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# Unit Power Lindley Distribution: Properties and Estimation

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**Highlights**

- This paper covers the definition of the unit power Lindley distribution.
- Properties of unit power Lindley distribution are derived.
- Different parameter estimation methods are used to estimate the unknown parameters.

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**Abstract**

This paper introduces the unit power Lindley distribution and presents its fundamental statistical properties, such as density and cumulative distribution functions, hazard rate functions, and, their characteristics, moments, and related measures. The parameters of this newly proposed distribution are estimated by using six different methods: maximum likelihood, least squares, weighted least squares, Cramér von Mises, Anderson Darling, and right-tail Anderson Darling. The performances of the considered estimation methods are compared through an extensive Monte Carlo simulation study. Additionally, two real datasets are modeled to demonstrate that the unit power Lindley distribution provides a significantly better fit than compared to commonly used unit distributions.



## 1. INTRODUCTION

Modeling bounded events, such as percentages, proportions, or rates, is essential in numerous practical scenarios. In such cases, the bounded statistical distributions, which lie on the interval (0,1), play a critical role. For instance, see [1-7]. Among these distributions, the Beta distribution has been widely utilized for modeling data within bounded intervals, see [8]. In recent years, the Topp-Leone [9] and Kumaraswamy (KM) [10] distributions have gained popularity for this purpose; see [11, 12]. Additionally, unit distributions, derived through transformations of well-known distributions, offer an alternative for modeling bounded data sets. For instance, unit Gamma [13], unit Weibull (UW) [14], unit Birnbaum-Saunders [15], unit Lindley (UL) [8], unit inverse Gaussian [16], new unit Lindley [17], unit Burr-XII [18], unit Chen [19], unit modified Burr-III [20], and unit log-log [21] distributions have been proposed by various authors. These studies delve into the statistical properties of the distributions, including characteristics, and moments, reliability measures, among others. Furthermore, several estimation methods have been employed to estimate the parameters of these models.

Lindley distribution is a very flexible distribution in reliability analysis which has so many extensions in the literature (see some examples [22-25]). In this study, we introduce the unit power Lindley (UPL) distribution derived from the transformation of the power Lindley (PL) distribution. The PL distribution, initially proposed by [26] as an extension of the Lindley distribution, see [27], has gained attention for its flexibility and computational simplicity. It has found applications in various fields including wind energy [28], reliability analysis [29], and quality control [30], among others. For further insights, refer to [31-33]. The PL distribution is advocated as a more flexible alternative to the Lindley distribution. Consequently,

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proposing a distribution with even more adaptable properties within the (0,1) range is necessary, thus serving as a potential alternative for the UL distribution.

The primary objective of this paper is to introduce a novel distribution, the UPL distribution, defined within the unit interval. To achieve this, we analyze various aspects of the UPL distribution, including the shapes of its density and cumulative distribution functions (cdf), the hazard rate function (hrf), the quantile function (qf), moments, related measures, and their corresponding 3D plots. Furthermore, the model parameters of the UPL distribution are estimated using several methods: maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér von Mises (CM), Anderson Darling (AD), and right tail Anderson Darling (RAD). The performances of these estimation techniques are evaluated through Monte Carlo simulation studies. Additionally, we compare the modeling performance of the UPL distribution with various distributions, namely KM, Beta, UL, and UW, which are bounded within the interval (0,1) and commonly used in unit modeling, by considering real data applications.

The remainder of the paper is structured as follows: Section 2 provides the definitions and properties of the UPL distribution. The estimation methods employed in this study are outlined in section 3. In section 4, we conduct an extensive Monte Carlo simulation study to compare the performances of the estimation methods. Additionally, real datasets are analyzed for illustrative purposes. Finally, concluding remarks and final comments are presented in section 5.

## 2. UPL DISTRIBUTION: DEFINITIONS AND PROPERTIES

The Lindley distribution was first defined by [34]. It has the following probability density function (pdf):

$$f(t;\beta) = \frac{\beta^2}{\beta+1}(1+t)e^{-\beta t}, t > 0 \text{ and } \beta > 0.$$

The Lindley distribution is extremely significant in reliability studies. Furthermore, [35] examined the distributional properties of the Lindley distribution and provided practical applications. However, the Lindley distribution may not be appropriate in some cases. To improve the flexibility of this distribution, [26] considered PL distribution by utilizing a power transformation as $Y = T^{\frac{1}{\alpha}}$. Consequently, the resulting distribution is known as the power Lindley (PL) distribution, and the pdf of $Y$ is

$$f_{PL}(y;\alpha,\beta) = \frac{\alpha\beta^2}{1+\beta}(1+y^\alpha)y^{\alpha-1}e^{-\beta y^\alpha}, y > 0, \alpha > 0 \; and \; \beta > 0.$$

Additionally, the cdf of $Y$ is

$$F_{PL}(y;\alpha,\beta) = 1 - \left(1 + \frac{\beta y^\alpha}{1+\beta}\right)e^{-\beta y^\alpha}.$$

Moreover, in this study, to derive a new distribution defined on the unit interval, we apply the $X = Y/(1+Y)$ transformation, resulting in the following pdf called UPL distribution:

$$f_{UPL}(x;\alpha,\beta) = \frac{\alpha\beta^2}{1+\beta}\left(1+\left(\frac{x}{1-x}\right)^\alpha\right)\left(\frac{x}{1-x}\right)^{\alpha-1} e^{-\beta\left(\frac{x}{1-x}\right)^\alpha}\left(\frac{1}{(1-x)^2}\right), \quad (1)$$

where $x \in (0,1)$, $\alpha > 0$, and $\beta > 0$.

**Definition 2.1.** If the random variable $X$ follows a UPL distribution with the parameters $\alpha$ and $\beta$ (denoted as $X \sim UPL(\alpha,\beta)$), its pdf is defined by the form provided in (1).

Additionally, the cdf of the UPL distribution is obtained as follows by integrating the pdf given in (1):

$$F_{UPL}(x;\alpha,\beta) = 1 - e^{-\beta\left(\frac{x}{1-x}\right)^{\alpha}}\left[1 + \left(\frac{1}{\beta+1}\left(\frac{x}{1-x}\right)^{\alpha}\right)\right].$$

The pdf and cdf plots of the UPL distribution for several parameter values of $\alpha$ and $\beta$ are demonstrated in Figure 1 (a) and (b), respectively. It is obvious from Figure 1 (a) that the pdf of the UPL distribution shows a single mode and typically positive skewness. As a result, the UPL distribution offers increased flexibility in modeling various datasets defined on the unit interval. It can be further seen from Figure 1 (b) that the cdf of UPL distribution is increasing and continuous.

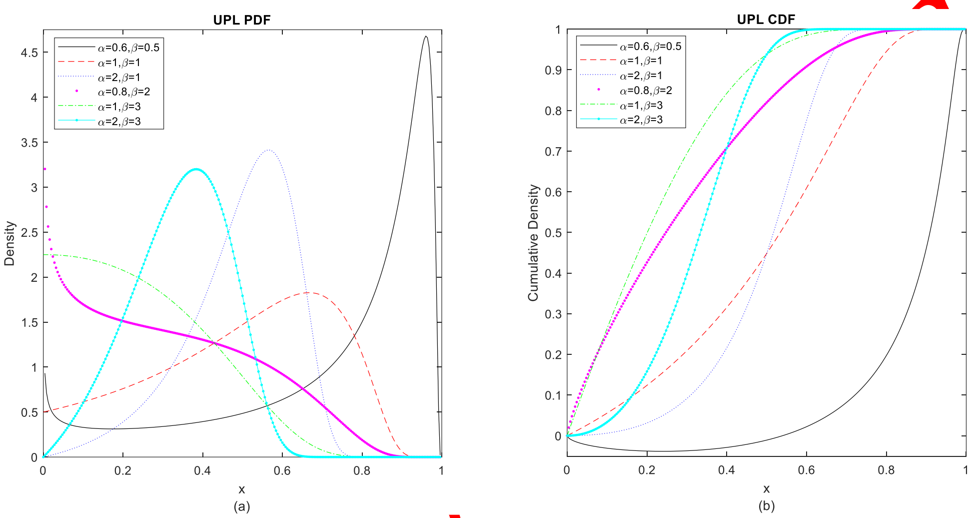



***Figure 1.*** *(a)The pdf plots of UPL distribution for different values of $\alpha$ and $\beta$,*

*(b) The cdf plots of UPL distribution for different values of $\alpha$ and $\beta$*



### 2.1. Hazard Rate Function

**Theorem 2.1.** The hrf of UPL distribution is

$$h_{UPL}(x;\alpha,\beta) = \frac{\alpha\beta^2 x^{\alpha-1}((1-x)^{\alpha} + x^{\alpha})}{(\beta+1)(1-x)^{2\alpha+1} + x^{\alpha}(1-x)^{\alpha+1}}. \quad (2)$$

**Proof.** The survival function of UPL distribution can be derived as follows:

$$S_{UPL}(x;\alpha,\beta) = 1 - F(x) = e^{-\beta\left(\frac{x}{1-x}\right)^{\alpha}}\left[1 + \left(\frac{1}{\beta+1}\left(\frac{x}{1-x}\right)^{\alpha}\right)\right]. \quad (3)$$

Then, the hrf can be easily derived using the formula $h(x) = \frac{f(x)}{S(x)}$, where $f(x)$ is the pdf of the UPL distribution given in (1) and $S(x)$ is given in (3).

