilde{v}}\Big\}_\zeta=\Big\{R(\Upsilon_1,\Upsilon_2)\omega+\frac{1}{4}R(R(\natural\varpi,\natural\omega)\Upsilon_2,\Upsilon_1)\varpi$$

$$-\frac{1}{4}R(R(\natural\varpi,\natural\omega)\Upsilon_1,\Upsilon_2)\varpi\Big\}_\zeta^{\widetilde{v}}+\frac{1}{2}\Big\{(\nabla_{\Upsilon_1}R)(\natural\varpi,\natural\omega)\Upsilon_2$$

$$-(\nabla_{\Upsilon_2}R)(\natural\varpi,\natural\omega)\Upsilon_1\Big\}_\zeta^{\widetilde{h}},$$

$$\Big\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\omega^{\widetilde{v}})\Upsilon_3^{\widetilde{h}}\Big\}_\zeta=\Big\{\frac{1}{4}R(R(\natural\varpi,\natural\omega)\Upsilon_3,\Upsilon_1)\varpi+\frac{1}{2}R(\Upsilon_1,\Upsilon_3)\omega\Big\}_\zeta^{\widetilde{v}}$$

$$+\frac{1}{2}\Big\{(\nabla_{\Upsilon_1}R)(\natural\varpi,\natural\omega)\Upsilon_3\Big\}_\zeta^{\widetilde{h}},$$

$$\Big\{\widetilde{R}(\Upsilon_1^{\widetilde{h}},\omega^{\widetilde{v}})\theta^{\widetilde{v}}\Big\}_\zeta=-\Big\{\frac{1}{2}R(\natural\omega,\natural\theta)\Upsilon_1+\frac{1}{4}R(\natural\varpi,\natural\omega)R(\natural\varpi,\natural\theta)\Upsilon_1\Big\}_\zeta^{\widetilde{h}},$$

$$\Big\{\widetilde{R}(\omega^{\widetilde{v}},\theta^{\widetilde{v}})\Upsilon_3^{\widetilde{h}}\Big\}_\zeta=\Big\{R(\natural\omega,\natural\theta)\Upsilon_3+\frac{1}{4}R(\natural\varpi,\natural\omega)R(\natural\varpi,\natural\theta)\Upsilon_3$$

$$-\frac{1}{4}R(\natural\varpi,\natural\theta)R(\natural\varpi,\natural\omega)\Upsilon_3\Big\}_\zeta^{\widetilde{h}},$$

$$\Big\{\widetilde{R}(\omega^{\widetilde{v}},\theta^{\widetilde{v}})\mu^{\widetilde{v}}\Big\}_\zeta=0,$$

for any $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathtt{TN})$ *and any* $\omega,\theta,\mu\in\Gamma(\mathtt{T^*N})$, *where* $\zeta\in\mathtt{T^*N}$ *and* ϖ *is a 1-form on* $\mathtt{N}$ *such that* $\varpi_{\widetilde{\pi}(\zeta)}=\zeta$.

Proof. Using Eq. (2) and Eq. (3), we obtain

$$♪_*\left(\Big\{\bar{R}(\natural_*\widetilde{\Upsilon}_1,\natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_3\Big\}_w\right)=\Big\{\widetilde{R}(\widetilde{\Upsilon}1,\widetilde{\Upsilon}_2)\widetilde{\Upsilon}_3\Big\}_\zeta, \tag{5}$$

such that $♪(w)=\zeta$ and for any vector fields $\overline{\Upsilon}_1,\overline{\Upsilon}_2,\overline{\Upsilon}_3$ on $\mathtt{TN}$ and any vector fields $\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2,\widetilde{\Upsilon}_3$ on $\mathtt{T^*N}$. Thus, by Eq. (5) and Theorem 1 in [11], we find the required formulae. □

3. Unit Cotangent Sphere Bundle

The unit tangent sphere bundle $\mathtt{T}_1\mathtt{N}$ of a Riemannian manifold $(\mathtt{N}, \mathfrak{g})$ consists of all unit tangent vectors to $\mathtt{N}$. As a hypersurface of $\mathtt{TN}$ it is given by

$$\mathtt{T}_1\mathtt{N} = \{\mathfrak{p} = (p, \mathtt{v}) \in \mathtt{TN} \,|\, \mathfrak{g}_p(\mathtt{v}, \mathtt{v}) = 1\}.$$

The vector field $N_{\mathfrak{p}} = \mathtt{v}^v$ is a unit normal of $\mathtt{T}_1\mathtt{N}$. In contrast with the horizontal lift of a vector field, the vertical lift is not in general tangent to $\mathtt{T}_1\mathtt{N}$ [3]; for this reason, it was defined the tangential lift of $\Upsilon_1 \in \mathtt{T}_p\mathtt{N}$ to $\mathfrak{p} \in \mathtt{T}_1\mathtt{N}$ as following [3]

$$\Upsilon^t_{1\,\mathfrak{p}} = \Upsilon^v_{1\,\mathfrak{p}} - \mathfrak{g}(\Upsilon_1, \mathtt{v}) N_{\mathfrak{p}} = (\Upsilon_1 - \mathfrak{g}(\Upsilon_1, \mathtt{v})\mathtt{v})^v_{\mathfrak{p}}.$$

Clearly, the tangent space $\mathtt{T}_{\mathfrak{p}}\mathtt{T}_1\mathtt{N}$ is spanned by vectors of the form Υ^h_1 and Υ^t_1, where there is $\Upsilon_1 \in \mathtt{T}_p\mathtt{N}$. To simplify notation, we will use $\overline{\Upsilon}_1$ for $\Upsilon_1 - \mathfrak{g}(\Upsilon_1, \mathtt{v})\mathtt{v}$, then $\Upsilon^t_1 = \overline{\Upsilon}^v_1$. The Riemannian metric $\bar{\mathfrak{g}}_S$ on the hypersurface $\mathtt{T}_1\mathtt{N}$ induced by $\mathfrak{g}_S$ on $\mathtt{TN}$ is uniquely determined by the formulae

$$\begin{aligned} \bar{\mathfrak{g}}_S(\Upsilon^h_1, \Upsilon^h_2) &= \mathfrak{g}_S(\Upsilon^h_1, \Upsilon^h_2), \\ \bar{\mathfrak{g}}_S(\Upsilon^t_1, \Upsilon^h_2) &= 0, \\ \bar{\mathfrak{g}}_S(\Upsilon^t_1, \Upsilon^t_2) &= \mathfrak{g}_S(\Upsilon^v_1, \Upsilon^v_2) - \mathfrak{g}_S(\Upsilon^v_1, N)\mathfrak{g}_S(\Upsilon^v_2, N). \end{aligned}$$

