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RESEARCH ARTICLE

Positive Toeplitz Operators from a Weighted Harmonic Bloch Space into Another

Ö. Faruk Doğan[1*]

[1]Namık Kemal University, Faculty of Arts and Sciences, Department of Mathematics, Süleymanpaşa, 59030 Tekirdağ, Türkiye

ABSTRACT
In the present paper, we define positive general Toeplitz operators between weighted harmonic Bloch spaces b_α^∞ on the unit ball of $\mathbb{R}^n$ for the full range of parameter $\alpha \in \mathbb{R}$, where symbols are positive Borel measures on the unit ball of $\mathbb{R}^n$. We characterize the boundedness and compactness of Toeplitz operators from one weighted harmonic Bloch space into another in terms of Carleson measures and vanishing Carleson measures. Recently, in Doğan (2022), positive symbols of bounded and compact general Toeplitz operators between harmonic Bergman-Besov spaces are completely characterized in term of Carleson measures and vanishing Carleson measures. Our results extend those known for harmonic Bloch space.

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1. INTRODUCTION

Let $n \geq 2$ be an integer and $\mathbb{B} = \mathbb{B}_n$ be the open unit ball of $\mathbb{R}^n$. Let ν be the normalized Lebesgue measure on $\mathbb{B}$. For $\alpha \in \mathbb{R}$, we define the weighted measures ν_α on $\mathbb{B}$ by

$$d\nu_\alpha(x) = \frac{1}{V_\alpha}(1 - |x|^2)^\alpha d\nu(x).$$

These measures are finite only when $\alpha > -1$ and in this case we select the constant V_α so that $\nu_\alpha(\mathbb{B}) = 1$. Naturally $V_0 = 1$. If $\alpha \leq -1$, we set $V_\alpha = 1$. For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_α by L_α^p and the corresponding norms by $\|\cdot\|_{L_\alpha^p}$.

Let $h(\mathbb{B})$ be the space of all complex-valued harmonic functions on $\mathbb{B}$ endowed with the topology of uniform convergence on compact subsets. For $\alpha > -1$, the harmonic Bergman space b_α^p is defined as $b_\alpha^p = L_\alpha^p \cap h(\mathbb{B})$ with norm $\|\cdot\|_{L_\alpha^p}$. When $p = 2$, b_α^2 is a reproducing kernel Hilbert space endowed with the inner product $[f, g]_{b_\alpha^2} = \int_{\mathbb{B}} f\overline{g}\, d\nu_\alpha(x)$ and with the reproducing kernel $R_\alpha(x, y)$ such that $f(x) = [f(.), R_\alpha(x, \cdot)]_{b_\alpha^2}$ for every $f \in b_\alpha^2$ and $x \in \mathbb{B}$. R_α is real-valued and symmetric in its variables. The homogeneous expansion of $R_\alpha(x, y)$ is given in the $\alpha > -1$ part of the formulas (2) and (3) below (see Djrbashian and Shamoian (1988), Gergün et al. (2016)).

The well-known harmonic Bloch space b is consists of all $f \in h(\mathbb{B})$ such that

$$\sup_{x\in\mathbb{B}}(1 - |x|^2)|\nabla f(x)|$$

is finite. Let L^∞ be the Lebesgue class of essentially bounded functions on $\mathbb{B}$. For $\alpha \in \mathbb{R}$ we define

$$L_\alpha^\infty = \{\varphi : (1 - |x|^2)^\alpha \varphi(x) \in L^\infty\},$$

so that $L_0^\infty = L^\infty$ and with norm $\|\varphi\|_{L_\alpha^\infty} = \|(1 - |x|^2)^\alpha \varphi(x)\|_{L^\infty}$. For $\alpha > 0$, the weighted harmonic Bloch space b_α^∞ is $h(\mathbb{B}) \cap L_\alpha^\infty$ endowed with the norm $\|\cdot\|_{L_\alpha^\infty}$, and is clearly imbedded in L_α^∞ by the inclusion i.

For $\alpha > 0$, the harmonic Bergman projection $Q_\alpha : L_\alpha^\infty \to b_\alpha^\infty$ is given by the integral operator

$$Q_\alpha f(x) = \frac{1}{V_\alpha}\int_{\mathbb{B}} R_\alpha(x, y) f(y)(1 - |y|^2)^\alpha d\nu(y) \quad (f \in L_\alpha^\infty). \tag{1}$$

It has a significant importance in the theory and the question when $Q_\alpha : L_\beta^\infty \to b_\beta^\infty$ is bounded is studied in many sources such

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as Choe et al. (2001); Jevtić and Pavlović (1999); Ligocka (1987) for $\beta = 0$ and Ren and Kähler (2003) with a different integral representation valid for $\beta > -1$. Then we define the Toeplitz operator ${}_{\alpha}T_{\theta} : b^{\infty}_{\alpha} \to b^{\infty}_{\alpha}$ with symbol $\theta \in L^1$ by ${}_{\alpha}T_{\theta} = Q_{\alpha} M_{\theta} i$, where M_{θ} is the operator of multiplication by θ. For a finite complex Borel measure μ on $\mathbb{B}$, the Toeplitz operator ${}_{\alpha}T_{\mu}$ is defined by

$$ {}_{\alpha}T_{\mu} f(x) = \frac{1}{V_{\alpha}} \int_{\mathbb{B}} R_{\alpha}(x,y) f(y)(1-|y|^2)^{\alpha} d\mu(y) $$

for $f \in L^{\infty}_{\alpha}$. The operator ${}_{\alpha}T_{\mu}$ is more general and reduces to ${}_{\alpha}T_{\theta}$ when $d\mu = \theta d\nu$. Toeplitz operators have been studied extensively on the harmonic Bergman spaces by many authors. Particularly, the boundedness and compactness of Toeplitz operators with positive symbols are completely characterized in term of Carleson measures as in Miao (1998), Miao (1997) on the ball and in Choe et al. (2004a) on smoothly bounded domains. The Boundedness and compactness of Toeplitz operators with positive symbols from a harmonic Bergman space into another are characterized in Choe et al. (2004b) on smoothly bounded domains and in Choe et al. (2002) on the half space.