Furthermore, the condition $\frac{\partial}{\partial x}h(x) > 0$, holds for all $\beta > 0$ and $a > 1$ indicating that the hrf is consistently increasing concerning $x$. Figure 2 illustrates the varying behaviour of the hrf across different parameter values. It can also be observed that the shape of the hrf resembles J-shaped.

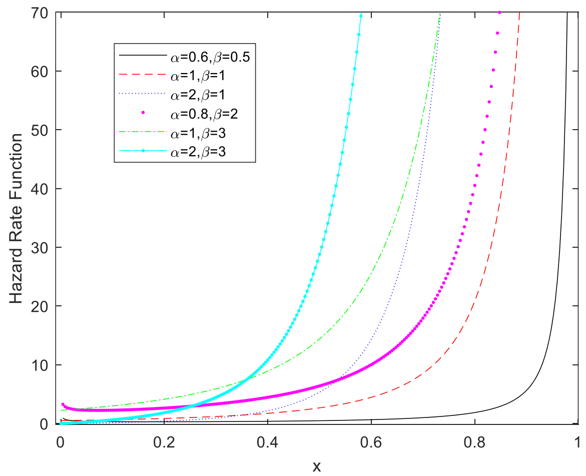



***Figure 2.*** *The hrf plots of UPL distribution for different values of α and β*

### 2.2. Quantile Function

**Theorem 2.2.** The qf of the UPL distribution for any $\alpha, \beta > 0$ is

$$Q_{UPL}(u;\alpha,\beta) = F_{UPL}^{-1}(u;\alpha,\beta) = \frac{\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-u)\right)\right]^{\frac{1}{\alpha}}}{1+\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-u)\right)\right]^{\frac{1}{\alpha}}}, \quad 0<u<1, \tag{4}$$

where $W_{-1}$ indicates the negative branch of the Lambert $W$ function (one can see [36] for details).

**Proof.** [26] derived the qf of the PL distribution as follows:

$$F_{PL}^{-1}(u;\alpha,\beta) = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-u)\right)\right]^{\frac{1}{\alpha}}. \tag{5}$$

Using the transformation $X = Y/(1+Y)$, the qf of the UPL distribution can be easily done by the formula $Q_{UPL}(u;\alpha,\beta) = F_{UPL}^{-1}(u;\alpha,\beta) = \frac{F_{PL}^{-1}(u;\alpha,\beta)}{1+F_{PL}^{-1}(u;\alpha,\beta)}$.

### 2.3. Moments and Related Measures

The $k$-th moment of the UPL distribution is expressed as

$$\mu_k' = E(X^k) = \int_0^1 x^k\left(\frac{\alpha\beta^2}{1+\beta}\left(1+\left(\frac{x}{1-x}\right)^{\alpha}\right)\left(\frac{x}{1-x}\right)^{\alpha-1} e^{-\beta\left(\frac{x}{1-x}\right)^{\alpha}}\left(\frac{1}{(1-x)^2}\right)\right)dx.$$

However, explicit expressions for the $k$-th moments of the UPL distribution are not available. The moments of UPL distribution can be computed thanks to numerical methods. By employing the numerical integration, we obtain the results for mean or expected value ($\mu_1'$), variance ($\sigma^2$), skewness ($\gamma_1$) and kurtosis ($\gamma_2$) for different values of $\alpha$ and $\beta$ as represented in Table 1. Here,

$$\mu_1' = E(X), \quad \sigma^2 = Var(X), \quad \gamma_1 = \frac{E(X-\mu_1')^3}{\sigma^3} \text{ and } \gamma_2 = \frac{E(X-\mu_1')^4}{\sigma^4}.$$

***Table 1.*** *Descriptive moments of UPL distribution for different values of $\alpha$ and $\beta$*

| | $\mu_1'$ | $\sigma^2$ | $\gamma_1$ | $\gamma_2$ |
|---|---|---|---|---|
| $\alpha = 0.6, \beta = 0.5$ | 0.7262 | 0.0717 | -1.2457 | 2.7020 |
| $\alpha = 1, \beta = 1$ | 0.5000 | 0.0482 | -0.4318 | 1.3651 |
| $\alpha = 2, \beta = 1$ | 0.4947 | 0.0171 | -0.7773 | 2.0124 |
| $\alpha = 0.8, \beta = 2$ | 0.3127 | 0.0464 | 0.3655 | 0.6968 |
| $\alpha = 1, \beta = 3$ | 0.2500 | 0.0272 | 0.4716 | 0.8638 |
| $\alpha = 2, \beta = 3$ | 0.3438 | 0.0142 | -0.2909 | 1.5132 |

Additionally, Figures 3 and 4 depict the 3D plots for the mean, variance, skewness, and kurtosis values of UPL distribution corresponding to various values of $\alpha$ and $\beta$. The findings from Table 1, Figure 3, and Figure 4 likely present the skewness and kurtosis values for different sets of parameters $\alpha$ and $\beta$, indicating how the shape of the UPL distribution changes with these parameters. Remarkably, the distribution tends to be platykurtic, as the computed kurtosis values are less than 3.

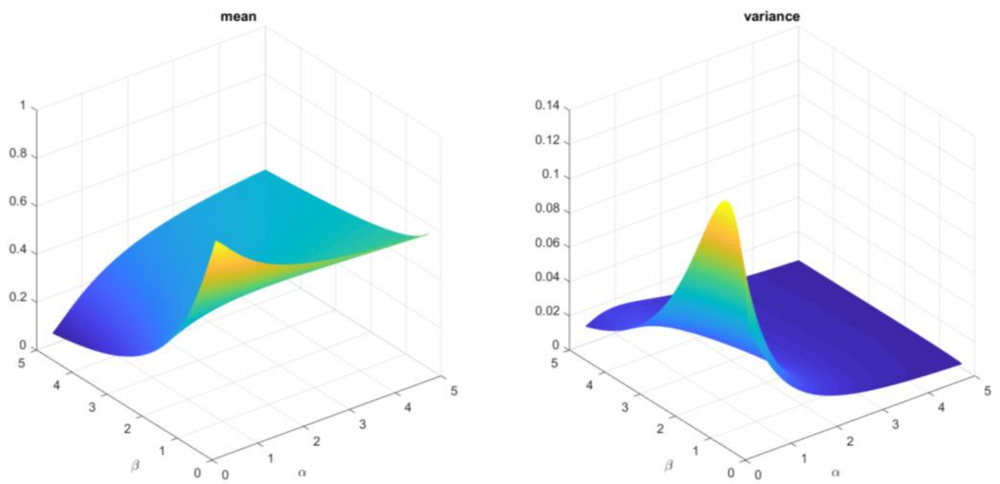



***Figure 3.*** *The 3D plots for the mean and variance of UPL distribution for different values of $\alpha$ and $\beta$*

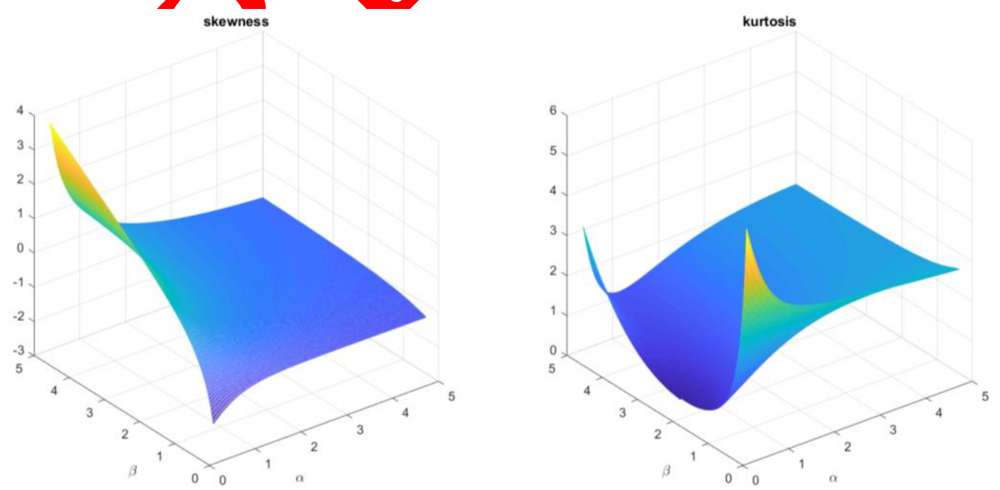



***Figure 4.*** *The 3D plots for the skewness and kurtosis of UPL distribution for different values of $\alpha$ and $\beta$*

## 3. PARAMETER ESTIMATION

In this section, the estimation methods used in this study are given which are the ML, LS, WLS, CM, AD, and RAD, respectively.