Now by analogy with unit tangent sphere bundle, we define the unit cotangent sphere bundle $\mathtt{T}^*_1\mathtt{N}$, as the set of all unit tangent covectors to $\mathtt{N}$. As a hypersurface of $\mathtt{T}^*\mathtt{N}$ it is defined by

$$\mathtt{T}^*_1\mathtt{N} = \{\mathfrak{q} = (p, \mathfrak{v}) \in \mathtt{T}^*\mathtt{N} \,|\, \mathfrak{g}^{-1}_p(\mathfrak{v}, \mathfrak{v}) = 1\}.$$

The vector field $\widetilde{N}_{\mathfrak{q}} = \mathfrak{v}^{\widetilde{v}}$ is a unit normal of $\mathtt{T}^*_1\mathtt{N}$. The horizontal lift of a vector field is tangent to $\mathtt{T}^*_1\mathtt{N}$, but in general the vertical lift is not tangent. Thus, for $\omega \in \mathtt{T}^*_p\mathtt{N}$ we define the tangential lift of ω to $\mathfrak{q} \in \mathtt{T}^*_1\mathtt{N}$ by

$$\omega^{\widetilde{t}}_{\mathfrak{q}} = \omega^{\widetilde{v}}_{\mathfrak{q}} - \mathfrak{g}^{-1}(\omega, \mathfrak{v})\widetilde{N}_{\mathfrak{q}} = (\omega - \mathfrak{g}^{-1}(\omega, \mathfrak{v})\mathfrak{v})^{\widetilde{v}}_{\mathfrak{q}}.$$

The tangent space $\mathtt{T}_{\mathfrak{q}}\mathtt{T}^*_1\mathtt{N}$ is spanned by vectors of the form Υ^h_1 and $\omega^{\widetilde{t}}$. For the sake of notation clarity, we will use $\overline{\omega}$ as a shorthand for $\omega - \mathfrak{g}^{-1}(\omega, \mathfrak{v})\mathfrak{v}$, then $\omega^{\widetilde{t}} = \overline{\omega}^{\widetilde{v}}$. The Riemannian metric $\tilde{\mathfrak{g}}_{\widetilde{S}}$ on the hypersurface $\mathtt{T}^*_1\mathtt{N}$ induced by $\mathfrak{g}_{\widetilde{S}}$ on $\mathtt{T}^*\mathtt{N}$ is uniquely determined by the formulae

$$\begin{aligned} \tilde{\mathfrak{g}}_{\widetilde{S}}(\Upsilon^{\widetilde{h}}_1, \Upsilon^{\widetilde{h}}_2) &= \mathfrak{g}_{\widetilde{S}}(\Upsilon^{\widetilde{h}}_1, \Upsilon^{\widetilde{h}}_2), \\ \tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \Upsilon^{\widetilde{h}}_2) &= 0, \\ \tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}}) &= \mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \theta^{\widetilde{v}}) - \mathfrak{g}_{\widetilde{S}}(\omega^{\widetilde{v}}, \widetilde{N})\mathfrak{g}_{\widetilde{S}}(\theta^{\widetilde{v}}, \widetilde{N}). \end{aligned} \tag{6}$$

We have the following

Theorem 1. *Let be an n-dimensional Riemannian manifold* $(\mathtt{N}, \mathfrak{g})$ *with Riemannian metric* $\mathfrak{g}$*. Subsequently, the musical isomorphisms generated by the metric* $\mathfrak{g}$ *represent isometric mappings between* $(\mathtt{T}_1\mathtt{N}, \bar{\mathfrak{g}}_S)$ *and* $(\mathtt{T}^*_1\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$*.*

Proof. From Eq. (2) and Eq. (3), we find

$$♪_*(\Upsilon_1^t) = (♪\Upsilon_1)^{\widetilde{t}}, \tag{7}$$

$$\natural_*(\omega^{\widetilde{t}}) = (\natural\omega)^t. \tag{8}$$

Thus

$$\begin{aligned} ♪^*(\tilde{\mathfrak{g}}_{\widetilde{S}})(\Upsilon_1^t, \Upsilon_2^t) &= \tilde{\mathfrak{g}}_{\widetilde{S}}(♪_*\Upsilon_1^t, ♪_*\Upsilon_2^t) \\ &= \tilde{\mathfrak{g}}_{\widetilde{S}}((♪\Upsilon_1)^{\widetilde{t}}, (♪\Upsilon_2)^{\widetilde{t}}) \\ &= \bar{\mathfrak{g}}_S(\Upsilon_1^t, \Upsilon_2^t), \end{aligned} \tag{9}$$

$$♪^*(\tilde{\mathfrak{g}}_{\widetilde{S}})(\Upsilon_1^t, \Upsilon_2^h) = 0 = \bar{\mathfrak{g}}_S(\Upsilon_1^t, \Upsilon_2^h), \tag{10}$$

and

$$\begin{aligned} ♪^*(\tilde{\mathfrak{g}}_{\widetilde{S}})(\Upsilon_1^h, \Upsilon_2^h) &= \tilde{\mathfrak{g}}_{\widetilde{S}}(♪_*\Upsilon_1^h, ♪_*\Upsilon_2^h) \\ &= \tilde{\mathfrak{g}}_{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}) \\ &= \bar{\mathfrak{g}}_S(\Upsilon_1^h, \Upsilon_2^h), \end{aligned} \tag{11}$$

then from Eq. (9)-(11), we find that $♪ : (\mathtt{T_1N}, \bar{\mathfrak{g}}_S) \to (\mathtt{T_1^*N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ is an isometry. In a similar way, we can also prove that $\natural : (\mathtt{T_1^*N}, \tilde{\mathfrak{g}}_{\widetilde{S}}) \to (\mathtt{T_1N}, \bar{\mathfrak{g}}_S)$ is an isometry. $\square$