The harmonic Bergman b^p_{α} and Bloch b^{∞}_{α} spaces can be extended to all real α. These are studied comprehensively in Gergün et al. (2016) and Doğan and Üreyen (2018), respectively. We call the extended set b^p_{α} $(\alpha \in \mathbb{R})$ harmonic Bergman-Besov spaces and the corresponding reproducing kernels $R_{\alpha}(x,y)$ $(\alpha \in \mathbb{R})$ harmonic Bergman-Besov kernels. The homogeneous expansion of $R_{\alpha}(x,y)$ can be expressed in terms of zonal harmonics $Z_k(x,y)$

$$ R_{\alpha}(x,y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x,y) \quad (\alpha \in \mathbb{R},\ x,y \in \mathbb{B}), \tag{2} $$

where (see (Gergün et al. 2009, Theorem 3.7), (Gergün et al. 2016, Theorem 1.3))

$$ \gamma_k(\alpha) := \begin{cases} \dfrac{(1+n/2+\alpha)_k}{(n/2)_k}, & \text{if } \alpha > -(1+n/2); \\[2ex] \dfrac{(k!)^2}{(1-(n/2+\alpha))_k (n/2)_k}, & \text{if } \alpha \le -(1+n/2), \end{cases} \tag{3} $$

and $(a)_b$ is the Pochhammer symbol. For further details about zonal harmonics, see (Axler et al. 2001, Chapter 5).

By using the radial differential operators D^t_s $(s,t \in \mathbb{R})$ introduced in Gergün et al. (2009) and Gergün et al. (2016), we can define the weighted harmonic Bloch spaces b^{∞}_{α} for all $\alpha \in \mathbb{B}$. These operators are compatible with reproducing kernels and yet mapping $h(\mathbb{B})$ onto itself. We present the fundamental properties of D^t_s in Section 2. The linear transformation I^t_s is defined by

$$ I^t_s f(x) := (1-|x|^2)^t D^t_s f(x), $$

for $f \in h(\mathbb{B})$.

Definition 1.1. For $\alpha \in \mathbb{R}$, we define the weighted harmonic Bloch space b^{∞}_{α} to consist of all $f \in h(\mathbb{B})$ for which $I^t_s f$ belongs to L^{∞}_{α} for some s and t satisfying (see Doğan and Üreyen (2018))

$$ \alpha + t > 0. \tag{4} $$

The quantity

$$ \|f\|_{b^{\infty}_{\alpha}} = \|I^t_s f\|_{L^{\infty}_{\alpha}} = \sup_{x \in \mathbb{B}} (1-|x|^2)^{\alpha+t} |D^t_s f(x)| < \infty, $$

defines a norm on b^{∞}_{α} for any such $s,t \in \mathbb{R}$.

Note that this definition is independent of s,t under (4), and the norms in these spaces are all equivalent. Therefore the operator I^t_s isometrically imbeds b^{∞}_{α} into L^{∞}_{α} for a given pair s,t if and only if (4) holds.

Harmonic Bergman-Besov projections Q_s that map Lebesgue classes boundedly onto weighted Bloch spaces b^{∞}_{α} can be identified exactly as in the case of $\alpha > 0$ by

$$ s > \alpha - 1. \tag{5} $$

Then I^t_s is a right inverse to Q_s. See Doğan and Üreyen (2018) for more details.

Let $\alpha \in \mathbb{R}$, s and t satisfing (5) and (4), and a measurable function θ on $\mathbb{B}$ be given. Then Q_s forces us to define Toeplitz operators on all b^{∞}_{α} as follows. We define the Toeplitz operator ${}_{s,t}T_{\theta} : b^{\infty}_{\alpha} \to b^{\infty}_{\alpha}$ with symbol θ by ${}_{s,t}T_{\theta} = Q_s M_{\theta} I^t_s$. Explicitly,

$$ {}_{s,t}T_{\theta} f(x) = \int_{\mathbb{B}} R_s(x,y)\theta(y) I^t_s f(y) d\nu_s(y) \quad (f \in b^{\infty}_{\alpha}). $$

We see that ${}_{s,t}T_{\theta}$ makes sense if $\theta \in L^1_{s-\alpha}$. When $\alpha > 0$, we can take $t = 0$ and a value of s satisfying (5) is $s = \alpha$. Then I^0_{α} is inclusion, and ${}_{s,t}T_{\theta}$ reduces to the classical Toeplitz operator ${}_{\alpha}T_{\theta} = Q_{\alpha} M_{\theta} i$ on b^{∞}_{α}, $\alpha > 0$. We use the word classical to mean a Toeplitz operator with $i = I^0_\alpha$. It is possible to take $s \neq \alpha$ also when $\alpha > -1$. Thus we have more general Toeplitz operators defined via I^t_s strictly on harmonic Bloch spaces too. It turns out that the properties of Toeplitz operators studied in this paper are independent of s, t under (5) and (4).

Since the integral form for ${}_{s,t}T_\theta$ is obtained, we can now define general Toeplitz operators on b^∞_α with symbol μ. Let α, and s and t satisfing (5) and (4) be given. We let

$$d\kappa(y) = (1 - |y|^2)^{s+t} \, d\mu(y)$$

and define

$$\begin{aligned} {}_{s,t}T_\mu f(x) &= \frac{1}{V_s} \int_{\mathbb{B}} R_s(x, y) I^t_s f(y)(1 - |y|^2)^s \, d\mu(y) \\ &= \frac{1}{V_s} \int_{\mathbb{B}} R_s(x, y) D^t_s f(y) d\kappa(y). \end{aligned}$$

The operator ${}_{s,t}T_\mu$ is more general and reduces to ${}_{s,t}T_\theta$ when $d\mu = \theta d\nu$. It makes sense when

$$d\psi(y) = (1 - |y|^2)^{-(\alpha+t)} d\kappa(y) = (1 - |y|^2)^{s-\alpha} d\mu(y)$$

is finite. Note that μ need not be finite in conformity with that α is unrestricted.