### 3.1. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size $n$ from $UPL(\alpha, \beta)$. the ML estimators of $\alpha$ and $\beta$ are obtained by following the likelihood function:

$$L(x;\alpha,\beta) = \prod_{i=1}^{n} \frac{\alpha\beta^2}{\beta+1}\left(1+\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right)\left(\frac{x_i}{1-x_i}\right)^{\alpha-1} e^{-\beta\left(\frac{x_i}{1-x_i}\right)^{\alpha}} \frac{1}{(1-x_i)^2}.$$

After taking the logarithm of this function, the log-likelihood function is derived as follows:

$$\begin{aligned}\ell(\alpha,\beta) = {} & n\log(\alpha) + 2n\log(\beta) - n\log(1+\beta) - \beta\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha} \\ & + \sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{1-x_i}\right)^{\alpha}\right) \\ & - \sum_{i=1}^{n}\log(1-x_i)^2 + (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{1-x_i}\right).\end{aligned}$$

The ML estimators of the corresponding parameters are obtained from the solutions of the likelihood equations:

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ell(\alpha,\beta) = {} & \frac{n}{\alpha} - \beta\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha}\log\left(\frac{x_i}{1-x_i}\right) \\ & + \sum_{i=1}^{n}\frac{\left(\frac{x_i}{1-x_i}\right)^{\alpha}\log\left(\frac{x_i}{1-x_i}\right)}{1+\left(\frac{x_i}{1-x_i}\right)^{\alpha}} + \sum_{i=1}^{n}\log\left(\frac{x_i}{1-x_i}\right) = 0,\end{aligned} \quad (5)$$

$$\frac{\partial}{\partial\beta}\ell(\alpha,\beta) = \frac{2n}{\beta} - \frac{n}{1+\beta} - \sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha} = 0. \quad (6)$$

Obviously, the ML estimator of $\beta$, say $\hat{\beta}$, is

$$\hat{\beta}(\alpha) = \frac{-\left(\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha} - n\right) + \sqrt{\left(\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha} - n\right)^2 + 8n\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha}}}{2\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha}}. \quad (7)$$

However, the ML estimator of $\alpha$, say $\hat{\alpha}$, is obtained from the solution of the following nonlinear equation

$$\frac{n}{\alpha} - \beta\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\alpha}\log\left(\frac{x_i}{1-x_i}\right) + \sum_{i=1}^{n}\frac{\left(\frac{x_i}{1-x_i}\right)^{\alpha}\log\left(\frac{x_i}{1-x_i}\right)}{1+\left(\frac{x_i}{1-x_i}\right)^{\alpha}} + \sum_{i=1}^{n}\log\left(\frac{x_i}{1-x_i}\right) = 0.$$

By using numerical methods, firstly we obtain $\hat{\alpha}$, then by incorporating it in (7), we derive $\hat{\beta}$.

Under regularity conditions, as sample size $n \to \infty$, the asymptotic distribution of $\boldsymbol{\theta} = (\alpha, \beta)$ goes to normal distribution

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{D} N(0, I^{-1}(\boldsymbol{\theta})),$$

where $I$ is the Fisher information matrix. Each element of this matrix is defined as $-E\left(\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right)$. In our case, since the calculation of the expectations is complicated, we use the observed Fisher information matrix, given as follows:

$$I(\boldsymbol{\theta}) = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix},$$

where

$$I_{11} = \frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \beta \sum_{i=1}^{n} \left(\frac{x_i}{1-x_i}\right)^{\alpha} \log\left(\frac{x_i}{1-x_i}\right)^2$$
$$+ \sum_{i=1}^{n} \frac{\log\left(\frac{x_i}{1-x_i}\right)^2 \left(\frac{x_i}{1-x_i}\right)^{\alpha}}{1+\left(\frac{x_i}{1-x_i}\right)^{\alpha}} - \sum_{i=1}^{n} \frac{\log\left(\frac{x_i}{1-x_i}\right)^2 \left(\frac{x_i}{1-x_i}\right)^{2\alpha}}{\left(\left(\frac{x_i}{1-x_i}\right)^{\alpha}+1\right)^2},$$
$$I_{22} = \frac{\partial^2 \ell(\alpha,\beta)}{\partial \beta^2} = -\frac{2n}{\beta^2} + \frac{n}{(1+\beta)^2},$$
$$I_{12} = \frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \left(\frac{x_i}{1-x_i}\right)^{\alpha} \log\left(\frac{x_i}{1-x_i}\right).$$

Thus, the asymptotic $100(1-\gamma)\%$ confidence interval for $\alpha$ and $\beta$ are

$$\hat{\alpha} \pm z_{\gamma/2}\mathrm{se}(\hat{\alpha}), \text{ and } \hat{\beta} \pm z_{\gamma/2}\mathrm{se}(\hat{\beta}),$$

where $z_{\gamma/2}$ is the $\gamma/2$th quantile of the standard normal distribution and $\mathrm{se}(\hat{\alpha})$ and $\mathrm{se}(\hat{\beta})$ are the standard error of $\hat{\alpha}$ and $\hat{\beta}$, respectively. It should be stated that the standard errors are derived from the square root of the diagonal of the inverse observed Fisher information matrix.

### 3.2. Least Squares Estimation

Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be ordered observations of a random sample of size $n$ from $UPL(\alpha, \beta)$. In order to derive the LS estimators of $\alpha$ and $\beta$, we use the method introduced by [37]:

$$S(\alpha,\beta) = \sum_{i=1}^{n} \left(F(x_{(i)}) - \frac{i}{n+1}\right)^2. \tag{8}$$

This method aims to minimize (8) which indicates the differences between expected and observed ordered cdf of $UPL(\alpha, \beta)$. Here, $\frac{i}{n+1}$, $(i = 1, .., n)$ are the expected values of $F(x_{(i)})$. We incorporate the cdf of $UPL(\alpha, \beta)$ into (8) and take derivatives with respect to $\alpha$ and $\beta$, obtain the following equations:

$$\frac{\partial S(\alpha,\beta)}{\partial \alpha} = \sum_{i=1}^{n} \left(1 - e^{-\beta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}} \left[1 + \left(\frac{1}{\beta+1}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\right)\right] - \frac{i}{n+1}\right) \delta_1(x_{(i)}, \alpha, \beta) = 0, \tag{9}$$

$$\frac{\partial S(\alpha,\beta)}{\partial \beta}=\sum_{i=1}^{n}\left(1-e^{-\beta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left[1+\left(\frac{1}{\beta+1}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\right)\right]-\frac{i}{n+1}\right)\delta_2\left(x_{(i)},\alpha,\beta\right)=0, \tag{10}$$

where

$$\delta_1(x,\alpha,\beta)=\left(\frac{x}{1-x}\right)^{\alpha} log\left(\frac{x}{1-x}\right)e^{-\beta\left(\frac{x}{1-x}\right)^{\alpha}}\left(\left(\beta+\frac{\beta}{\beta+1}\left(\frac{x}{1-x}\right)^{\alpha}\right)-\frac{1}{\beta+1}\right), \tag{11}$$

$$\delta_2(x,\alpha,\beta)=\frac{\partial F(x)}{\partial \beta}=e^{-\beta\left(\frac{x}{1-x}\right)^{\alpha}}\left(\frac{x}{1-x}\right)^{\alpha}\left(1+\frac{\beta}{\beta+1}\left(\frac{x}{1-x}\right)^{\alpha}+\frac{1}{(\beta+1)^2}\right). \tag{12}$$

Obviously, (9)-(10) cannot be solved explicitly. Therefore, we resort to iterative methods to obtain LS estimates of $\alpha$ and $\beta$.

### 3.3. Weighted Least Squares Estimation

The WLS estimators of $\alpha$ and $\beta$ are derived by minimizing the following equation with respect to the corresponding parameters:

$$W(\alpha,\beta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F\left(x_{(i)}\right)-\frac{i}{n+1}\right)^2. \tag{13}$$

We incorporate the cdf of $UPL(\alpha,\beta)$ into (13) and take derivatives concerning $\alpha$ and $\beta$, obtain the following equations:

$$\sum_{i=1}^{n}\frac{1}{i(n-i+1)}\left(1-e^{-\beta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left[1+\left(\frac{1}{\beta+1}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\right)\right]-\frac{i}{n+1}\right)\delta_1\left(x_{(i)},\alpha,\beta\right)=0, \tag{14}$$

$$\sum_{i=1}^{n}\frac{1}{i(n-i+1)}\left(1-e^{-\beta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left[1+\left(\frac{1}{\beta+1}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\right)\right]-\frac{i}{n+1}\right)\delta_2\left(x_{(i)},\alpha,\beta\right)=0. \tag{15}$$

Here, $\delta_1\left(x_{(i)},\alpha,\beta\right)$ and $\delta_2\left(x_{(i)},\alpha,\beta\right)$ are similar as in (11) and (12), respectively. By solving (14)-(15) numerically, we obtain the WLS estimates of $\alpha$ and $\beta$.

### 3.4. Cramér-von Mises Estimation

In this subsection and the following subsections, we use minimum distance methods proposed by [38, 39]. These methods work based on minimizing the goodness of fit statistics, see [40].