By virtue of Eq. (2) and Eq. (7) and the formulae (3.2)-(3.3) in [3], the Lie brackets of vector fields on $\mathtt{T_1^*N}$ involving tangential lifts are given as follows:

$$\left[\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{t}}\right]_\nu = (\nabla_{\Upsilon_1}\omega)_\nu^{\widetilde{t}} \ , \tag{12}$$

$$\left[\omega^{\widetilde{t}}, \theta^{\widetilde{t}}\right]_\nu = \mathfrak{g}^{-1}(\omega, \vartheta)\theta_\nu^{\widetilde{t}} - \mathfrak{g}^{-1}(\theta, \vartheta)\omega_\nu^{\widetilde{t}} \ ,$$

for any $\Upsilon_1 \in \Gamma(\mathtt{TN})$ and any $\omega, \theta \in \Gamma(\mathtt{T^*N})$, here $\nu = (x, q) \in \mathtt{T_1^*N}$ and ϑ is a 1-form on $\mathtt{N}$ such that $\vartheta_{\widehat{\pi}(\nu)} = \nu$ where $\widehat{\pi} : \mathtt{T_1^*N} \longrightarrow \mathtt{N}$ is the natural projection. Using Eq. (2), Eq. (3), Eq. (7), Eq. (8) and the Levi-Civita connection $\overline{\nabla}^S$ of $\bar{\mathfrak{g}}_S$ given by Proposition 3.1 in [3], we obtain the following:

Proposition 2. *The Levi-Civita connection* $\widetilde{\nabla}^{\widetilde{S}}$ *of Riemannian metric* $\tilde{\mathfrak{g}}_{\widetilde{S}}$ *is entirely described by*

$$\begin{aligned} (\widetilde{\nabla}^{\widetilde{S}}_{\Upsilon_1^{\tilde{h}}}\Upsilon_2^{\widetilde{h}})_\nu &= (\nabla_{\Upsilon_1}\Upsilon_2)_\nu^{\widetilde{h}} - \frac{1}{2}(R(\Upsilon_1, \Upsilon_2)\vartheta)_\nu^{\widetilde{t}}, \\ (\widetilde{\nabla}^{\widetilde{S}}_{\Upsilon_1^{\tilde{h}}}\theta^{\widetilde{t}})_\nu &= (\nabla_{\Upsilon_1}\theta)_\nu^{\widetilde{t}} + \frac{1}{2}(R(\natural\vartheta, \natural\theta)\Upsilon_1)_\nu^{\widetilde{h}}, \\ (\widetilde{\nabla}^{\widetilde{S}}_{\omega^{\tilde{t}}}\Upsilon_2^{\widetilde{h}})_\nu &= \frac{1}{2}(R(\natural\vartheta, \natural\omega)\Upsilon_2)_\nu^{\widetilde{h}}, \\ (\widetilde{\nabla}^{\widetilde{S}}_{\omega^{\tilde{t}}}\theta^{\widetilde{t}})_\nu &= -\mathfrak{g}^{-1}(\theta, \vartheta)\omega_\nu^{\widetilde{t}}, \end{aligned} \tag{13}$$

for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ *and any* $\omega, \theta \in \Gamma(\mathtt{T^*N})$*, here* $\nu = (p, \mathfrak{v}) \in \mathtt{T_1^*N}$ *and* ϑ *is a 1-form on* $\mathtt{N}$ *such that* $\vartheta_{\widehat{\pi}(\nu)} = \nu$*.*

Proposition 3. *The curvature tensor* $\widetilde{R}^{\widetilde{S}}$ *of* $(\mathtt{T}_1^*\mathtt{N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ *is entirely described by*

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}})\Upsilon_3^{\widetilde{h}}\right\}_\nu = \left\{R(\Upsilon_1, \Upsilon_2)\Upsilon_3 + \frac{1}{4}R(\natural\vartheta, R(\Upsilon_3, \Upsilon_2)\natural\vartheta)\Upsilon_1 + \frac{1}{4}R(\natural\vartheta, R(\Upsilon_1, \Upsilon_3)\natural\vartheta)\Upsilon_2 + \frac{1}{2}R(\natural\vartheta, R(\Upsilon_1, \Upsilon_2)\natural\vartheta)\Upsilon_3\right\}_\nu^{\widetilde{h}} + \frac{1}{2}\left\{(\nabla_{\Upsilon_3}R)(\Upsilon_1, \Upsilon_2)\vartheta\right\}_\nu^{\widetilde{t}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}})\omega^{\widetilde{t}}\right\}_\nu = \left\{R(\Upsilon_1, \Upsilon_2)(\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\vartheta) + \frac{1}{4}R(R(\natural\vartheta, \natural\omega)\Upsilon_2, \Upsilon_1)\vartheta - \frac{1}{4}R(R(\natural\vartheta, \natural\omega)\Upsilon_1, \Upsilon_2)\vartheta\right\}_\nu^{\widetilde{t}} + \frac{1}{2}\left\{(\nabla_{\Upsilon_1}R)(\natural\vartheta, \natural\omega)\Upsilon_2 - (\nabla_{\Upsilon_2}R)(\natural\vartheta, \natural\omega)\Upsilon_1\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{t}})\Upsilon_3^{\widetilde{h}}\right\}_\nu = \left\{\frac{1}{2}R(\Upsilon_1, \Upsilon_3)(\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\vartheta) + \frac{1}{4}R(R(\natural\vartheta, \natural\omega)\Upsilon_3, \Upsilon_1)\vartheta\right\}_\nu^{\widetilde{v}} + \frac{1}{2}\left\{(\nabla_{\Upsilon_1}R)(\natural\vartheta, \natural\omega)\Upsilon_3\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \omega^{\widetilde{t}})\theta^{\widetilde{t}}\right\}_\nu = -\left\{\frac{1}{2}R(\natural\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\natural\vartheta, \natural\theta - \mathfrak{g}^{-1}(\theta, \vartheta)\natural\vartheta)\Upsilon_1 + \frac{1}{4}R(\natural\vartheta, \natural\omega)R(\natural\vartheta, \natural\theta)\Upsilon_1\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}})\Upsilon_3^{\widetilde{h}}\right\}_\nu = \left\{R(\natural\omega - \mathfrak{g}^{-1}(\omega, \vartheta)\natural\vartheta, \natural\theta - \mathfrak{g}^{-1}(\theta, \vartheta)\natural\vartheta)\Upsilon_3 + \frac{1}{4}[R(\natural\vartheta, \natural\omega), R(\natural\vartheta, \natural\theta)]\Upsilon_3\right\}_\nu^{\widetilde{h}},$$

$$\left\{\widetilde{R}^{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}})\mu^{\widetilde{t}}\right\}_\nu = -\tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \mu^{\widetilde{t}})\theta_\nu^{\widetilde{t}} + \tilde{\mathfrak{g}}_{\widetilde{S}}(\mu^{\widetilde{t}}, \theta^{\widetilde{t}})\omega_\nu^{\widetilde{t}},$$

for any $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in \Gamma(\mathtt{TN})$ *and any* $\omega, \theta, \mu \in \Gamma(\mathtt{T}^*\mathtt{N})$*, here* $\nu = (p, \mathfrak{v}) \in \mathtt{T}_1^*\mathtt{N}$ *and* ϑ *is a 1-form on* $\mathtt{N}$ *such that* $\vartheta_{\widehat{\pi}(\nu)} = \nu$.

