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Uncorrelation Sets for Jointly Normal Random Variables

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ABSTRACT

Let ξ_1 *and* ξ_2 *be random variables with finite moments of all orders. The set* $U:=\{(j,l):\ \mathbf{E}(\xi_1{}^j\xi^l{}_2) = \mathbf{E}(\xi_1{}^j)\mathbf{E}(\xi_2{}^l)\}$ *is called the uncorrelation set of* ξ_1 *and* ξ_2. *In this paper we describe possible uncorrelation sets of jointly normal random variables.*

Key Words: *Random variables, normal distribution, uncorrelatedness*

1. INTRODUCTION

The concept of independence is a fundamental one in Probability Theory and Mathematical Statistics. Generalizations of independence from various points of view have been studied by a great number of authors. The mostly known generalizations are uncorrelatedness of two random variables and convolutional independence.

Uncorrelatedness of random variables ξ_1 and ξ_2 is defined by the condition

$$\mathbf{E}(\xi_1\xi_2) = \mathbf{E}(\xi_1)\mathbf{E}(\xi_2),$$

provided that mathematical expectations of the random variables ξ_1 and ξ_2 exist.

Random variables ξ_1 and ξ_2 with distribution functions F_{ξ_1} and F_{ξ_2} are said to be convolutionally independent, if the distribution function of their sum satisfies the following condition

$$F_{\xi_1+\xi_2}(x) = F_{\xi_1} * F_{\xi_2}(x) = \int_{-\infty}^{\infty} F_{\xi_1}(x-s)dF_{\xi_2}(s), \quad x \in \mathbf{R}.$$

It is commonly known (cf [5] v.II, Ch.II.4, p.51) that convolutional independence is a less restrictive condition than independence but a more restrictive one than uncorrelatedness. Convolutional independence and its relation to independence were studied by G. Dall'Aglio ([2], [3]).

Comparison of different generalizations of independence leads naturally to the idea of constructing measures of independence. As examples of some widely used measures we should mention Spearman's ρ, Kendall's τ, Kolmogorov's K measures (cf. e.g. [10]). (They are also known as measures of dependence).

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In [7] new measures of independence were introduced. In distinction to the measures mentioned above these new ones are based on properties of moments of random variables. Other developments on measures of independence and their connection with moments of random variables can be found in e.g. [1] and [4].

1. STATEMENT OF RESULTS

To formulate our results we need the following definition.

Definition. Let ξ_1 and ξ_2 be random variables with finite moments of all orders. We say that $U \subseteq \boldsymbol{N}^2$ is an *uncorrelation set* of ξ_1 and ξ_2 if

$$\mathbf{E}(\xi_1{}^j \xi^l{}_2) = \mathbf{E}(\xi_1{}^j)\mathbf{E}(\xi_2{}^l) \quad \text{for } (j,l) \in U,$$

and

$$\mathbf{E}(\xi_1{}^j \xi^l{}_2) \neq \mathbf{E}(\xi_1{}^j)\mathbf{E}(\xi_2{}^l) \quad \text{for } (j,l) \setminus U.$$

An uncorrelation set shows which powers of random variables are uncorrelated. Obviously, if ξ_1 and ξ_2 are independent, then their uncorrelation set $U = \boldsymbol{N}^2$. We may think that uncorrelation sets provide a measure of independence for random variables in the following sense: the wider an uncorrelation set is, the more independent random variables are.

The following general theorem concerning uncorrelation sets was proved in [7].

Theorem 1. for any subset $U \subseteq \boldsymbol{N}^2$ there exist random variables ξ_1 and ξ_2 such that U is their uncorrelation set.

This means that uncorrelatedness of any set of powers of random variables does not imply uncorrelatedness of any other powers. In particular, we may take any $(k,m) \in \boldsymbol{N}^2$ and set $U = \boldsymbol{N}^2 \setminus (k,m)$. There exist random variables ξ_1 and ξ_2 such that

$$\mathbf{E}(\xi_1{}^j \xi^l{}_2) = \mathbf{E}(\xi_1{}^j)\mathbf{E}(\xi_2{}^l) \qquad (j,l) \neq (k,m).$$

The statement of Theorem 1 does not remain true if we prescribe distributions of random variables. For example, admissible uncorrelation sets for random variables with uniform distributions on [0,1] were studied in [6].