The CM estimators of $\alpha$ and $\beta$ are derived by minimizing the following equation with respect to the corresponding parameters:

$$C(\alpha,\beta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F\left(x_{(i)}\right)-\frac{2i-1}{2n}\right)^2. \tag{16}$$

We incorporate the cdf of $UPL(\alpha,\beta)$ into (16) and take derivatives according to $\alpha$ and $\beta$, obtain the following equations:

$$\sum_{i=1}^{n}\left(1-e^{-\beta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left[1+\left(\frac{1}{\beta+1}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\right)\right]-\frac{2i-1}{2n}\right)\delta_1\left(x_{(i)},\alpha,\beta\right)=0, \quad (17)$$

$$\sum_{i=1}^{n}\left(1-e^{-\beta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}}\left[1+\left(\frac{1}{\beta+1}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^{\alpha}\right)\right]-\frac{2i-1}{2n}\right)\delta_2\left(x_{(i)},\alpha,\beta\right)=0. \quad (18)$$

Here, $\delta_1\left(x_{(i)},\alpha,\beta\right)$ and $\delta_2\left(x_{(i)},\alpha,\beta\right)$ are similar as in (11) and (12), respectively. By solving (17)-(18) numerically, we obtain the CV estimates of $\alpha$ and $\beta$.

### 3.5. Anderson Darling Estimation

The AD estimators of $\alpha$ and $\beta$ are derived by minimizing the following equation concerning the corresponding parameters:

$$A(\alpha,\beta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[F\left(x_{(i)}\right)\left(1-F\left(x_{(i^*)}\right)\right)\right], \quad (19)$$

where $i^*=n-i+1$. We incorporate the cdf of $UPL(\alpha,\beta)$ into (19) and take derivatives with respect to $\alpha$ and $\beta$, obtain the following equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\delta_1\left(x_{(i)},\alpha,\beta\right)}{F\left(x_{(i)},\alpha,\beta\right)}-\frac{\delta_1\left(x_{(i^*)},\alpha,\beta\right)}{1-F\left(x_{(i^*)},\alpha,\beta\right)}\right]=0, \quad (20)$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\delta_2\left(x_{(i)},\alpha,\beta\right)}{F\left(x_{(i)},\alpha,\beta\right)}-\frac{\delta_2\left(x_{(i^*)},\alpha,\beta\right)}{1-F\left(x_{(i^*)},\alpha,\beta\right)}\right]=0. \quad (21)$$

Here, $\delta_1\left(x_{(i)},\alpha,\beta\right)$ and $\delta_2\left(x_{(i)},\alpha,\beta\right)$ are similar as in (11) and (12), respectively. By solving (17)-(18) numerically, we obtain the AD estimates of $\alpha$ and $\beta$.

### 3.6. Right-Tail Anderson Darling Estimation

The RAD estimators of $\alpha$ and $\beta$ are derived by minimizing the following equation with respect to the corresponding parameters:

$$R(\alpha,\beta)=\frac{n}{2}-2\sum_{i=1}^{n}\left(F\left(x_{(i)}\right)\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1-F\left(x_{(i^*)}\right)\right), \quad (22)$$

where $i^*=n-i+1$. We incorporate the cdf of $UPL(\alpha,\beta)$ into (22) and take derivatives according to $\alpha$ and $\beta$, then obtain the following equations:

$$-2\sum_{i=1}^{n}\frac{\delta_1\left(x_{(i)},\alpha,\beta\right)}{F\left(x_{(i)},\alpha,\beta\right)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_1\left(x_{(i^*)},\alpha,\beta\right)}{1-F\left(x_{(i^*)},\alpha,\beta\right)}=0, \quad (23)$$

$$-2\sum_{i=1}^{n}\frac{\delta_2\left(x_{(i)},\alpha,\beta\right)}{F\left(x_{(i)},\alpha,\beta\right)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_2\left(x_{(i^*)},\alpha,\beta\right)}{1-F\left(x_{(i^*)},\alpha,\beta\right)}=0. \quad (24)$$

Here, $\delta_1\left(x_{(i)},\alpha,\beta\right)$ and $\delta_2\left(x_{(i)},\alpha,\beta\right)$ are similar as in (11) and (12), respectively. By solving (23)-(24) numerically, we find the AD estimates of $\alpha$ and $\beta$.

## 4. NUMERICAL STUDIES

This section covers an extensive Monte Carlo simulation study which is for evaluating the performance of the numerous estimation methods, namely ML, LS, WLS, CV, AD, and RAD, and real data analysis to demonstrate the implementation of the UPL distribution. Additionally, we provide $100(1-\gamma)$ confidence intervals (CI) by using average width (AW) and coverage probabilities (CP) for the ML estimators. The computational details for the simulation study, particularly the procedure for generating random numbers from the UPL distribution with parameters α and β, are outlined below:

***Computational details:***

(i) Random number generation procedure for the UPL distribution with parameters $\alpha$ and $\beta$:

- Generate $U_i \sim Uniform(0,1)$ $i=1,2,\ldots,n$.
- Compute $X_i=\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(-\frac{\beta+1}{e^{\beta+1}}(1-U_i)\right)\right]^{1/\alpha}$, $i=1,2,\ldots,n$. Then, $\frac{X_i}{X_i+1}$ yields the UPL-distributed random sample.

(ii) The number of replications is set to $N=1000$ for the simulations.

(iii) The true parameter values for the simulation are taken as:

$(\alpha,\beta)=(1,1),(1,3),(2,1),(2,3),(0.8,2),(0.6,0.5)$.

We note that these true parameter values are also used for the pdf plots in Figure 1, demonstrating various modelling types.

(iv) The sample size for the simulation is designated as $n=25,50,100$ and $200$.

(v) The simulation study and the real data example are conducted using the MATLAB R2023a software.

(vi) For the performance comparison in the simulation study, bias and mean squared error (MSE) measures are provided as follows:

$Bias(\hat{\alpha})=\bar{\alpha}-\alpha,$ $\qquad Bias(\hat{\beta})=\bar{\beta}-\beta,$

$MSE(\hat{\alpha})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i-\alpha)^2,$ $\qquad MSE(\hat{\beta})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i-\beta)^2.$

Here, $\bar{\alpha}=\frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i$, $\bar{\beta}=\frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_i$, and $\hat{\alpha}_i$ and $\hat{\beta}_i$ represent the $i$-th simulated estimate.

(vii) The same true parameter values are used for computing the $100(1-\gamma)$ CI and CP, with the significance level set as $\gamma=0.05$.

(viii) The "*fminsearch*" optimization function in MATLAB is utilized for parameter estimation in all computations. One can see [41] some details of algorithms used in *"fminsearch"*.

### 4.1. Simulation Study

This comprehensive simulation study offers valuable insights into the comparative performance of estimation methods, allowing for comparing their efficiency in the case of different sample sizes and parameter settings for the UPL distribution. Tables 2-7 present detailed simulation study results, including biases and MSE values for all estimates obtained using ML, LS, WLS, CV, AD, and RAD methods. Notably, as observed in the tables, MSE values tend to decrease with larger sample sizes, indicating the consistency of all estimation methods. It is worth noting that the tables highlight the smallest MSE values using bold labels, indicating superior performance in estimation. Specific insights from each table are summarized below:

In most cases in Table 2, the AD method yields the smallest MSE values for parameters $\alpha = 1$ and $\beta = 1$. According to Table 3 for parameter values $\alpha = 1$ and $\beta = 3$, the ML method produces the smallest MSE values, with AD and RAD methods closely following. Results for parameter values $\alpha = 2$ and $\beta = 1$ in Table 4 indicate the AD method as the most suitable among the compared methods. Furthermore, the ML method is identified as the most efficient based on MSE values for parameter values $\alpha = 2$ and $\beta = 3$ in Table 5. When we check the results of Tables 6 and 7, the ML method consistently exhibits the smallest MSE values, particularly for parameter values $\alpha = 0.6$ and $\beta = 0.5$, and $\alpha = 0.8$ and $\beta = 2$.

Additionally, Table 8 provides the CP and the AW values for the ML method. We have the following from this table. CP values closely align with the nominal value of 95%, indicating accurate estimation, and AW values decrease as the number of samples increases, as expected.