The *holomorphic* counterpart of our characterizations from a Dirichlet space into itself have been obtained in Alpay and Kaptanoğlu (2007). Recently, in Doğan (2022), positive symbols of bounded and compact general Toeplitz operators between harmonic Bergman-Besov spaces are completely characterized in term of Carleson measures. In the present paper, we consider the positive Toeplitz operator ${}_{s,t}T_\mu$ and characterize those that are bounded and compact from a weighted harmonic Bloch space $b^\infty_{\alpha_1}$ into another $b^\infty_{\alpha_2}$ for $\alpha_1, \alpha_2 \in \mathbb{R}$. Our main tool is Carleson measure.

Suppose μ is a positive Borel measure on $\mathbb{B}$. For $\alpha > -1$, we say that μ is a α-Carleson measure if the inclusion $i : b^p_\alpha \to L^p(\mu)$ is bounded, that is, if

$$\left(\int_{\mathbb{B}} |f(x)|^p \, d\mu(x) \right)^{1/p} \lesssim \|f\|_{b^p_\alpha}, \qquad (f \in b^p_\alpha).$$

As is usual with Carleson measure theorems, the property of being an α-Carleson measure is independent of p, because Theorem 3.1 is true for any p. However, it depends on $\alpha > -1$. So for a fixed α, an α-Carleson measure for one b^p_α is a Carleson measure for all b^p_α with the same α. We can now state our main result.

Theorem 1.2. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$. Suppose that $\alpha_1 + t > 0$, $\alpha_2 + t > 0$ and*

$$s > \alpha_i - 1, \qquad i = 1, 2. \tag{6}$$

Let

$$\gamma = s + t + \alpha_1 - \alpha_2.$$

Let μ be a positive Borel measure on $\mathbb{B}$ and $d\kappa(y) = (1 - |y|^2)^{s+t} d\mu(y)$. Then the following are equivalent:

(i) ${}_{s,t}T_\mu : b^\infty_{\alpha_1} \to b^\infty_{\alpha_2}$ is bounded.
(ii) κ is a γ-Carleson measure.

In order to characterize compact Toeplitz operators ${}_{s,t}T_\mu$ with positive μ from weighted harmonic Bloch spaces $b^\infty_{\alpha_1}$ into another $b^\infty_{\alpha_2}$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$, we present the notion of vanishing α-Carleson measures. If, for any sequence $\{f_k\}$ in b^p_α with $f_k \to 0$ uniformly on each compact subset of $\mathbb{B}$ and $\|f_k\|_{b^p_\alpha} \leq 1$, where

$$\lim_{k \to \infty} \int_{\mathbb{B}} |f_k(x)|^p \, d\mu(x) = 0,$$

then $\mu \geq 0$ is called vanishing α-Carleson measure. One can see from Theorem 3.3 that the notion of vanishing α-Carleson measures on b^p_α is also independent of p.

Theorem 1.3. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$. Let s, t, γ and κ be as in Theorem 1.2. Then the following are equivalent:*

(i) ${}_{s,t}T_\mu : b^\infty_{\alpha_1} \to b^\infty_{\alpha_2}$ is compact.
(ii) κ is a vanishing γ-Carleson measure.

The proofs of our results are inspired by the work of Pau and Zhao (2015), where bounded and compact classical Toeplitz operators between holomorphic weighted Bergman spaces are characterized.

We briefly summarize the notation and some preliminary material in Section 2. Section 3 is devoted to recall some characterizations of (vanishing) α-Carleson measures. We will give the proof of our main results, Theorems 1.2 and 1.3, in Section 4.

Throughout the paper, for two positive expressions A and B, $A \lesssim B$ means that there exists a positive constant C, whose exact value is inessential, such that $A \leq CB$. We also use $A \sim B$ to mean both $A \lesssim B$ and $B \lesssim A$.

2. NOTATION AND PRELIMINARIES

The Pochhammer symbol $(a)_b$ is defined by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)},$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function. Stirling formula provides

$$\frac{(a)_c}{(b)_c} \sim c^{a-b} \quad (c \to \infty). \tag{7}$$

2.1. Pseudo-hyperbolic Metric

For any $a \in \mathbb{B}$ with $a \neq 0$, the Möbius transformation on $\mathbb{B}$ that exchanges the points 0 and a is

$$\varphi_a(x) = \frac{(1-|a|^2)(a-x) + |a-x|^2 a}{[x,a]^2}.$$

Here we use the abbreviation

$$[x,a] = \sqrt{1 - 2x \cdot a + |x|^2|a|^2},$$

where $x \cdot a$ denotes the usual inner product in $\mathbb{R}^n$. Note that $[x,x] = 1 - |x|^2$. The pseudo-hyperbolic distance on $\mathbb{B}$ between $x, y \in \mathbb{B}$ is defined by

$$\rho(x,y) = |\varphi_x(y)| = \frac{|x-y|}{[x,y]}.$$

We need the following lemma from (Choe et al. 2008, Lemma 2.2).

Lemma 2.1. *If $a, x, y \in \mathbb{B}$, then*

$$\frac{1-\rho(x,y)}{1+\rho(x,y)} \leq \frac{[x,a]}{[y,a]} \leq \frac{1+\rho(x,y)}{1-\rho(x,y)}.$$

The following lemma shows that if $x, y \in \mathbb{B}$ are close in the pseudo-hyperbolic metric, then certain quantities are comparable. Its proof clearly follows from Lemma 2.1.