Proof. Let $\bar{R}^S$ be the curvature tensor of $(\mathtt{T_1N}, \bar{\mathfrak{g}}_S)$. Using Eq. (2) and Eq. (7), we obtain

$$\sharp_*\left(\left\{\bar{R}^S(\natural_*\widetilde{\Upsilon}_1, \natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_3\right\}_v\right) = \left\{\widetilde{R}^{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2)\widetilde{\Upsilon}_3\right\}_\nu, \tag{14}$$

such that $\sharp(v) = \nu$ and for any vector fields $\overline{\Upsilon}_1, \overline{\Upsilon}_2, \overline{\Upsilon}_3$ on $\mathtt{T_1N}$ and any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3$ on $\mathtt{T_1^*N}$. Thus, the required formulae follow from Eq. (14) and Proposition 3.2 in [3]. □

Theorem 2. *The pair* $(\mathtt{T_1^*N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ *is locally symmetric if and only if the base manifold* $(\mathtt{N}, \mathfrak{g})$ *is flat or* $\mathtt{N}$ *is a* 2*-dimensional manifold with a constant curvature* 1*.*

Proof. It's clear that

$$\sharp_*\left[(\overline{\nabla}^S_{\natural_*\widetilde{W}}\bar{R}^S)(\natural_*\widetilde{\Upsilon}_1, \natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_3\right] = (\widetilde{\nabla}^{\widetilde{S}}_{\widetilde{W}}\widetilde{R}^S)(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2)\widetilde{\Upsilon}_3,$$

and

$$\natural_*\left[(\widetilde{\nabla}^{\widetilde{S}}_{\sharp_*\overline{W}}\widetilde{R}^S)(\sharp_*\overline{\Upsilon}_1, \sharp_*\overline{\Upsilon}_2)\sharp_*\overline{\Upsilon}_3\right] = (\overline{\nabla}^S_{\overline{W}}\bar{R}^S)(\overline{\Upsilon}_1, \overline{\Upsilon}_2)\overline{\Upsilon}_3,$$

for any vector fields $\overline{\Upsilon}_1, \overline{\Upsilon}_2, \overline{\Upsilon}_3$ on $\mathtt{T_1N}$ and any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3$ on $\mathtt{T_1^*N}$. Therefore $(\mathtt{T_1^*N}, \tilde{\mathfrak{g}}_{\widetilde{S}})$ is locally symmetric if and only if $(\mathtt{T_1N}, \bar{\mathfrak{g}}_S)$ is locally symmetric, combining this fact with the main result in [2] we deduce the required assertion. □

3.1. **An Almost Contact Structure on $\mathtt{T_1^*N}$.** We first recall some notions on almost contact structure, for more details we refer to [2]. Let $\mathtt{N}^{2n+1}$ be an odd-dimensional smooth manifold, we say that $\mathtt{N}^{2n+1}$ has an almost contact structure if the relations

$$\mu(\varsigma) = 1 \quad \text{and} \quad \mathcal{F}^2\Upsilon_1 = -\Upsilon_1 + \mu(\Upsilon_1)\varsigma$$

hold on $\mathtt{N}^{2n+1}$, where ς is a vector field, μ is a 1-form, and $\mathcal{F}$ is a (1,1)-tensor field on $\mathtt{N}^{2n+1}$.

Then there exists a compatible Riemannian metric $\mathfrak{g}$

$$\mathfrak{g}(\mathcal{F}\Upsilon_1, \mathcal{F}\Upsilon_2) = \mathfrak{g}(\Upsilon_1, \Upsilon_2) - \mu(\Upsilon_1)\mu(\Upsilon_2)$$

for all vector fields Υ_1 and Υ_2 on $\mathtt{N}$. We call $(\mu, \varsigma, \mathcal{F}, \mathfrak{g})$ an almost contact metric manifold, ς being known as its characteristic vector field. For an almost contact metric manifold $\mathtt{N}$, its fundamental 2-form Φ is defined by $\Phi(\Upsilon_1, \Upsilon_2) = \mathfrak{g}(\mathcal{F}\Upsilon_1, \Upsilon_2)$. If

$$\Phi = d\mu,$$

$\mathtt{N}$ is called a contact metric manifold. A contact metric manifold for which ς is a Killing vector field (resp. harmonic vector field) is called a K-contact manifold (resp. H-contact manifold). Recall that a unit vector field Υ_1 on $\mathtt{N}$ is harmonic if and only if $\bar{\Delta}\Upsilon_1$ is parallel to Υ_1, where $\bar{\Delta}\Upsilon_1$ is the rough Laplacian of Υ_1 (see [8]). In [14] Perrone showed that a contact metric manifold is H-contact if and only if the characteristic vector field ς is an eigenvector of the Ricci operator.

A contact metric structure is called Sasakian structure if it is normal. Recall that an almost contact structure $(\mu, \varsigma, \mathcal{F}, \mathfrak{g})$ is said to be normal if

$$N(\Upsilon_1, \Upsilon_2) = [\mathcal{F}, \mathcal{F}](\Upsilon_1, \Upsilon_2) + 2d\mu(\Upsilon_1, \Upsilon_2)\varsigma = 0,$$

for all $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$, here $N(\Upsilon_1, \Upsilon_2)$ is $(1, 2)$-tensor field and $[\mathcal{F}, \mathcal{F}]$ is the Nijenhuis torsion of $\mathcal{F}$,

$$[\mathcal{F}, \mathcal{F}](\Upsilon_1, \Upsilon_2) = \mathcal{F}^2[\Upsilon_1, \Upsilon_2] + [\mathcal{F}\Upsilon_1, \mathcal{F}\Upsilon_2] - \mathcal{F}[\mathcal{F}\Upsilon_1, \Upsilon_2] - \mathcal{F}[\Upsilon_1, \mathcal{F}\Upsilon_2].$$

A powerful characterization for Sasakian manifolds is the following: An almost contact metric manifold $(\mu, \varsigma, \mathcal{F}, \mathfrak{g})$ is Sasakian if and only if

$$(\nabla_{\Upsilon_1}\mathcal{F})\Upsilon_2 = \mathfrak{g}(\Upsilon_1, \Upsilon_2)\varsigma - \mu(\Upsilon_2)\Upsilon_1; \quad \Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN}),$$

where ∇ is the Levi-Civita connection of $(\mathtt{N}, \mathfrak{g})$.