***Table 2.*** *Bias and MSE values for $\hat{\alpha}$ and $\hat{\beta}$*

| | | | | $\alpha = 1,\ \beta = 1$ | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | ML | LS | WLS | CV | AD | RAD |
| $n = 25$ | $\hat{\alpha}$ | 0.0637 | -0.0031 | 0.0066 | 0.0597 | 0.0148 | 0.0353 |
| | | (0.0351) | (0.0370) | (0.0323) | (0.0499) | **(0.0269)** | (0.0331) |
| | $\hat{\beta}$ | -0.0020 | 0.0096 | 0.0093 | 0.0041 | 0.0080 | 0.0002 |
| | | (0.0360) | (0.0184) | (0.0183) | (0.0206) | **(0.0180)** | (0.0194) |
| $n = 50$ | $\hat{\alpha}$ | 0.0266 | -0.0056 | 0.0018 | 0.0237 | 0.0044 | 0.0106 |
| | | (0.0131) | (0.0168) | (0.0136) | (0.0185) | **(0.0122)** | (0.0131) |
| | $\hat{\beta}$ | -0.0023 | 0.0044 | 0.0033 | 0.0013 | 0.0032 | 0.0007 |
| | | (0.0164) | (0.0086) | **(0.0083)** | (0.0090) | (0.0084) | (0.0090) |
| $n = 100$ | $\hat{\alpha}$ | 0.0180 | 0.0027 | 0.0079 | 0.0173 | 0.0077 | 0.0112 |
| | | **(0.0063)** | (0.0083) | (0.0067) | (0.0088) | **(0.0063)** | (0.0069) |
| | $\hat{\beta}$ | -0.0051 | -0.0010 | -0.0017 | -0.0026 | -0.0015 | -0.0028 |
| | | (0.0086) | (0.0046) | (0.0045) | (0.0047) | **(0.0045)** | (0.0047) |
| $n = 200$ | $\hat{\alpha}$ | 0.0061 | -0.0019 | 0.0009 | 0.0054 | 0.0004 | 0.0035 |
| | | (0.0027) | (0.0034) | (0.0027) | (0.0035) | **(0.0026)** | (0.0029) |
| | $\hat{\beta}$ | 0.0021 | 0.0026 | 0.0024 | 0.0017 | 0.0026 | 0.0014 |
| | | (0.0043) | (0.0023) | **(0.0022)** | (0.0023) | **(0.0022)** | (0.0024) |

* The MSE values of the estimates are given in parentheses

***Table 3.*** *Bias and MSE values for $\hat{\alpha}$ and $\hat{\beta}$*

| | | | | $\alpha = 1,\ \beta = 3$ | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | ML | LS | WLS | CV | AD | RAD |
| $n = 25$ | $\hat{\alpha}$ | 0.0440 | 0.0127 | 0.0249 | 0.0781 | 0.0359 | 0.0650 |
| | | **(0.0279)** | (0.0437) | (0.0405) | (0.0570) | (0.0363) | (0.0489) |
| | $\hat{\beta}$ | 0.0372 | -0.2512 | -0.2252 | -0.0482 | -0.1985 | -0.1537 |
| | | **(0.1720)** | (0.6312) | (0.5761) | (0.8172) | (0.4944) | (0.5275) |
| $n = 50$ | $\hat{\alpha}$ | 0.0202 | 0.0108 | 0.0195 | 0.0425 | 0.0237 | 0.0353 |
| | | **(0.0136)** | (0.0201) | (0.0170) | (0.0234) | (0.0160) | (0.0188) |
| | $\hat{\beta}$ | 0.0498 | -0.3272 | -0.3076 | -0.2399 | -0.2960 | -0.2808 |
| | | **(0.1265)** | (0.2788) | (0.2412) | (0.2591) | (0.2285) | (0.2210) |
| $n = 100$ | $\hat{\alpha}$ | 0.0130 | 0.0203 | 0.0272 | 0.0360 | 0.0265 | 0.0349 |
| | | **(0.0067)** | (0.0106) | (0.0090) | (0.0119) | (0.0084) | (0.0101) |
| | $\hat{\beta}$ | 0.0408 | -0.3304 | -0.3140 | -0.2884 | -0.3150 | -0.3031 |
| | | **(0.0768)** | (0.1980) | (0.1718) | (0.1792) | (0.1682) | (0.1618) |
| $n = 200$ | $\hat{\alpha}$ | 0.0088 | 0.0250 | 0.0285 | 0.0328 | 0.0275 | 0.0351 |
| | | **(0.0034)** | (0.0056) | (0.0048) | (0.0062) | (0.0046) | (0.0060) |
| | $\hat{\beta}$ | 0.0232 | -0.3325 | -0.3217 | -0.3119 | -0.3234 | -0.3126 |
| | | **(0.0391)** | (0.1537) | (0.1392) | (0.1421) | (0.1392) | (0.1324) |

* The MSE values of the estimates are given in parentheses

***Table 4.*** *Bias and MSE values for* $\hat{\alpha}$ *and* $\hat{\beta}$

| | | | | $\alpha = 2,\quad \beta = 1$ | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | ML | LS | WLS | CV | AD | RAD |
| $n = 25$ | $\hat{\alpha}$ | 0.1007 | -0.0198 | -0.0070 | 0.1031 | 0.0118 | 0.0386 |
| | | (0.1245) | (0.1558) | (0.1333) | (0.1930) | **(0.1037)** | (0.1192) |
| | $\hat{\beta}$ | 0.0059 | 0.0165 | 0.0155 | 0.0120 | 0.0142 | 0.0087 |
| | | (0.0356) | (0.0192) | **(0.0185)** | (0.0221) | **(0.0185)** | (0.0199) |
| $n = 50$ | $\hat{\alpha}$ | 0.0422 | -0.0300 | -0.0128 | 0.0282 | -0.0058 | 0.0078 |
| | | (0.0530) | (0.0591) | (0.0505) | (0.0631) | **(0.0459)** | (0.0499) |
| | $\hat{\beta}$ | 0.0062 | 0.0112 | 0.0100 | 0.0082 | 0.0099 | 0.0072 |
| | | (0.0171) | **(0.0086)** | **(0.0086)** | (0.0090) | **(0.0086)** | (0.0091) |
| $n = 100$ | $\hat{\alpha}$ | 0.0242 | -0.0088 | 0.0024 | 0.0203 | 0.0018 | 0.0080 |
| | | (0.0241) | (0.0319) | (0.0256) | (0.0334) | **(0.0239)** | (0.0267) |
| | $\hat{\beta}$ | 0.0010 | 0.0042 | 0.0032 | 0.0026 | 0.0034 | 0.0022 |
| | | (0.0082) | (0.0044) | **(0.0042)** | (0.0045) | **(0.0042)** | (0.0046) |
| $n = 200$ | $\hat{\alpha}$ | 0.0143 | -0.0052 | 0.0018 | 0.0092 | 0.0008 | 0.0088 |
| | | **(0.0120)** | (0.0155) | (0.0121) | (0.0158) | **(0.0120)** | (0.0132) |
| | $\hat{\beta}$ | 0.0004 | 0.0012 | 0.0011 | 0.0004 | 0.0013 | -0.0003 |
| | | (0.0042) | (0.0023) | **(0.0022)** | (0.0023) | **(0.0022)** | (0.0024) |

* The MSE values of the estimates are given in parentheses

***Table 5.*** *Bias and MSE values for* $\hat{\alpha}$ *and* $\hat{\beta}$

| | | | | $\alpha = 2,\quad \beta = 3$ | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | ML | LS | WLS | CV | AD | RAD |
| $n = 25$ | $\hat{\alpha}$ | 0.1146 | 0.0409 | 0.0649 | 0.1734 | 0.0850 | 0.1461 |
| | | **(0.1407)** | (0.1949) | (0.1810) | (0.2550) | (0.1559) | (0.2197) |
| | $\hat{\beta}$ | 0.1617 | -0.2791 | -0.2528 | -0.0761 | -0.2490 | -0.2040 |
| | | **(0.3773)** | (0.8552) | (0.7858) | (1.1870) | (0.4746) | (0.5428) |
| $n = 50$ | $\hat{\alpha}$ | 0.0466 | 0.0297 | 0.0479 | 0.0933 | 0.0528 | 0.0868 |
| | | **(0.0587)** | (0.0862) | (0.0744) | (0.1007) | (0.0693) | (0.0893) |
| | $\hat{\beta}$ | 0.0947 | -0.3232 | -0.3012 | -0.2356 | -0.2929 | -0.2687 |
| | | **(0.1830)** | (0.2926) | (0.2506) | (0.2762) | (0.2403) | (0.2319) |
| $n = 100$ | $\hat{\alpha}$ | 0.0301 | 0.0455 | 0.0578 | 0.0771 | 0.0580 | 0.0754 |
| | | **(0.0268)** | (0.0413) | (0.0353) | (0.0467) | (0.0340) | (0.0403) |
| | $\hat{\beta}$ | 0.0585 | -0.3169 | -0.3019 | -0.2744 | -0.3006 | -0.2881 |
| | | **(0.0902)** | (0.1952) | (0.1698) | (0.1778) | (0.1671) | (0.1606) |
| $n = 200$ | $\hat{\alpha}$ | 0.0147 | 0.0509 | 0.0568 | 0.0666 | 0.0547 | 0.0707 |
| | | **(0.0125)** | (0.0228) | (0.0191) | (0.0251) | (0.0181) | (0.0216) |
| | $\hat{\beta}$ | 0.0264 | -0.3267 | -0.3168 | -0.3059 | -0.3187 | -0.3067 |
| | | **(0.0391)** | (0.1510) | (0.1364) | (0.1396) | (0.1364) | (0.1286) |