Lemma 2.2. *Let $0 < \delta < 1$. Then*

$$[x,y] \sim 1 - |x|^2 \sim 1 - |y|^2,$$

for all $x, y \in \mathbb{B}$ with $\rho(x,y) < \delta$.

For $x \in \mathbb{B}$, and $0 < \delta < 1$, the pseudo-hyperbolic ball with center x and radius δ is given by $E_\delta(x)$. We note that $E_\delta(x)$ is an Euclidean ball with center at c and radius r, where

$$c = \frac{(1-\delta^2)x}{1-\delta^2|x|^2} \qquad \text{and} \qquad r = \frac{(1-|x|^2)\delta}{1-\delta^2|x|^2}.$$

So, we have $\nu(E_\delta(x)) \sim (1-|x|^2)^n$ for fixed $0 < \delta < 1$. More generally, for $\alpha \in \mathbb{R}$, by Lemma 2.2

$$\nu_\alpha(E_\delta(x)) = \frac{1}{V_\alpha} \int_{E_\delta(x)} (1-|y|^2)^\alpha \, d\nu(y) \sim (1-|x|^2)^\alpha \nu(E_\delta(x)) \sim (1-|x|^2)^{\alpha+n}. \tag{8}$$

Let $\{a_k\}$ be a sequence of points in $\mathbb{B}$ and $0 < \delta < 1$. We say that $\{a_k\}$ is δ-separated if $\rho(a_j, a_k) \geq \delta$ for all $j \neq k$. See Luecking (1993) for a proof of the following lemma.

Lemma 2.3. *For fixed $0 < \delta < 1$, There exists a sequence of points $\{a_k\}$ in $\mathbb{B}$ such that the following hold.*

(i) $\{a_k\}$ *is δ-separated.*

(ii) $\displaystyle\bigcup_{k=1}^{\infty} E_\delta(a_k) = \mathbb{B}$.

(iii) There exists a positive integer N such that each $x \in \mathbb{B}$ is contained in at most N of the balls $E_\delta(a_k)$.

From now on, whenever we use representation like $\widehat{\mu}_{\alpha,\delta}(a_k)$, the sequence $\{a_k\} = \{a_k(\delta)\}$ will refer to the sequence chosen in Lemma 2.3 at all times.

2.2. The Radial Differential Operators D_s^t

If $f \in h(\mathbb{B})$, then f has a homogeneous expansion $f(x) = \sum_{k=0}^{\infty} f_k(x)$ with homogeneous harmonic polynomials f_k of degree k. The series converges absolutely and uniformly on compact subsets of $\mathbb{B}$.

For every $\alpha \in \mathbb{R}$, $R_\alpha(x, y)$ is harmonic as a function of either of its variables on $\overline{\mathbb{B}}$. We have by (7)

$$\gamma_k(\alpha) \sim k^{1+\alpha} \quad (k \to \infty) \tag{9}$$

for every $\alpha \in \mathbb{R}$. By using the coefficients $\gamma_k(\alpha)$ in the Bergman-Besov kernels, we define the radial differential operators D_s^t of order t.

Definition 2.4. Let $f = \sum_{k=0}^{\infty} f_k \in h(\mathbb{B})$ be given by its homogeneous expansion. For $s, t \in \mathbb{R}$ we define D_s^t of order t by

$$D_s^t f := \sum_{k=0}^{\infty} \frac{\gamma_k(s+t)}{\gamma_k(s)} f_k.$$

By (9), $\gamma_k(s+t)/\gamma_k(s) \sim k^t$ for any s, t. For every $s \in \mathbb{R}$, $D_s^0 = I$, the identity. The additive property $D_{s+t}^z D_s^t = D_s^{z+t}$ of D_s^t implies that it is invertible with two-sided inverse

$$D_{s+t}^{-t} D_s^t = D_s^t D_{s+t}^{-t} = I. \tag{10}$$

For every $s, t \in \mathbb{R}$, the operator $D_s^t : h(\mathbb{B}) \to h(\mathbb{B})$ is continuous in the topology of uniform convergence on compact subsets (see (Gergün et al. 2016, Theorem 3.2)). The operator D_s^t is constructed so that in all cases

$$D_s^t R_s(x, y) = R_{s+t}(x, y), \tag{11}$$

where differentiation is performed on one of the variables.

One of the most crucial properties about the map D_s^t is that it enables us to pass from one Bloch space to another. Moreover, we have the following isomorphism. For a proof see (Doğan and Üreyen 2018, Proposition 4.6).

Lemma 2.5. *Given α, for any $s, t \in \mathbb{R}$, the map $D_s^t : b_\alpha^\infty \to b_{\alpha+t}^\infty$ is an isomorphism.*

The following duality result is (Doğan and Üreyen 2018, Theorem 5.4).

Theorem 2.6. *Let $q \in \mathbb{R}$. Pick s', t' such that*

$$s' > q,$$
$$q + t' > -1.$$

The dual of b_q^1 can be identified with b_α^∞ (for any $\alpha \in \mathbb{R}$) under the pairing

$$\langle f, g \rangle = \int_{\mathbb{B}} I_{s'}^{t'} f \, \overline{I_{t'+q+\alpha}^{s'-q-\alpha} g} \, d\nu_{q+\alpha}, \qquad (f \in b_q^1, \ g \in b_\alpha^\infty).$$

2.3. Estimates on Harmonic Bergman-Besov Kernels

In case $\alpha > -1$, Bergman Kernels $R_\alpha(x, y)$ are real-valued and well-studied by many authors. The curious reader is referred to Gergün et al. (2016) for extension of these properties to all $\alpha \in \mathbb{R}$.

We have the following pointwise upper bounds on the Bergman-Besov kernels. For a proof see Coifman and Coifman (1980); Ren (2003) when $\alpha > -1$ and Gergün et al. (2016) when $\alpha \in \mathbb{R}$.