Next, it's well known from [17] that the unit tangent sphere bundle $\mathtt{T_1N}$ has a standard contact metric structure $(\varsigma', \mu', \mathcal{F}', \bar{\mathfrak{g}}'_S) = (2\varsigma, \frac{1}{2}\mu, \mathcal{F}, \frac{1}{4}\bar{\mathfrak{g}}_S)$, where ς, μ and $\mathcal{F}$ are given by

$$\varsigma = -JN = \mathtt{v}^i\left(\frac{\partial}{\partial x^i}\right)^h,$$

$$\mu(\Upsilon_1^t) = 0, \quad \mu(\Upsilon_1^h) = \mathfrak{g}(\Upsilon_2, \mathtt{v}), \tag{15}$$

$$\mathcal{F}(\Upsilon_1^t) = -\Upsilon_1^h + \mathfrak{g}(\Upsilon_1, \mathtt{v})\varsigma, \quad \mathcal{F}(\Upsilon_1^h) = \Upsilon_1^t, \tag{16}$$

here $(p, \mathtt{v}) \in \mathtt{TN}$ and $\Upsilon_1 \in \Gamma(\mathtt{TN})$. Note that ς is the geodesic flow.

3.1.1. *An almost Kählerian structure on* $\mathtt{T^*N}$. Let $(\mathtt{N}, \mathfrak{g})$ be a Riemannian manifold of dimension n and $(\mathtt{T^*N}, \mathfrak{g}_{\widetilde{S}})$ its cotangent bundle endowed with the Sasaki metric. On $\mathtt{T^*N}$ we define the structure $\widetilde{J}$ by

$$\begin{cases} \widetilde{J}(\Upsilon_1^{\widetilde{h}}) = (\flat\Upsilon_1)^{\widetilde{v}}, \\ \widetilde{J}(\omega^{\widetilde{v}}) = -(\sharp\omega)^{\widetilde{h}}, \end{cases} \tag{17}$$

for any $\Upsilon_1 \in \Gamma(\mathtt{TN})$ and $\omega \in \Gamma(\mathtt{T^*N})$. It is clear that $(\mathtt{T^*N}, \widetilde{J})$ is an almost complex manifold. Moreover, since

$$\mathfrak{g}_{\widetilde{S}}(\widetilde{J}(\Upsilon_1^h), \omega^v) = \omega(\Upsilon_1) = -\mathfrak{g}_{\widetilde{S}}(\Upsilon_1^h, \widetilde{J}(\omega^v)),$$

$$\mathfrak{g}_{\widetilde{S}}(\widetilde{J}(\omega^v), \theta^v) = 0 = -\mathfrak{g}_{\widetilde{S}}(\omega^v, \widetilde{J}(\theta^v)),$$

and

$$\mathfrak{g}_{\widetilde{S}}(\widetilde{J}(\Upsilon_1^h), \Upsilon_2^h) = 0 = -\mathfrak{g}_{\widetilde{S}}(\Upsilon_1^h, \widetilde{J}(\Upsilon_2^h)),$$

for any $\Upsilon_1, \Upsilon_2 \in \Gamma(\mathtt{TN})$ and any covectors ω and θ on $\mathtt{N}$, then $(\mathtt{T^*N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ is an almost hermitian manifold. Furthermore, the 2-form $\Omega_{\widetilde{S}}$ defined by:

$$\Omega_{\widetilde{S}} = \mathfrak{g}_{\widetilde{S}}(\widetilde{J}\cdot, \cdot)$$

is closed. In fact, we know

$$d\Omega_{\widetilde{S}} = 0 \Leftrightarrow \mathfrak{g}_{\widetilde{S}}((\widetilde{\nabla}_{\widetilde{\Upsilon}_1}\widetilde{J})\widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3) + \mathfrak{g}_{\widetilde{S}}((\widetilde{\nabla}_{\widetilde{\Upsilon}_2}\widetilde{J})\widetilde{\Upsilon}_3, \widetilde{\Upsilon}_1) + \mathfrak{g}_{\widetilde{S}}((\widetilde{\nabla}_{\widetilde{\Upsilon}_3}\widetilde{J})\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2) = 0,$$

for any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2, \widetilde{\Upsilon}_3$ on $\mathtt{T}^*\mathtt{N}$. Using the algebraic Bianchi identity, Eq. (4) and Eq. (17), we get $d\Omega_{\widetilde{S}} = 0$. Hence, we may state the following:

Theorem 3. *Let be an n-dimensional Riemannian manifold $(\mathtt{N}, \mathfrak{g})$ with Riemannian metric $\mathfrak{g}$ be. Then $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ is an almost Kählerian manifold.*

Theorem 4. *Let be an n-dimensional Riemannian manifold $(\mathtt{N}, \mathfrak{g})$ with Riemannian metric $\mathfrak{g}$ be. The musical isomorphisms $\flat$ and $\natural$ are holomorphic maps between $(\mathtt{TN}, \mathfrak{g}_S, J_S)$ and $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$. Moreover, $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ is a Kählerian manifold if and only if $(\mathtt{N}, \mathfrak{g})$ is flat.*