* The MSE values of the estimates are given in parentheses

***Table 6.*** *Bias and MSE values for* $\hat{\alpha}$ *and* $\hat{\beta}$

| | | | | $\alpha = 0.8,\quad \beta = 2$ | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | ML | LS | WLS | CV | AD | RAD |
| $n = 25$ | $\hat{\alpha}$ | 0.0509 | 0.0304 | 0.0375 | 0.0853 | 0.0446 | 0.0627 |
| | | **(0.0223)** | (0.0311) | (0.0274) | (0.0428) | (0.0235) | (0.0316) |
| | $\hat{\beta}$ | 0.0661 | -0.1830 | -0.1772 | -0.1139 | -0.1700 | -0.1642 |
| | | (0.1345) | (0.1514) | (0.1387) | (0.1685) | **(0.1270)** | (0.1300) |
| $n = 50$ | $\hat{\alpha}$ | 0.0256 | 0.0296 | 0.0347 | 0.0561 | 0.0365 | 0.0480 |
| | | **(0.0093)** | (0.0147) | (0.0126) | (0.0181) | (0.0116) | (0.0141) |
| | $\hat{\beta}$ | 0.0452 | -0.1882 | -0.1820 | -0.1568 | -0.1792 | -0.1759 |
| | | **(0.0639)** | (0.0890) | (0.0817) | (0.0854) | (0.0771) | (0.0769) |
| $n = 100$ | $\hat{\alpha}$ | 0.0109 | 0.0272 | 0.0309 | 0.0402 | 0.0306 | 0.0372 |
| | | **(0.0041)** | (0.0072) | (0.0062) | (0.0084) | (0.0060) | (0.0067) |
| | $\hat{\beta}$ | 0.0217 | -0.1971 | -0.1919 | -0.1822 | -0.1916 | -0.1895 |
| | | **(0.0262)** | (0.0607) | (0.0564) | (0.0564) | (0.0559) | (0.0552) |
| $n = 200$ | $\hat{\alpha}$ | 0.0068 | 0.0281 | 0.0303 | 0.0345 | 0.0299 | 0.0354 |
| | | **(0.0022)** | (0.0041) | (0.0037) | (0.0046) | (0.0036) | (0.0043) |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\hat{\beta}$ | 0.0098 | -0.2032 | -0.1993 | -0.1960 | -0.1993 | -0.1976 |
| | | **(0.0128)** | (0.0522) | (0.0495) | (0.0496) | (0.0493) | (0.0486) |

* The MSE values of the estimates are given in parentheses

***Table 7.*** *Bias and MSE values for $\hat{\alpha}$ and $\hat{\beta}$*

| | | $\alpha = 0.6,$ | $\beta = 0.5$ | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | | ML | LS | WLS | CV | AD | RAD |
| $n = 25$ | $\hat{\alpha}$ | 0.0307 | -0.0863 | -0.0820 | -0.0630 | -0.0439 | -0.0609 |
| | | (0.0096) | (0.0121) | (0.0106) | (0.0091) | (0.0085) | **(0.0082)** |
| | $\hat{\beta}$ | -0.0134 | 0.1839 | 0.1831 | 0.1700 | 0.1497 | 0.1662 |
| | | **(0.0134)** | (0.0390) | (0.0382) | (0.0341) | (0.0301) | (0.0333) |
| $n = 50$ | $\hat{\alpha}$ | 0.0125 | -0.0830 | -0.0790 | -0.0716 | -0.0696 | -0.0670 |
| | | **(0.0037)** | (0.0091) | (0.0080) | (0.0074) | (0.0073) | (0.0063) |
| | $\hat{\beta}$ | -0.0036 | 0.1815 | 0.1809 | 0.1746 | 0.1726 | 0.1702 |
| | | **(0.0068)** | (0.0358) | (0.0353) | (0.0333) | (0.0333) | (0.0320) |
| $n = 100$ | $\hat{\alpha}$ | 0.0048 | -0.0810 | -0.0780 | -0.0753 | -0.0779 | -0.0684 |
| | | **(0.0017)** | (0.0076) | (0.0069) | (0.0067) | (0.0069) | (0.0056) |
| | $\hat{\beta}$ | -0.0002 | 0.1788 | 0.1792 | 0.1753 | 0.1791 | 0.1702 |
| | | **(0.0033)** | **(0.0333)** | (0.0333) | (0.0320) | (0.0333) | (0.0304) |
| $n = 200$ | $\hat{\alpha}$ | 0.0050 | -0.0776 | -0.0756 | -0.0748 | -0.0768 | -0.0665 |
| | | **(0.0009)** | (0.0066) | (0.0061) | (0.0061) | (0.0063) | (0.0050) |
| | $\hat{\beta}$ | -0.0035 | 0.1748 | 0.1754 | 0.1724 | 0.1763 | 0.1667 |
| | | **(0.0017)** | (0.0310) | (0.0314) | (0.0304) | (0.0317) | (0.0285) |

* The MSE values of the estimates are given in parentheses

***Tablo 8.*** *CP and AW values for $\hat{\alpha}$ and $\hat{\beta}$*

| | | | $\hat{\alpha}$ | | $\hat{\beta}$ | |
|---|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | $n$ | CP | AW | CP | AW |
| 1 | 1 | 25 | 0.9510 | 0.1238 | 0.9450 | 0.1422 |
| | | 50 | 0.9510 | 0.0607 | 0.9430 | 0.0707 |
| | | 100 | 0.9520 | 0.0299 | 0.9550 | 0.0353 |
| | | 200 | 0.9570 | 0.0148 | 0.9480 | 0.0177 |
| 1 | 3 | 25 | 0.9350 | 0.1346 | 0.9620 | 0.4889 |
| | | 50 | 0.9390 | 0.0648 | 0.9460 | 0.2235 |
| | | 100 | 0.9480 | 0.0320 | 0.9470 | 0.1089 |
| | | 200 | 0.9520 | 0.0158 | 0.9510 | 0.0533 |
| 2 | 1 | 25 | 0.9520 | 0.2492 | 0.9370 | 0.1421 |
| | | 50 | 0.9490 | 0.1215 | 0.9470 | 0.0705 |
| | | 100 | 0.9470 | 0.0596 | 0.9590 | 0.0353 |
| | | 200 | 0.9510 | 0.0297 | 0.9540 | 0.0176 |
| 2 | 3 | 25 | 0.9490 | 0.2683 | 0.9620 | 0.4776 |
| | | 50 | 0.9600 | 0.1297 | 0.9550 | 0.2244 |
| | | 100 | 0.9530 | 0.0634 | 0.9670 | 0.1080 |
| | | 200 | 0.9540 | 0.0315 | 0.9530 | 0.0534 |
| 0.8 | 2 | 25 | 0.9490 | 0.1062 | 0.9600 | 0.2697 |
| | | 50 | 0.9670 | 0.0512 | 0.9570 | 0.1293 |
| | | 100 | 0.9470 | 0.0253 | 0.9530 | 0.0638 |
| | | 200 | 0.9530 | 0.0125 | 0.9570 | 0.0317 |
| 0.6 | 0.5 | 25 | 0.9480 | 0.0692 | 0.9110 | 0.0919 |
| | | 50 | 0.9490 | 0.0338 | 0.9410 | 0.0464 |
| | | 100 | 0.9420 | 0.0167 | 0.9530 | 0.0231 |
| | | 200 | 0.9540 | 0.0083 | 0.9560 | 0.0116 |

### 4.2. Real Data Examples

This section includes the application of two real data sets from the metal industry to illustrate the modelling performance of the UPL distribution. The first dataset consists of measures on burrs with a hole diameter of 12 mm and a sheet thickness of 3.15 mm, for 50 observations. The second dataset contains measures on burrs with a hole diameter of 9 mm and a sheet thickness of 2 mm for 50 observations. Note that [42] used extreme value distribution to model the corresponding data sets. The datasets are available in Tables 9 and 10.

***Table 9.*** *Dataset 1*

| |
|---|
| 0.04 0.02 0.06 0.12 0.14 0.08 0.22 0.12 0.08 0.26 0.24 0.04 0.14 0.16 0.08 0.26 0.32 0.28 0.14 0.16 0.24 0.22 0.12 |
| 0.18 0.24 0.32 0.16 0.14 0.08 0.16 0.24 0.16 0.32 0.18 0.24 0.22 0.16 0.12 0.24 0.06 0.02 0.18 0.22 0.14 0.06 0.04 |
| 0.14 0.26 0.18 0.16 |

***Table 10.*** *Dataset 2*

| |
|---|
| 0.06 0.12 0.14 0.04 0.14 0.16 0.08 0.26 0.32 0.22 0.16 0.12 0.24 0.06 0.02 0.18 0.22 0.14 0.22 0.16 0.12 0.24 0.06 |
| 0.02 0.18 0.22 0.14 0.02 0.18 0.22 0.14 0.06 0.04 0.14 0.22 0.14 0.06 0.04 0.16 0.24 0.16 0.32 0.18 0.24 0.22 0.04 |
| 0.14 0.26 0.18 0.16 |

According to the results of the Monte Carlo simulation study, it is realised that the ML method demonstrates superior performances among the others. Therefore, in the application part, only the ML method is used to estimate the parameters of the UPL distribution. Moreover, to compare the modelling performance of the UPL distribution, the well-known unit distributions, namely, Beta, KM, UL, and UW distributions are used. It should be stated that the ML estimates of the parameters of the corresponding distributions are obtained as well.

To assess the modelling performance of the Beta, KM, UL, UW, and UPL distributions, the values for the Akaike information criterion (AIC) [43], Bayesian information criterion (BIC) [44], and Effective determination information criterion (EDC) [45] are calculated by using the following formula:

$$-2\ell + mc_n\,,$$

where $\ell$ represents the maximized log-likelihood, $m$ is the number of parameters to be estimated in the model and $c_n$ is the penalty term. Specially, $c_n = 2$ for AIC, $c_n = \log(n)$ for BIC and $c_n = 0.2\sqrt{n}$ for EDC. These criteria provide a quantitative basis for comparing the models, considering both goodness of fit and model complexity.