Lemma 2.7. *Let $\alpha \in \mathbb{R}$. For all $x, y \in \mathbb{B}$,*

$$|R_\alpha(x, y)| \lesssim \begin{cases} \dfrac{1}{[x, y]^{\alpha+n}}, & \text{if } \alpha > -n; \\ 1 + \log \dfrac{1}{[x, y]}, & \text{if } \alpha = -n; \\ 1, & \text{if } \alpha < -n. \end{cases}$$

The next result shows that the first part of the above estimate continues to hold when x and y are close enough in the pseudo-hyperbolic distance. It can be proved in just the same way as (Miao 1998, Proposition 5).

Lemma 2.8. *Assume $\alpha > -n$. Then there exists a $\delta \in (0, 1)$ such that*

$$R_\alpha(x, y) \sim \frac{1}{(1 - |x|^2)^{\alpha+n}}$$

whenever $x \in \mathbb{B}$ and $y \in E_\delta(x)$.

3. CARLESON MEASURES

Carleson measures on more general domains have been well studied by many authors. In this subsection we will recollect some characterizations of (vanishing) α-Carleson measures for b^p_α ($\alpha > -1$) in terms of the averaging functions.

Let $0 < \delta < 1$, the averaging function $\widehat{\mu}_\delta$ of μ is defined by

$$\widehat{\mu}_\delta(x) = \frac{\mu(E_\delta(x))}{\nu(E_\delta(x))} \qquad (x \in \mathbb{B}).$$

Also, for general case $\alpha \in \mathbb{R}$ we define

$$\widehat{\mu}_{\alpha,\delta}(x) := \frac{\mu(E_\delta(x))}{\nu_\alpha(E_\delta(x))} \qquad (x \in \mathbb{B}).$$

By (8), $\widehat{\mu}_{\alpha,\delta}(x) \sim \mu(E_\delta(x))/(1-|x|^2)^{\alpha+n}$.

Now, we cite the next characterization of Carleson measures in terms of averaging functions which justify the fact that the notion of α-Carleson measures on b^p_α depend only on α.

Theorem 3.1. *Assume μ is a positive Borel measure on $\mathbb{B}$, $0 < p < \infty$ and $\alpha > -1$. The following are equivalent:*

(a) μ is a α-Carleson measure.
(b) $\widehat{\mu}_{\alpha,\delta} \lesssim 1$ for some (every) $0 < \delta < 1$.

Notice that the condition (b) is equivalent to

$$\mu(E_\delta(x)) \lesssim (1-|x|^2)^{\alpha+n} \quad \text{for some (every) } 0 < \delta < 1.$$

Proof. For the case $\alpha = 0$, equivalence of (a) and (b) is given in (Choe et al. 2004a, Theorem 3.5) for bounded smooth domains. The proof works just as well for general α too.

We also need the following proposition. Its proof is similar to that of (Doğan 2022, Proposition 3.6), but for the sake of completeness, we give the simplified version of it.

Proposition 3.2. *Let μ be a positive Borel measure on $\mathbb{B}$. Let $\alpha_1 > 0$ and $-1 < \alpha_2 < \infty$ and let*

$$\varrho = \alpha_1 + \alpha_2.$$

If μ is a ϱ-Carleson measure, then

$$\int_{\mathbb{B}} |f(x)||g(x)|\, d\mu(x) \lesssim \|f\|_{b^\infty_{\alpha_1}} \|g\|_{b^1_{\alpha_2}} \qquad (f \in b^\infty_{\alpha_1}, g \in b^1_{\alpha_2}).$$

Proof. First, for $f \in b^\infty_{\alpha_1}$,

$$\int_{\mathbb{B}} |f(x)||g(x)|\, d\mu(x) \le \|f\|_{b^\infty_{\alpha_1}} \int_{\mathbb{B}} |g(x)|(1-|x|^2)^{-\alpha_1}\, d\mu(x).$$

Next, by Theorem 3.1, if μ is a $\varrho = \alpha_1 + \alpha_2$ Carleson measure, that is, $\mu(E_\delta(x)) \lesssim (1-|x|^2)^{\alpha_1+\alpha_2+n}$, then $(1-|x|^2)^{-\alpha_1}\, d\mu(x)$ is an α_2-Carleson measure since by Lemma 2.2,

$$\int_{E_\delta(x)} (1-|y|^2)^{-\alpha_1} d\mu(y) \sim (1-|x|^2)^{-\alpha_1} \mu(E_\delta(x)) \lesssim (1-|x|^2)^{\alpha_2+n}.$$

Thus by the definition of a Carleson measure

$$\int_{\mathbb{B}} |g(x)|(1-|x|^2)^{-\alpha_1}\, d\mu(x) \lesssim \|g\|_{b^1_{\alpha_2}}$$

for all $g \in b^1_{\alpha_2}$, which concludes the proof.

We next present a characterization of vanishing α-Carleson measures.

Theorem 3.3. *Let μ be a positive Borel measure on $\mathbb{B}$, $0 < p < \infty$ and $\alpha > -1$. The following are equivalent:*

(a) μ is a vanishing α-Carleson measure.
(b) $\lim_{|x|\to 1^-} \widehat{\mu}_{\alpha,\varepsilon}(x) = 0$ for some (every) $0 < \varepsilon < 1$.
(c) $\lim_{k\to\infty} \widehat{\mu}_{\alpha,\delta}(a_k) = 0$ for some (every) $0 < \delta < 1$.

Proof. For the case $\alpha = 0$, equivalence of (a), (b) and (c) is given in (Choe et al. 2004b, Theorem 3.5) for bounded smooth domains. It works just as well for general α too.