Proof. From Eq. (1), Eq. (2), Eq. (3) and Eq. (17) we obtain $\flat_* J = \widetilde{J}\flat_*$ and $\natural_*\widetilde{J} = J\natural_*$, it follows that $\flat$ and $\natural$ are holomorphic maps. Thus, by a direct computations we get

$$\flat_*(\overline{\nabla}_{\natural_*\widetilde{\Upsilon}_1} J)\natural_*\widetilde{\Upsilon}_2 = (\widetilde{\nabla}_{\widetilde{\Upsilon}_1}\widetilde{J})\widetilde{\Upsilon}_2,$$

and

$$\natural_*(\widetilde{\nabla}_{\flat_*\overline{\Upsilon}_1}\widetilde{J})\flat_*\overline{\Upsilon}_2 = (\overline{\nabla}_{\overline{\Upsilon}_1} J)\overline{\Upsilon}_2,$$

for any vector fields $\overline{\Upsilon}_1, \overline{\Upsilon}_2$ on $\mathtt{TN}$ and any vector fields $\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2$ on $\mathtt{T}^*\mathtt{N}$, then $(\mathtt{T}^*\mathtt{N}, \mathfrak{g}_{\widetilde{S}}, \widetilde{J})$ is a Kählerian manifold if and only if $(\mathtt{TN}, \mathfrak{g}_S, J)$ is, or equivalently $(\mathtt{N}, \mathfrak{g})$ is flat. □

3.1.2. *An almost contact structure on* $\mathtt{T}^*_1\mathtt{N}$. With the help of the almost complex structure $\widetilde{J}$, we can define a unit vector field $\widetilde{\varsigma}$, a 1-form $\widetilde{\mu}$ and a $(1,1)$-tensor field $\widetilde{\mathcal{F}}$ on $\mathtt{T}^*\mathtt{N}$, as given below:

$$\widetilde{\varsigma} = -\widetilde{J}\widetilde{N}, \quad \widetilde{\mathcal{F}} = \widetilde{J} - \widetilde{\mu} \otimes \widetilde{N}.$$

Explicitly $\widetilde{\varsigma}$, $\widetilde{\mu}$ and $\widetilde{\mathcal{F}}$ are given by

$$\widetilde{\varsigma} = \mathfrak{v}^i (\frac{\partial}{\partial x^i})^{\widetilde{h}}, \tag{18}$$

$$\widetilde{\mu}(\omega^{\widetilde{t}}) = 0, \quad \widetilde{\mu}(\Upsilon_1^{\widetilde{h}}) = \mathfrak{g}^{-1}(\flat\Upsilon_1, \mathfrak{v}), \tag{19}$$

$$\widetilde{\mathcal{F}}(\omega^{\widetilde{t}}) = -(\natural\omega)^{\widetilde{h}} + \mathfrak{g}^{-1}(\omega, \mathfrak{v})\widetilde{\varsigma}, \quad \widetilde{\mathcal{F}}(\Upsilon_1^{\widetilde{h}}) = (\flat\Upsilon_1)^{\widetilde{t}}. \tag{20}$$

Note that $\widetilde{\varsigma}$ is the cogeodesic flow.

Proposition 4. *$(\mathtt{T}^*_1\mathtt{N}, \widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \widetilde{\mathfrak{g}}'_{\widetilde{S}})$ is an almost contact metric manifold, where we have $(\widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \widetilde{\mathfrak{g}}'_{\widetilde{S}}) = (2\widetilde{\varsigma}, \frac{1}{2}\widetilde{\mu}, \widetilde{\mathcal{F}}, \frac{1}{4}\widetilde{\mathfrak{g}}_{\widetilde{S}})$.*

Proof. By definition, we shall show that $(\widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \widetilde{\mathfrak{g}}'_{\widetilde{S}})$ satisfies

$$\widetilde{\mu}'(\widetilde{\varsigma}') = 1, \quad \widetilde{\mathcal{F}}'^2 = -I + \widetilde{\mu}' \otimes \widetilde{\varsigma}' \quad \text{and} \quad \widetilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\mathcal{F}}'\widetilde{\Upsilon}_1, \widetilde{\mathcal{F}}'\widetilde{\Upsilon}_2) = \widetilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2) - \widetilde{\mu}'(\widetilde{\Upsilon}_1)\widetilde{\mu}'(\widetilde{\Upsilon}_2)$$

for all vector fields $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_2$ on $\mathtt{T}^*_1\mathtt{N}$. From Eq. (18)-(20), we yield

$$\widetilde{\mu}'(\widetilde{\varsigma}') = 1, \quad \widetilde{\mathcal{F}}'(\widetilde{\varsigma}') = 0,$$

$$\begin{aligned}\widetilde{\mathcal{F}}'^2(\omega^{\widetilde{t}}) &= -\widetilde{\mathcal{F}}'((\natural\omega)^{\widetilde{h}}) \\ &= -\omega^{\widetilde{t}} \end{aligned} \tag{21}$$

and

$$\begin{aligned}\widetilde{\mathcal{F}}'^2(\Upsilon_1^{\widetilde{h}}) &= \widetilde{\mathcal{F}}'((\sharp\Upsilon_1)^{\widetilde{t}}) \\ &= -\Upsilon_1^{\widetilde{h}} + \mathfrak{g}^{-1}(\sharp\Upsilon_1, \mathfrak{v})\widetilde{\varsigma} \\ &= -\Upsilon_1^{\widetilde{h}} + \widetilde{\mu}'(\Upsilon_1^{\widetilde{h}})\widetilde{\varsigma}'. \end{aligned} \tag{22}$$

By Eq. (21) and Eq. (22), we see that $\widetilde{\phi}'^2 = -I + \widetilde{\mu}' \otimes \widetilde{\varsigma}'$. By virtue of Eq. (3) and Eq. (20), it follows that

$$\begin{aligned}\tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\phi}'(\omega^{\widetilde{t}}), \widetilde{\phi}'(\theta^{\widetilde{t}})) &= \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}}(\widetilde{\phi}'(\omega^{\widetilde{t}}), \widetilde{\phi}'(\theta^{\widetilde{t}})) \\ &= \frac{1}{4}(\mathfrak{g}^{-1}(\omega,\theta) - \mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(\theta,\mathfrak{v})) \\ &= \frac{1}{4}(\tilde{\mathfrak{g}}_{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}}) - \widetilde{\mu}(\omega^{\widetilde{t}})\widetilde{\mu}(\theta^{\widetilde{t}})) \\ &= \tilde{\mathfrak{g}}'_{\widetilde{S}}(\omega^{\widetilde{t}}, \theta^{\widetilde{t}}) - \widetilde{\mu}'(\omega^{\widetilde{t}})\widetilde{\mu}'(\theta^{\widetilde{t}}), \end{aligned} \tag{23}$$