In this paper we present a complete description of uncorrelation sets for jointly normal random variables with zero mean.

Let ξ_1 and ξ_2 be *jointly normal* random variables, that is random variables with joint probability density

$$\rho(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-\eta_1)^2}{\sigma_1^2} - 2r\frac{(x-\eta_1)(y-\eta_2)}{\sigma_1\sigma_2} + \frac{(y-\eta_2)^2}{\sigma_2^2}\right]\right\}$$

If $\eta_1 = \eta_2 = 0$, then ξ_1 and ξ_2 are said to be *jointly normal with zero mean.* The coefficient r is a correlation coefficient of ξ_1 and ξ_2. It is commonly known that if $r = 0$, that is the random variables ξ_1 and ξ_2 are uncorrelated, then they are independent. In terms of uncorrelation sets this may be formulated as follows:

$$(1,1) \in U \Rightarrow U = \mathbf{N}^2,$$

where U denotes an uncorrelation set of ξ_1 and ξ_2. Therefore, an uncorrelation set of jointly normal random variables cannot be arbitrary and we face the problem to describe possible uncorrelation sets for jointly normal random variables.

Our results are based on the usage of the following fact (cf. [8] , [9]).

Price's Theorem. Let ξ_1 and ξ_2 be jointly normal random variables with a joint probability density $\rho(x, y)$ and a covariance μ. Suppose that *g(x,y)* is a function satisfying

$$\lim_{(x,y)\to\infty} g(x, y)\rho(x, y) = 0.$$

Consider the mathematical expectation

$$\mathbf{E}[g(\xi_1\xi_2)] =: \mathrm{I}(\mu).$$

If *g(x,y)* is smooth enough, then

$$I^{(n)}(\mu) = \mathbf{E}\left[\frac{\partial^{2n} g(\xi_1\xi_2)}{\partial\xi_1^n \partial\xi_2^n}\right].$$

Using Price's theorem we conclude that the mathematical expectation $\mathbf{E}(\xi_1{}^j \xi^l{}_2)$ of jointly normal random variables is a polynomial in μ which we denote by $I_{(j,l)}(\mu)$. We can readily see that the following conditions are equivalent

$$(j,l) \in U \Leftrightarrow I_{(j,l)}(\mu) = I_{(j,l)}(0),$$

since $\mu = 0$ implies independence of ξ_1 and ξ_2. Investigation of the behaviour of the polynomials $I_{(j,l)}(\mu)$. for different values of *(j,l)* allows us to get the following assertion.

Theorem 2. Let ξ_1 and ξ_2 be jointly normal random variables with zero mean and U denote their uncorrelation set. If $(j,l) \in U$ and $j \equiv l \pmod 2$, then ξ_1 and ξ_2 are independent, and thus $U = \boldsymbol{N}^2$.

Remark. If the numbers *j* and *l* are both *odd*, the statement is true for any jointly normal random variables, not necessarily with zero mean.

On the other hand, the following statement holds.

Theorem 3. Let ξ_1 and ξ_2 be jointly normal random variables with zero mean and U denote their uncorrelation set. If j T l (mod 2), then $(j,l) \in U$.

Summarising the last two theorems, we get a complete description of admissible uncorrelation sets for jointly normal random variables with zero mean. Namely, there are exactly two possible uncorrelation sets: $\boldsymbol{N}^2$ (in this case random variables are independent); and the set $\{(j,l)$: j T l (mod 2) $\}$.

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Birleşik Normal Rasgele Değişkenler için Korelasyonsuzluk Kümeleri

ÖZET

ξ_1 ve ξ_2, tüm momentleri sonlu olan rasgele değişkenler olsunlar. $U:=\{(j,l)$: $\mathbf{E}(\xi_1{}^j\xi^l{}_2) = \mathbf{E}(\xi_1{}^j)\mathbf{E}(\xi_2{}^l)\}$kümesine ξ_1 ve ξ_2'nin *korelasyonsuzluk kümesi* denir. Bu makalede birleşik normal rasgele değişkenlerin mümkün korelasyonsuzluk kümelerinin tarifi verilir.

Anahtar Kelimeler: *Rasgele değişkenler, normal dağılımı, korelasyonsuzluk*