4. BOUNDEDNESS AND COMPACTNESS OF TOEPLITZ OPERATORS

Our goal in this section is to prove Theorems 1.2 and 1.3. Before that we introduce a helpful relation for transforming certain problems for general Toeplitz operators between b^∞_α, $\alpha \in \mathbb{R}^n$ to similar problems for classical Toeplitz operators between b^∞_α when $\alpha > 0$. The harmonic and holomorphic Bergman-Besov-space versions are in Doğan (2022) and Alpay and Kaptanoğlu (2007), respectively.

Theorem 4.1. *We have $D^t_s({}_{s,t}T_\mu) = ({}_{s+t}T_\kappa)D^t_s$, where*

$$ {}_{s+t}T_\kappa f(x) = \frac{1}{V_s}\int_{\mathbb{B}} R_{s+t}(x,y)f(y)d\kappa(y) $$

is the classical Toeplitz operator from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$. Consequently,

$$ ({}_{s,t}T_\mu) = D^{-t}_{s+t}({}_{s+t}T_\kappa)D^t_s, \qquad ({}_{s+t}T_\kappa) = D^t_s({}_{s,t}T_\mu)D^{-t}_{s+t}. $$

Proof. By differentiation under the integral sign and (11), we have

$$ \begin{aligned} D^t_s({}_{s,t}T_\mu f)(x) &= \frac{1}{V_s}\int_{\mathbb{B}} R_{s+t}(x,y)D^t_s f(y)d\kappa(y) \\ &= ({}_{s+t}T_\kappa)(D^t_s f)(x) \qquad (f \in b^\infty_{\alpha_1}). \end{aligned} $$

For the other statements, we note that $(D^t_s)^{-1} = D^{-t}_{s+t}$ by (10).

By Theorem 4.1, ${}_{s,t}T_\mu$ is bounded from $b^\infty_{\alpha_1}$ to $b^\infty_{\alpha_2}$ if and only if ${}_{s+t}T_\kappa$ is bounded from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$. With all these preliminary works, we have laid the groundwork for proving our main results.

4.1. Proof of Theorem 1.2

(i) Implies (ii). Let ${}_{s,t}T_\mu : b^\infty_{\alpha_1} \to b^\infty_{\alpha_2}$ be bounded. First note that $[x,y] \gtrsim (1-|x|^2)$ and $[x,y] \gtrsim (1-|y|^2)$ for $x, y \in \mathbb{B}$. Then fix $x \in \mathbb{B}$ and consider $R_{s+t}(x,.)$. Under the condition $n+s+t > \alpha_1 + t$ provided by (6), it is elementary to verify using Lemma 2.7 that $R_{s+t}(x,.) \in b^\infty_{\alpha_1+t}$ with

$$ \|R_{s+t}(x,.)\|_{b^\infty_{\alpha_1+t}} \lesssim \sup_{y\in\mathbb{B}} \frac{(1-|y|^2)^{\alpha_1+t}}{[x,y]^{n+s+t}} \lesssim \sup_{y\in\mathbb{B}} \frac{(1-|y|^2)^{\alpha_1+t}}{(1-|y|^2)^{\alpha_1+t}(1-|x|^2)^{n+s-\alpha_1}} = (1-|x|^2)^{\alpha_1-(n+s)}. $$

Take $\delta = \delta_0$ where δ_0 is the number made available by Lemma 2.8. We have by Lemma 2.2 and Lemma 2.8

$$ \begin{aligned} \kappa(E_\delta(x)) &\lesssim \frac{V_{\alpha_1}}{V_s}(1-|x|^2)^{2(n+s+t)}\int_{E_\delta(x)} |R_{s+t}(x,y)|^2 d\kappa(y) \\ &\lesssim \frac{V_{\alpha_1}}{V_s}(1-|x|^2)^{2(n+s+t)}\int_{\mathbb{B}} |R_{s+t}(x,y)|^2 d\kappa(y) \\ &= (1-|x|^2)^{2(n+s+t)}\,{}_{s+t}T_\kappa[R_{s+t}(x,.)](x), \end{aligned} $$

and therefore

$$ \begin{aligned} \widehat{\kappa}_{\gamma,\delta}(x) &= \frac{\kappa(E_\delta(x))}{\nu_\gamma(E_\delta(x))} \\ &\lesssim (1-|x|^2)^{2(n+s+t)-(n+\gamma)}\,{}_{s+t}T_\kappa[R_{s+t}(x,.)](x). \end{aligned} $$

On the other hand, by the definition of b^∞_α, $\alpha > 0$, the boundedness of the Toeplitz operator ${}_{s+t}T_\kappa$ and an inequality above, we obtain

$$ \begin{aligned} {}_{s+t}T_\kappa[R_{s+t}(x,.)](x) &= |{}_{s+t}T_\kappa[R_{s+t}(x,.)](x)| \\ &\lesssim (1-|x|^2)^{-t-\alpha_2}\|{}_{s+t}T_\kappa[R_{s+t}(x,.)]\|_{b^\infty_{\alpha_2+t}} \\ &\lesssim (1-|x|^2)^{-t-\alpha_2}\|{}_{s+t}T_\kappa\|\|R_{s+t}(x,.)\|_{b^\infty_{\alpha_1+t}} \\ &\lesssim (1-|x|^2)^{-t-\alpha_2-n-s+\alpha_1}\|{}_{s+t}T_\kappa\|, \end{aligned} $$

where $\|{}_{s+t}T_\kappa\|$ denotes the operator norm of ${}_{s+t}T_\kappa : b^\infty_{\alpha_1+t} \to b^\infty_{\alpha_2+t}$. By bringing these estimates together, we conclude that

$$ \widehat{\kappa}_{\gamma,\delta}(x) \lesssim \|{}_{s+t}T_\kappa\|. $$

By Theorem 3.1 this means that κ is a γ-Carleson measure.