and

$$\begin{aligned}\tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\phi}'(\Upsilon_1^{\widetilde{h}}), \widetilde{\phi}'(\Upsilon_2^{\widetilde{h}})) &= \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}}(\widetilde{\phi}'(\Upsilon_1^{\widetilde{h}}), \widetilde{\phi}'(\Upsilon_2^{\widetilde{h}})) \\ &= \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}}((\sharp\Upsilon_1)^{\widetilde{t}}, (\sharp\Upsilon_2)^{\widetilde{t}}) \\ &= \frac{1}{4}(\mathfrak{g}^{-1}(\sharp\Upsilon_1, \sharp\Upsilon_2) - \mathfrak{g}^{-1}(\sharp\Upsilon_1,\mathfrak{v})\mathfrak{g}^{-1}(\sharp\Upsilon_2,\mathfrak{v})) \\ &= \tilde{\mathfrak{g}}'_{\widetilde{S}}(\Upsilon_1^{\widetilde{h}}, \Upsilon_2^{\widetilde{h}}) - \widetilde{\mu}'(\Upsilon_1^{\widetilde{h}})\widetilde{\mu}'(\Upsilon_2^{\widetilde{h}}). \end{aligned} \tag{24}$$

From Eq. (23) and Eq. (24), we see that

$$\tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\mathcal{F}}'(\widetilde{\Upsilon}_1), \widetilde{\mathcal{F}}'(\widetilde{\Upsilon}_2)) = \tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2) - \widetilde{\mu}'(\widetilde{\Upsilon}_1)\widetilde{\mu}'(\widetilde{\Upsilon}_2),$$

for all vector fields $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_2$ on $\mathtt{T}_1^*\mathtt{N}$. Therefore $(\mathtt{T}_1^*\mathtt{N}, \widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \tilde{\mathfrak{g}}'_{\widetilde{S}})$ is an almost contact metric manifold. □

Proposition 5. $(\mathtt{T}_1^*\mathtt{N}, \widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \tilde{\mathfrak{g}}'_{\widetilde{S}})$ *is a contact metric manifold, where we have* $(\widetilde{\varsigma}', \widetilde{\mu}', \widetilde{\mathcal{F}}', \tilde{\mathfrak{g}}'_{\widetilde{S}}) = (2\widetilde{\varsigma}, \frac{1}{2}\widetilde{\mu}, \widetilde{\mathcal{F}}, \frac{1}{4}\tilde{\mathfrak{g}}_{\widetilde{S}})$.

Proof. By using Eq. (20), we yield

$$\begin{aligned}\tilde{\mathfrak{g}}'_{\widetilde{S}}(\omega^{\widetilde{t}}, \widetilde{\mathcal{F}}'(\Upsilon_1^{\widetilde{h}})) &= \tilde{\mathfrak{g}}'_{\widetilde{S}}(\omega^{\widetilde{t}}, (\sharp\Upsilon_1)^{\widetilde{t}}) \\ &= \frac{1}{4}\{\mathfrak{g}^{-1}(\omega, \sharp\Upsilon_1) - \mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(\sharp\Upsilon_1,\mathfrak{v})\}. \end{aligned}$$

On the other side, from the definition of the vertical lift to $\mathtt{T^*N}$ we get

$$\omega^{\widetilde{v}}(\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}))=\mathfrak{g}^{-1}(♪\Upsilon_1,\omega),$$

and

$$q^{\widetilde{v}}(\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}))=\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}).$$

Thus, we obtain

$$\omega^{\widetilde{t}}(\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}))=\mathfrak{g}^{-1}(\omega,♪\Upsilon_1)-\mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v}), \tag{25}$$

it follows from Eq. (12), Eq. (19) and Eq. (25) that

$$\begin{aligned} d\widetilde{\mu}'(\omega^{\widetilde{t}},\Upsilon_1^{\widetilde{h}}) &= \frac{1}{2}\{\omega^{\widetilde{t}}\widetilde{\mu}'(\Upsilon_1^{\widetilde{h}})-\Upsilon_1^{\widetilde{h}}\widetilde{\mu}'(\omega^{\widetilde{t}})-\widetilde{\mu}'([\omega^{\widetilde{t}},\Upsilon_1^{\widetilde{h}}])\} \\ &= \frac{1}{4}\{\omega^{\widetilde{t}}\widetilde{\mu}(\Upsilon_1^{\widetilde{h}})\} \\ &= \frac{1}{4}\{\mathfrak{g}^{-1}(\omega,♪\Upsilon_1)-\mathfrak{g}^{-1}(\omega,\mathfrak{v})\mathfrak{g}^{-1}(♪\Upsilon_1,\mathfrak{v})\}. \end{aligned}$$

Then we get the contact metric structure $(\widetilde{\varsigma}',\widetilde{\mu}',\widetilde{\phi}',\tilde{\mathfrak{g}}'_{\widetilde{S}})$ on $\mathtt{T_1^*N}$. □

Theorem 5. *The contact metric structure on* $\mathtt{T_1^*N}$ *is* K*-contact if and only if the contact metric structure on* $\mathtt{T_1N}$ *is.*

Proof. As ♪ and ♮ are isometries between $(\mathtt{T_1N},\bar{\mathfrak{g}}_S)$ and $(\mathtt{T_1^*N},\tilde{\mathfrak{g}}_{\widetilde{S}})$, we deduce that

$$♪^*(L_{\widetilde{\varsigma}'}\tilde{\mathfrak{g}}_{\widetilde{S}})(\overline{\Upsilon}_1,\overline{\Upsilon}_2)=(L_{♮_*\widetilde{\varsigma}'}♪^*\tilde{\mathfrak{g}}_{\widetilde{S}})(\overline{\Upsilon}_1,\overline{\Upsilon}_2)=(L_{\varsigma'}\bar{\mathfrak{g}}_S)(\overline{\Upsilon}_1,\overline{\Upsilon}_2), \tag{26}$$

and

$$♮^*(L_{\varsigma'}\bar{\mathfrak{g}}_S)(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2)==(L_{♪_*\varsigma'}♮^*\bar{\mathfrak{g}}_S)(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2)=(L_{\widetilde{\varsigma}'}\tilde{\mathfrak{g}}_{\widetilde{S}})(\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2), \tag{27}$$

for any vector fields $\overline{\Upsilon}_1,\overline{\Upsilon}_2$ on $\mathtt{T_1N}$ and any vector fields $\widetilde{\Upsilon}_1,\widetilde{\Upsilon}_2$ on $\mathtt{T_1^*N}$. Then, from Eq. (26) and Eq. (27) we get our assertion. □