Furthermore, the Kolmogorov-Smirnov (KS) test statistic is employed to assess the suitability of each distribution for the given datasets. The KS test statistic is calculated through the following steps:
**Step 1:** Sort the data from the smallest to the largest values: $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$.
**Step 2:** Calculate the KS test statistic using the formula:

$$D = \max_{i=1,2,\dots,n} \{|F_n(x) - F(x)|\},$$

where, $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$ represents the empirical distribution and $F(x)$ is the distribution function.

It's worth noting that a smaller value of D indicates a better fit between the empirical data and the distribution being tested. It should be noted that the KS test statistics and p-values are computed via the *"kstest2"* function in MATLAB.

In Table 11, the ML estimates, KS test statistics with p-values, and values of the maximized log-likelihood ($\ell$), AIC, BIC, and EDC, are presented for KM, Beta, UL, UW, and UPL distributions. The results given in Table 11 indicate that, for Dataset 1, the UPL distribution exhibits the smallest information criterion values and the lowest KS test statistic with the highest p-value. Consequently, the UPL distribution appears as the most suitable model for Dataset 1. To support this conclusion, Figure 5 includes the histogram of Dataset 1 along with the fitted density lines for the KM, Beta, UL, UW, and UPL distributions. According to this visual illustration, the UPL distribution demonstrates the best fit among the considered distributions.

Additionally, Quantile-Quantile (Q-Q) plots for Dataset 1 are derived from the KM, Beta, UL, UW, and UPL distributions in Figure 6. According to Figure 6, it is evident that the Q-Q plot for the UPL distribution closely follows the expected straight line, indicating a superior fit compared to the other distributions for Dataset 1. This visual confirmation aligns with the results obtained from the statistical measures and further supports the conclusion that the UPL distribution offers the best fit for the given Dataset 1.

***Table 11.*** *Estimation results of related distributions given for Dataset 1*

| | | UPL | KM | Beta | UL | UW |
|---|---|---|---|---|---|---|
| Estimates | $\hat{\alpha}$ | 1.7646 | 2.0774 | 2.6826 | 5.5829 | 0.0876 |
| | $\hat{\beta}$ | 14.1279 | 33.1374 | 13.8658 | - | 3.0519 |
| Information Criteria | $\ell$ | **56.9368** | 56.0687 | 54.6067 | 47.5774 | 48.6626 |
| | AIC | **-109.8736** | -108.1374 | -105.2133 | -93.1548 | -93.3252 |
| | BIC | **-106.0496** | -104.3133 | -101.3893 | -87.3307 | -89.5012 |
| | EDC | **-111.0452** | -109.3089 | -106.3849 | -92.3263 | -94.4968 |
| Test Statistics | KS | **0.0967** | 0.1000 | 0.1400 | 0.2310 | 0.1800 |
| | p-value | **0.9671** | 0.9541 | 0.6779 | 0.1222 | 0.3584 |

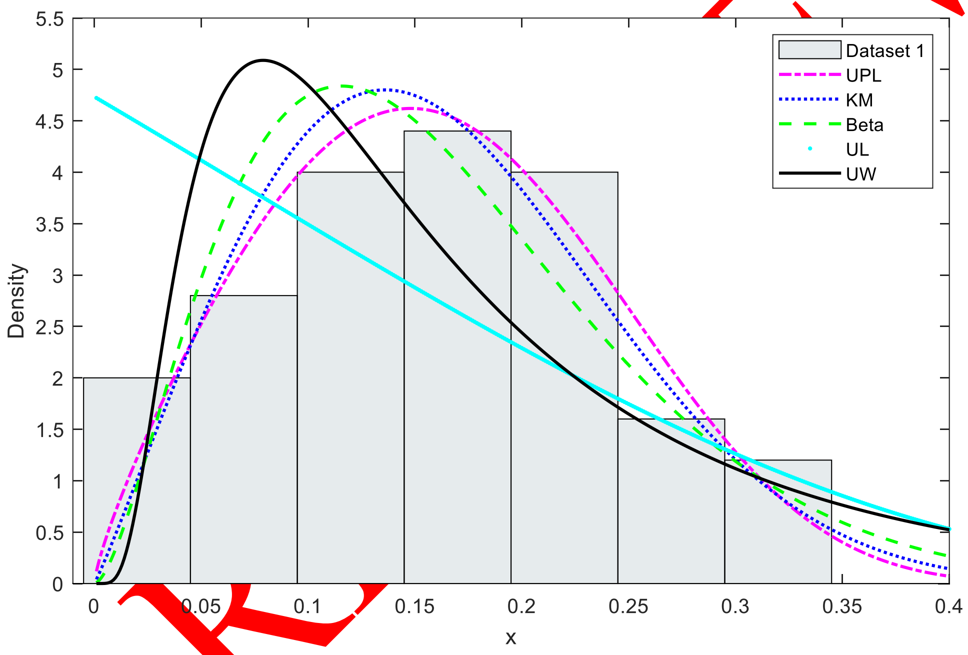
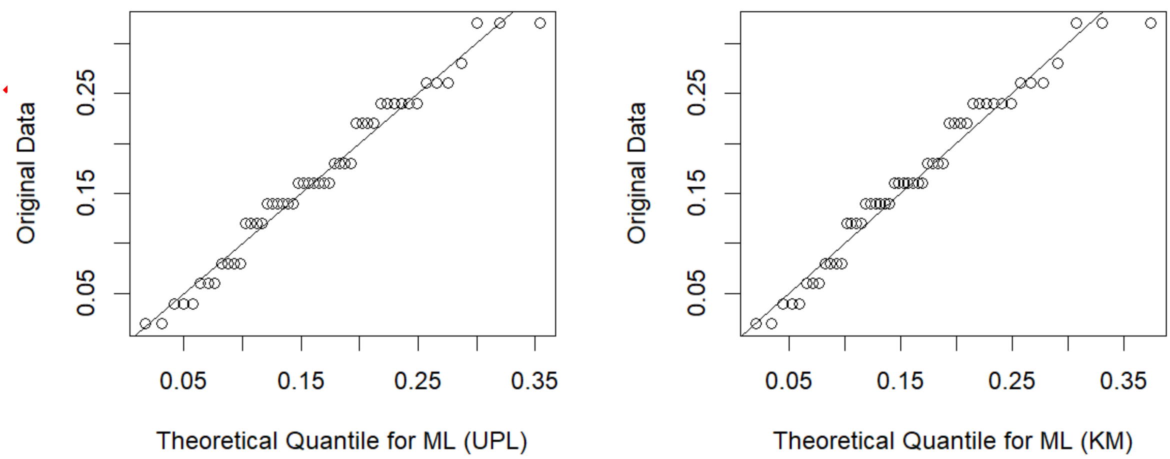



***Figure 5.*** *Histogram of Dataset 1 along with the fitted pdfs obtained from UPL, KM, Beta, UL, and UW*



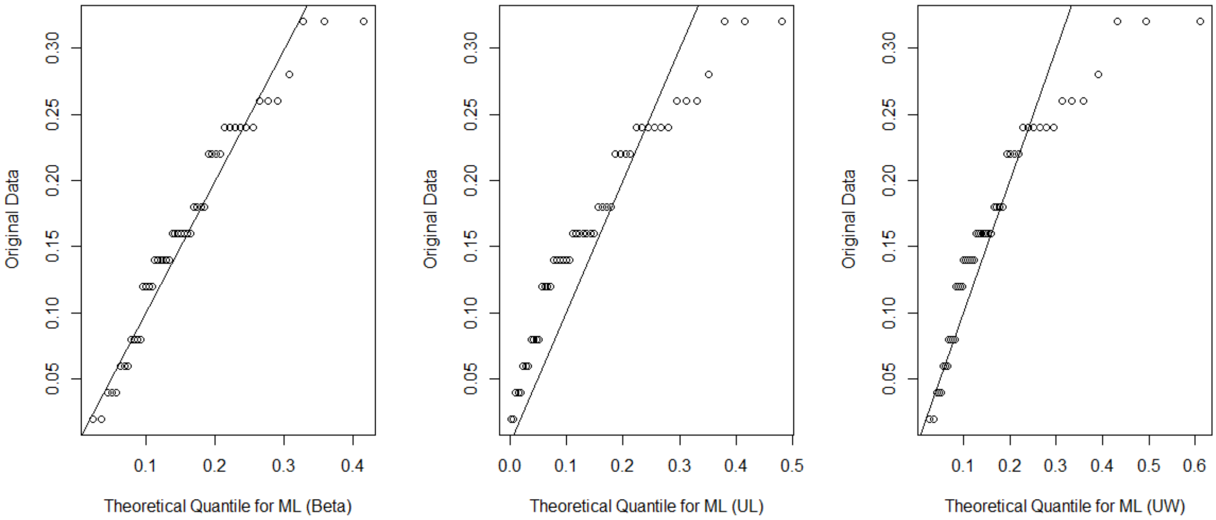



***Figure 6.*** *Q-Q plots of the UPL, KM, Beta, UL, and UW for Dataset 1*

Table 12 presents the estimation results for Dataset 2 given in Table 10, including the ML estimates for the UPL, KM, Beta, UL, and UW distributions, as well as the KS test statistics with p-values, and the values of the maximized log-likelihood ($\ell$), AIC, BIC, and EDC. The results reveal that the UPL distribution yields the smallest information criterion values and the lowest KS test statistic with the highest p-value. Consequently, it is concluded that the UPL distribution fits as the best model for Dataset 2. This conclusion is further supported by Figure 7, which includes estimated pdf graphs for the UPL, KM, Beta, UL, and UW distributions, along with a histogram of Dataset 2. As indicated by this visual representation, the UPL distribution demonstrates the best fit among the considered distributions for Dataset 2.