(ii) Implies (i). Next, suppose κ is a γ-Carleson measure. Let

$$ \alpha'_2 = s - \alpha_2 > -1. \tag{12} $$

Since $\alpha_2' > -1$ and $\alpha_2 + t > 0$, applying Theorem 2.6 (with $q = \alpha_2'$, $\alpha = \alpha_2 + t$, $s' = \alpha_2' + \alpha_2 + t$ and $t' = 0$), we get that the dual of $b^1_{\alpha_2'}$ can be identified with $b^\infty_{\alpha_2+t}$ under each of the pairings

$$[f, g]_{b^2_{s+t}} = \int_{\mathbb{B}} f(x)\overline{g(x)}\, d\nu_{s+t}(x).$$

Let $f \in b^\infty_{\alpha_1+t}$ and $h \in b^1_{\alpha_2'}$. Fubini theorem and the reproducing formula (1.5) of Gergün et al. (2016), since $\alpha_2' > -1$ and $\alpha_2' < s + t$ by $\alpha_2 + t > 0$, yield

$$\begin{aligned}[][h, {}_{s+t}T_\kappa f]_{b^2_{s+t}} &= \frac{1}{V_s}\int_{\mathbb{B}} h(y) \int_{\mathbb{B}} R_{s+t}(x, y)\overline{f(x)} d\kappa(x)\, d\nu_{s+t}(y)\\ &= \frac{1}{V_s}\int_{\mathbb{B}} \left(\int_{\mathbb{B}} R_{s+t}(x, y) h(y) d\nu_{s+t}(y)\right)\overline{f(x)}\, d\kappa(x)\\ &= \frac{1}{V_s}\int_{\mathbb{B}} h(x)\overline{f(x)}\, d\kappa(x).\end{aligned}$$

The γ in the statement of the theorem is

$$\gamma = \alpha_1 + t + \alpha_2'.$$

Thus, by Proposition 3.2,

$$|[h, {}_{s+t}T_\kappa f]_{b^2_{s+t}}| \lesssim \int_{\mathbb{B}} |h(x)||f(x)|\, d\kappa(x) \lesssim \|f\|_{b^\infty_{\alpha_1+t}} \|h\|_{b^1_{\alpha_2'}}.$$

By duality we have

$$\begin{aligned}\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} &\lesssim \sup_{\|h\|_{b^1_{\alpha_2'}} \le 1} |[h, {}_{s+t}T_\kappa f_k]_{b^2_{s+t}}|\\ &\lesssim \|f\|_{b^\infty_{\alpha_1+t}}.\end{aligned}$$

Hence ${}_{s+t}T_\kappa$ is bounded from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$.

4.2. Proof of Theorem 1.3

Before going to the proof, it is worth noting that by Theorem 4.1, ${}_{s,t}T_\mu$ is compact from $b^\infty_{\alpha_1}$ to $b^\infty_{\alpha_2}$ if and only if ${}_{s+t}T_\kappa$ is compact from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$. For the proof of Theorem 1.3 we need the following lemma.

Lemma 4.2. *Let $0 < \alpha_1, \alpha_2 < \infty$ and s, t, γ and κ be as in Theorem 1.2. Let ${}_{s+t}T_\kappa$ be a bounded linear operator from $b^\infty_{\alpha_1}$ into $b^\infty_{\alpha_2}$. Then ${}_{s+t}T_\kappa$ is compact if and only if $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2}} \to 0$ as $k \to \infty$ whenever $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1}$ that converges to 0 uniformly on compact subsets of $\mathbb{B}$.*

Proof. The necessity being obvious we will only prove the sufficiency part of the equivalence above. Suppose $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1}$. Note that if $\alpha > 0$, we have by (Doğan and Üreyen 2018, Corollary 5.3)

$$|u(x)| \lesssim \frac{\|u\|_{b^\infty_\alpha}}{(1 - |x|^2)^\alpha} \tag{13}$$

for all $u \in b^\infty_\alpha$ and $x \in \mathbb{B}$. Accordingly, it is uniformly bounded on each compact subset of $\mathbb{B}$ by (13) and thus it is a normal family (see (Axler et al. 2001, Theorem 2.6)). That is, there exists a subsequence of $\{f_k\}$ that converges uniformly on compact subsets of $\mathbb{B}$ to a bounded harmonic function f on $\mathbb{B}$; for simplicity we denote this subsequence by $\{f_k\}$ as well. The sequence $\{f_k - f\}$ is therefore bounded in $b^\infty_{\alpha_1}$ and converges to 0 uniformly on compact subsets of $\mathbb{B}$. By assumption $\|{}_{s+t}T_\kappa(f_k - f)\|_{b^\infty_{\alpha_2}} \to 0$ as $k \to \infty$. This implies that the subsequence $\{{}_{s+t}T_\kappa f_k\}$ converges in $b^\infty_{\alpha_2}$ (to ${}_{s+t}T_\kappa f$). The proof is complete.

(i) Implies (ii). Since ${}_{s+t}T_\kappa$ is compact, then $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0$ whenever $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1+t}$ that converges to 0 uniformly on compact subsets of $\mathbb{B}$ by Lemma 4.2. Let $\{a_k\} \subset \mathbb{B}$ with $|a_k| \to 1^-$ and consider the functions

$$f_k(x) = (1 - |a_k|^2)^{n+s-\alpha_1} R_{s+t}(x, a_k).$$

Under the assumptions on s and Lemma 2.7, since $[x, y] \gtrsim (1-|x|^2)$ and $[x, y] \gtrsim (1-|y|^2)$ for $x, y \in \mathbb{B}$, we get $\sup_k \|f_k\|_{b^\infty_{\alpha_1+t}} < \infty$, and it is clear that f_k converges to 0 uniformly on compact subsets of $\mathbb{B}$. Thus $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0$. Therefore, proceeding as in (i)

Implies (ii) in Theorem 1.2, for any $\delta > 0$, we obtain

$$\begin{aligned}\widehat{\kappa}_{\gamma,\delta}(a_k) &\lesssim (1-|a_k|^2)^{2(n+s+t)-(n+\gamma)}|{}_{s+t}T_\kappa[R_{s+t}(a_k,.)](a_k)|\\ &= (1-|a_k|^2)^{2(n+s+t)-(n+\gamma)-(n+s-\alpha_1)}|{}_{s+t}T_\kappa f_k(a_k)|\\ &= (1-|a_k|^2)^{\alpha_2+t}|{}_{s+t}T_\kappa f_k(a_k)|\\ &\lesssim \|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0.\end{aligned}$$

Hence, by Theorem 3.3, the measure κ is a vanishing γ-Carleson measure.