Theorem 6. *The contact metric structure on* $\mathtt{T_1^*N}$ *is Sasakian if and only if the contact metric structure on* $\mathtt{T_1N}$ *is.*

Proof. Let $(\varsigma',\mu',\mathcal{F}',\bar{\mathfrak{g}}'_S)$ (resp. $(\widetilde{\varsigma}',\widetilde{\mu}',\widetilde{\mathcal{F}}',\tilde{\mathfrak{g}}'_{\widetilde{S}})$ be the standard contact metric structure of $\mathtt{T_1N}$ (resp. $\mathtt{T_1^*N}$). From Eq. (2), Eq. (3), Eq. (7), Eq. (8), Eq. (15), Eq. (16), Eq. (19) and Eq. (20) we have

$$\begin{aligned} ♪_*\mathcal{F}' &= \widetilde{\mathcal{F}}'♪_*, \\ ♮_*\widetilde{\mathcal{F}}' &= \mathcal{F}'♮_*, \\ ♪^*\widetilde{\mu}' &= \mu', \\ ♮^*\mu' &= \widetilde{\mu}'. \end{aligned}$$

Hence ♪ is $(\mathcal{F}',\widetilde{\mathcal{F}}')$-holomorphic map and ♮ is $(\widetilde{\mathcal{F}}',\mathcal{F}')$-holomorphic map. Therefore, we get

$$♪_*((\overline{\nabla}^S_{\overline{\Upsilon}_1}\mathcal{F}')\overline{\Upsilon}_2)=(\widetilde{\nabla}^{\widetilde{S}}_{♪_*\overline{\Upsilon}_1}\widetilde{\mathcal{F}}')♪_*\overline{\Upsilon}_2,$$

and

$$\natural_*(\widetilde{\nabla}^{\widetilde{S}}_{\widetilde{\Upsilon}_1}\widetilde{\mathcal{F}}')\widetilde{\Upsilon}_2 = (\overline{\nabla}^{S}_{\natural_*\widetilde{\Upsilon}_1}\mathcal{F}')\natural_*\widetilde{\Upsilon}_2.$$

Thus, we obtain

$$\begin{aligned}\sharp_*((\overline{\nabla}^{S}_{\natural_*\widetilde{\Upsilon}_1}\mathcal{F}')\natural_*\widetilde{\Upsilon}_2 - \bar{\mathfrak{g}}'_S(\natural_*\widetilde{\Upsilon}_1, \natural_*\widetilde{\Upsilon}_2)\varsigma' + \mu'(\natural_*\widetilde{\Upsilon}_2)\natural_*\widetilde{\Upsilon}_1) = (&\widetilde{\nabla}^{\widetilde{S}}_{\widetilde{\Upsilon}_1}\widetilde{\mathcal{F}}')\widetilde{\Upsilon}_2\\ &- \tilde{\mathfrak{g}}'_{\widetilde{S}}(\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_2)\widetilde{\varsigma}'\\ &+ \widetilde{\mu}'(\widetilde{\Upsilon}_2)\widetilde{\Upsilon}_1,\end{aligned}$$

and

$$\begin{aligned}\natural_*((\widetilde{\nabla}^{\widetilde{S}}_{\sharp_*\overline{\Upsilon}_1}\widetilde{\mathcal{F}}')\sharp_*\overline{\Upsilon}_2 - \tilde{\mathfrak{g}}'_{\widetilde{S}}(\sharp_*\overline{\Upsilon}_1, \sharp_*\overline{\Upsilon}_2)\varsigma' + \widetilde{\mu}'(\sharp_*\overline{\Upsilon}_2)\sharp_*\overline{\Upsilon}_1) = (&\overline{\nabla}^{S}_{\overline{\Upsilon}_1}\mathcal{F}')\overline{\Upsilon}_2\\ &- \bar{\mathfrak{g}}'_S(\overline{\Upsilon}_1, \overline{\Upsilon}_2)\varsigma'\\ &+ \mu'(\overline{\Upsilon}_2)\overline{\Upsilon}_1,\end{aligned}$$

then, the contact metric structure on $\mathtt{T}_1^*\mathtt{N}$ is Sasakian if and only if the contact metric structure on $\mathtt{T}_1\mathtt{N}$ is. □

Theorem 7. *The contact metric structure present on $\mathtt{T}_1^*\mathtt{N}$ is categorized as K-contact if and only if the Riemannian manifold $(\mathtt{N}, \mathfrak{g})$ possesses a constant curvature of 1. In such instances, the structure established on $\mathtt{T}_1^*\mathtt{N}$ is denoted as Sasakian.*

Proof. Combining Theorems 5 and 6 with Theorem 8 in [17], we get our assertion. □

Finally, recall that a Riemannian manifold $(\mathtt{N}, \mathfrak{g})$ of dimension n is said to be 2-stein if there exist two functions α_1, $\alpha_2 : \mathtt{N} \longrightarrow \mathbb{R}$ such that for every $p \in \mathtt{N}$ and every vector Υ_1 tangent to $\mathtt{N}$ at p we have

$$\mathrm{Tr}(R_{\Upsilon_1}) = \alpha_1(p)\,\|\Upsilon_1\|^2, \qquad \mathrm{Tr}(R^2_{\Upsilon_1}) = \alpha_2(p)\,\|\Upsilon_1\|^4,$$

where R_{Υ_1} is the Jacobi operator [6].

Theorem 8. *The contact metric structure on $\mathtt{T}_1^*\mathtt{N}$ is H-contact if and only if $(\mathtt{N}, \mathfrak{g})$ is 2-stein.*

Proof. It is obvious that the Ricci operators $\widetilde{Q}(\widetilde{\varsigma})$ on $\mathtt{T}_1\mathtt{N}$ and $\overline{Q}(\varsigma)$ on $\mathtt{T}_1^*\mathtt{N}$ are related by:

$$\sharp_*\overline{Q}(\varsigma) = \widetilde{Q}(\widetilde{\varsigma}),$$

and

$$\natural_*\widetilde{Q}(\widetilde{\varsigma}) = \overline{Q}(\varsigma).$$

Thus, it follows from the main Theorem in [13] that the contact metric structure on $\mathtt{T}_1^*\mathtt{N}$ is H-contact if and only if $(\mathtt{N}, \mathfrak{g})$ is 2-stein. □

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