Figure 8 presents Q-Q plots for Dataset 2 derived from the UPL, KM, Beta, UL, and UW distributions. As observed in Figure 8, the Q-Q plot for the UPL distribution closely aligns with the expected straight line, supporting the conclusion that the UPL distribution offers the best fit for Dataset 2. This visual representation, combined with the statistical results, provides additional support for the findings based on information criteria and the KS test statistic.

***Table 12.*** *Estimation results of related distributions given for Dataset 2*

| | | UPL | KM | Beta | UL | UW |
|---|---|---|---|---|---|---|
| Estimates | $\hat{\alpha}$ | 1.6996 | 1.9606 | 2.4004 | 6.0324 | 0.0791 |
| | $\hat{\beta}$ | 14.9204 | 31.3795 | 13.5218 | - | 2.9937 |
| Information Criteria | $\ell$ | **58.5944** | 57.5214 | 55.9312 | 50.4449 | 50.0217 |
| | AIC | **-113.1888** | -111.0429 | -107.8624 | -98.8897 | -96.0434 |
| | BIC | **-109.3648** | -107.2189 | -104.0384 | -93.0657 | -92.2194 |
| | EDC | **-114.3604** | -112.2145 | -109.0340 | -98.0613 | -97.2150 |
| Test Statistics | KS | **0.1494** | 0.1600 | 0.1800 | 0.2514 | 0.2200 |
| | p-value | **0.6030** | 0.5077 | 0.3584 | 0.0730 | 0.1546 |

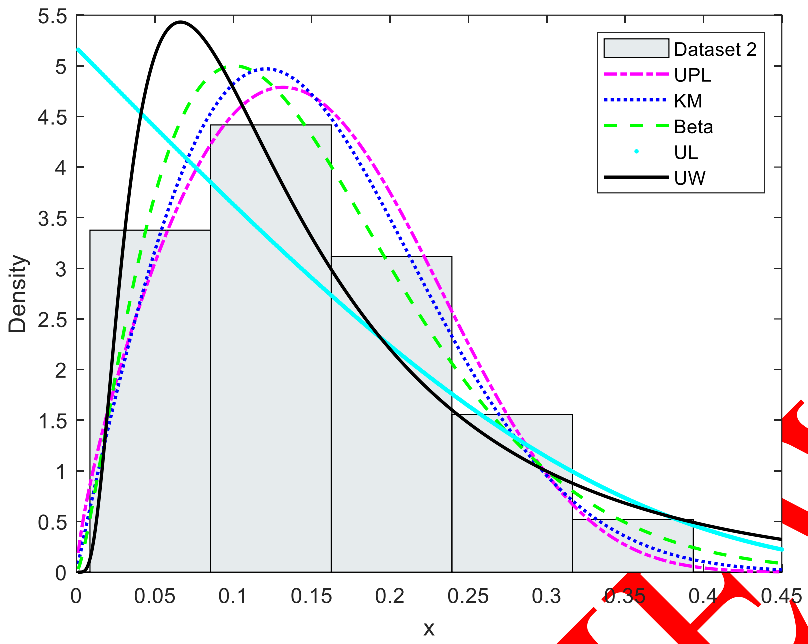



***Figure 7.*** *Histogram of Dataset 2 along with the fitted pdfs obtained from UPL, KM, Beta, UL, and UW*

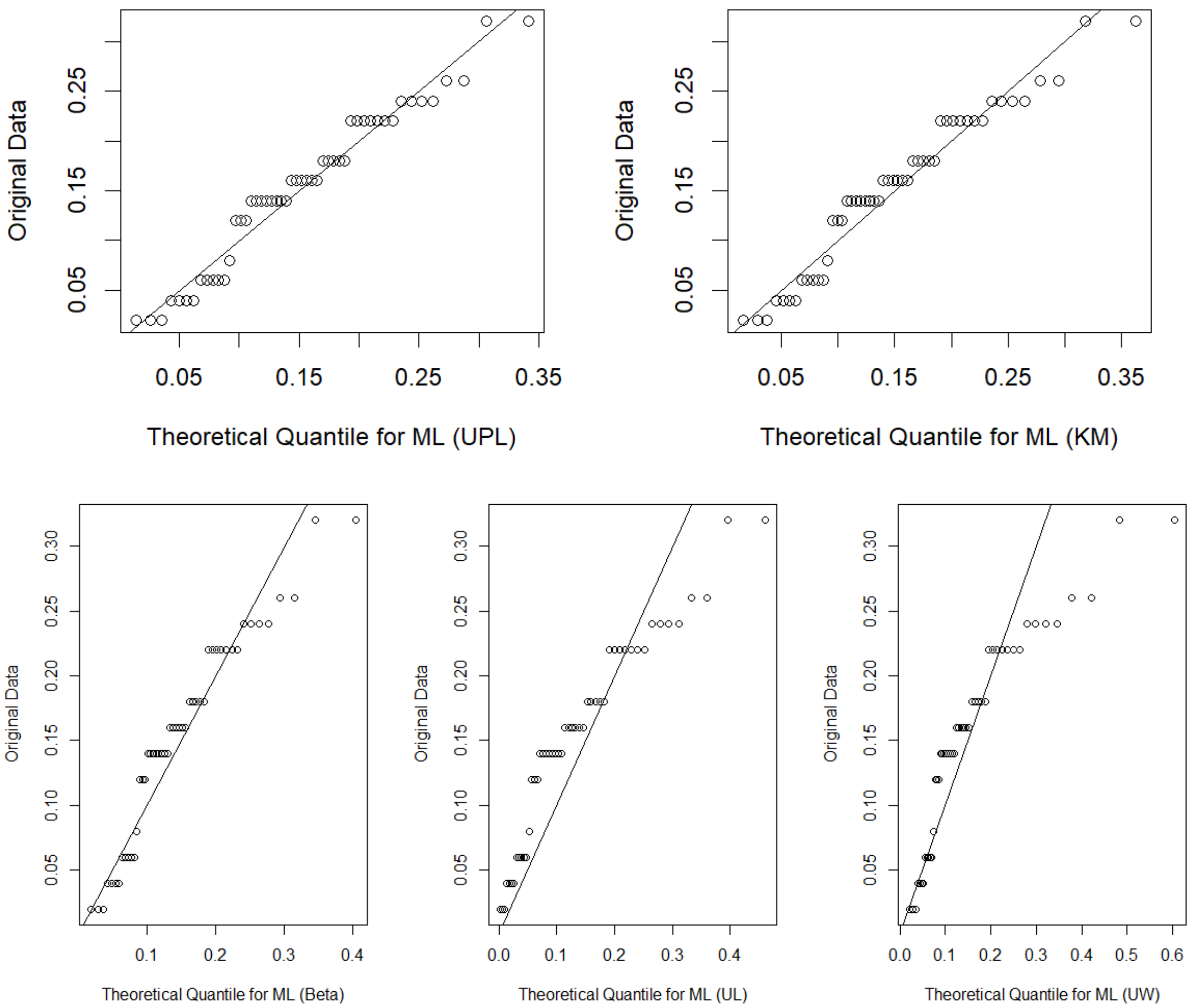



***Figure 8.*** *Q-Q plots of the UPL, KM, Beta, UL, and UW for Dataset 2*

## 5. RESULTS

In this paper, we propose a new unit distribution, named the UPL distribution, by transforming the PL distribution. In order to specify the statistical characteristic of this distribution, firstly we draw the pdf, cdf, and hrf plots. It is seen that the UPL distribution can be positively or negatively skewed concerning different parameter settings and generally platykurtic. The hrf of the UPL distribution displays a j-shaped pattern, indicating its versatility in modeling various dataset characteristics.

The parameters of UPL distribution are estimated by using the ML, LS, WLS, CM, AD, and RAD methods. The performances of these methods are compared via Monte-Carlo simulation study. Moreover, the asymptotic confidence intervals of the parameters are constructed by using the asymptotic properties of ML estimators. The simulation study concludes that the ML estimators generally outperform the other methods. Consequently, in real data applications, we utilize the ML method to estimate the parameters of the UPL distribution.

The modeling performance of UPL distribution is compared via KM, Beta, UL, and UW distributions. These comparisons are based on information criteria such as AIC, BIC EDC, and KS test statistics with p-values. This analysis indicates that the UPL distribution demonstrates superior performance in terms of both information criteria and KS test statistics. Therefore, we propose that the UPL distribution can serve as an effective alternative unit distribution for modeling datasets bounded within the interval (0,1) due to its flexible modeling performance.

## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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