(ii) Implies (i). Finally, assume that κ is a vanishing γ-Carleson measure. In particular, it is a γ-Carleson measure and thus ${}_{s+t}T_\kappa : b^\infty_{\alpha_1+t} \to b^\infty_{\alpha_2+t}$ is bounded by Theorem 1.2. To show that the operator ${}_{s+t}T_\kappa$ is compact, we must prove that $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0$ whenever $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1+t}$ converging to 0 uniformly on compact subsets of $\mathbb{B}$ by Lemma 4.2. Similarly, as in the proof of Theorem 1.2, by duality we have (the number α_2' being the one defined by (12)

$$\begin{aligned}\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} &\lesssim \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} |[h, {}_{s+t}T_\kappa f_k]_{b^2_{s+t}}|\\ &\le \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} \int_{\mathbb{B}} |f_k(x)||h(x)|d\kappa(x).\end{aligned}$$

Let $0 < \delta < 1$. Since $E_{\delta/2}(x)$ is also a Euclidean ball with center at $c = (1-(\delta/2)^2)x/(1-(\delta/2)^2|x|^2)$ and its radius behaves like $1-|x|^2$ when $\delta/2$ is fixed, (Doğan 2020, Lemma 3.3) implies that

$$|f_k(x)h(x)| \lesssim \frac{1}{r^n}\int_{B(x,r)} |f_k(y)h(y)|d\nu(y),$$

whenever $B(x,r) = \{y : |y-x| < r\} \subset E_{\delta/2}(x)$ for all $x \in \mathbb{B}$. This directly leads to the estimate

$$|f_k(x)h(x)| \lesssim \frac{1}{(1-|x|^2)^{n+\gamma}}\int_{E_{\delta/2}(x)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y) \quad (x \in \mathbb{B}).$$

Note that $E_{\delta/2}(x) \subset E_\delta(a)$ for $a \in \mathbb{B}$ and $x \in E_{\delta/2}(a)$. Let $E_{\delta/2}(a_i)$ be the balls related to the sequence $\{a_i\} = \{a_i(\delta/2)\}$ in Lemma 2.3. So we obtain

$$\begin{aligned}|f_k(x)h(x)| &\lesssim \frac{1}{(1-|x|^2)^{n+\gamma}}\int_{E_{\delta/2}(x)} |f(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\\ &\lesssim \frac{1}{(1-|x|^2)^{n+\gamma}}\int_{E_\delta(a_i)} |f(y)h(y)|(1-|y|^2)^\gamma d\nu(y), \quad x \in E_{\delta/2}(a_i)\end{aligned}$$

for $i = 1, 2, \ldots$. Then Lemma 2.3 and Lemma 2.2 yield

$$\begin{aligned}&\int_{\mathbb{B}} |f_k(x)h(x)|\, d\kappa(x)\\ &\lesssim \sum_{i=1}^\infty \int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x)\\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + \sum_{i\ge j}\int_{E_\delta(a_i)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\int_{E_{\delta/2}(a_i)} \frac{d\kappa(x)}{(1-|x|^2)^{n+\gamma}}\\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + \sum_{i\ge j}\frac{\kappa(E_{\delta/2}(a_i))}{(1-|a_i|^2)^{n+\gamma}}\int_{E_\delta(a_i)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + \sup_{i\ge j}\widehat{\kappa}_{\gamma,\delta}(a_i)\sum_{i\ge j}\int_{E_\delta(a_i)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + N\sup_{i\ge j}\widehat{\kappa}_{\gamma,\delta}(a_i)\int_{\mathbb{B}} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\end{aligned}$$

for any j where N denotes the number provided by Lemma 2.3. Fix j and let $k \to \infty$. Since f_k converges to 0 uniformly on each

$E_{\delta/2}(a_i)$, the $i < j$ terms go to 0. The result is

$$\limsup_{k} \sup_{\|h\|_{b^1_{\alpha_2'}} \le 1} \int_{\mathbb{B}} |f_k(x)h(x)|\, d\kappa(x) \lesssim \sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \sup_{k} \sup_{\|h\|_{b^1_{\alpha_2'}} \le 1} \int_{\mathbb{B}} |f_k(y)h(y)|(1-|y|^2)^{\gamma} d\nu(y)$$
$$\lesssim \sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \sup_{k} \|f_k\|_{b^\infty_{\alpha_1+t}} \sup_{\|h\|_{b^1_{\alpha_2'}} \le 1} \int_{\mathbb{B}} |h(y)| d\nu_{\alpha_2'}(x)$$
$$\lesssim \sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \sup_{k} \|f_k\|_{b^\infty_{\alpha_1+t}}$$

for each j. Now let $j \to \infty$. Since $b^\infty_{\alpha_1+t}$-norms of f_k are bounded and $\sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \to 0$ by assumption, it follows that

$$\sup_{\|h\|_{b^1_{\alpha_2'}} \le 1} \int_{\mathbb{B}} |f_k(x)||h(x)| d\kappa(x) \to 0.$$

Thus, $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+p_2 t}} \to 0$, finishing the proof.

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LIST OF AUTHOR ORCIDS

Ö. Faruk Doğan https://orcid.org/0000-0002-0168-1456

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