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Research Article

n-Dimensional Lattice Path Enumeration

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Abstract: Let $V_n := \{(h_1, h_2, \dots, h_n) \ : h_i \in \{0,1\}, 1 \le i \le n\} - \{(0)\}$ be a set of integer vectors. We enumerate lattice paths that only uses vectors in V_n. Unlike most lattice path enumeration problems, the number of dimensions isn't fixed and the vector set is dependent on the dimension. This requires us to follow a different approach in explicitly expressing the number of lattice paths from origin to any point in n-dimensional space. We notice that a special case of this problem corresponds to Fubini numbers, which count the number of weak orderings of a set consisting of n elements. Then, we find the recursive relation of this sequence. Finally, we develop an algorithm that can be used to find the number of paths between any two points that do not touch the lattice points in $\mathbb{R}$. The crucial part of our algorithm is that it doesn't rely on finding all paths and checking each path for usage of restricted points.

Keywords: Lattice paths, forbidden paths, binary paths, enumeration in n dimensions.

1. Introduction

In the literature, lattice path is defined as; one of the shortest paths from one point to another in a model that consists of horizontal and vertical paths that intersect each other perpendicularly. Various researches have been carried out on the "lattice path" for many years. These studies gained momentum, especially after the 19th century and the most comprehensive studies on the subject have been made in recent years. We refer the reader [5] for a history of lattice path enumeration. A Hamiltonian path is a path that visits each vertex of a graph exactly once.

A Hamiltonian loop is a loop that visits each vertex exactly once. A graph containing a Hamilton cycle is also called a Hamiltonian graph. Determining whether such paths and loops exist in graphs is called the Hamiltonian path problem. In the study of E. Goodman and T.V. Narayana in 1969 [3], lattice paths were examined by including cross-steps. In 1976, B.R. Handa and S.G. Mohanty [4] conducted studies on lattice paths in high dimensions. Similarly, in a 1982

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article by A. Itai, C.H. Papadimitriou, and J.L. Szwarcfiter [6], the applications of Hamiltonian paths, cycles, and graphs in grid graphs were examined.

The Delannoy number refers to the paths used in mathematics to get from the south-west corner of a grid to the northeast corner in just simple steps (north, east and north-east). Most of Delannoy's work between 1886 and 1898 solved different mathematical problems using a chessboard. C. Krattenhaler and S.G. Mohanty [8], C. Krattenhaler [7], G. Mohanty [9] has done many studies on lattice paths. J.M. Autebert, M. Latapy and S.R. Schwer [1] brought their work "The lattice of Delannoy paths" to the literature in English and French. Later in 2003, J.M. Autebert and S.R. Schwer [2] expanded this concept (Delannoy path) to n-dimensional space and defined it over a particular type of alphabet (S-Alphabet) in their study called "On Generalized Delannoy Paths".

We enumerate lattice paths in an n-dimensional space for a fixed set of vectors $V_n := \{(h_1, h_2, \ldots, h_n) \ : h_i \in \{0,1\}, 1 \le i \le n\} - \{(0)\}$. In [10, 11] lattice paths are studied in n-dimensions. Our goal is to find a formula that gives the number of paths from the origin to the point $(l_1, l_2, \ldots, l_n)$ using only the vectors in V_n for $n \ge 2$. We usually refer to these vectors as *steps*. Figure 1 gives concrete examples of such lattice paths in 2 and 3-dimensional spaces. For example, when $n = 2$ we get the set of steps $V_2 = \{(1,0),(0,1),(1,1)\}$ which has been studied many times.

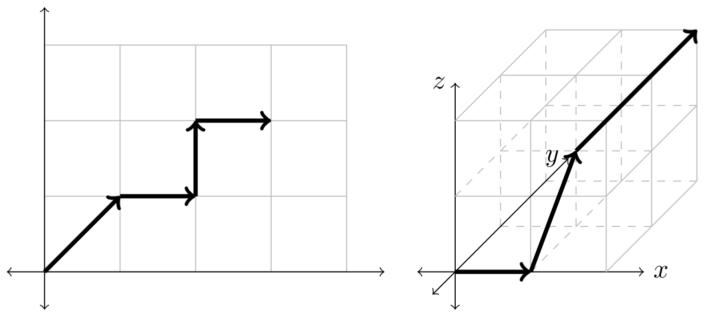

Figure 1: Left: A lattice path terminating at $(3,2)$ - Right: A lattice path terminating at $(2,2,2)$ consisting of vectors $(1,0,0)$, $(0,1,1)$ and $(1,1,1)$

The formula we found is

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!} \prod_{i=1}^{n} \left\lfloor x^{-\left[\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right)+r_i-l_i\right]^2} \right\rfloor \tag{1}$$

with

$$f_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} r_{S+v}, & \text{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v} \right), & \text{if } i \neq p+1 \end{cases},$$

$S = \sum_{s=1}^{k} \binom{n}{s}$, $m_0 = i$, $g_0 = p$, $g_t = g_{t-1} - m_{t-1} + m_t + 1$ and $x > 1$.

First, we want to note that n is the number of dimensions in our space. This formula finds all possible paths (including the ones that do not terminate at the desired point) in n-dimensional space, then chooses the ones that terminate at the desired point $(l_1, l_2, \dots, l_n)$.

2. Finding All Paths in n-Dimensional Space

There are $2^n - 1$ steps in V_n. Each a_j with $1 \leq j \leq 2^n - 1$ represents a different step in V_n. r_j is the number of a_j steps used in a path. A bundle of steps is an unordered group of steps that do not have to be different. Given all values of r_j for $1 \leq j \leq 2^n - 1$, we can form exactly one bundle of steps. For example for $n = 2$ and $(r_1, r_2, r_3) = (3, 1, 0)$, we get the bundle (a_1, a_1, a_1, a_3).

Lemma 2.1 *The number of all possible paths in n-dimensional space can be found with*

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!}. \tag{2}$$

Proof All possible bundle of steps comes from the sums $\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty}$. For a given bundle of steps, $\frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!}$ finds all possible permutations of that bundle. □

3. Finding Paths That Terminate at the Desired Point

In this section, we find a formula that determines whether a path terminates at the desired point or not. Determining the terminal point of a path is the same as determining the terminal point of the bundle that the path was created. It follows because all arrangements of a bundle of steps terminate at the same point. The following part of our formula

$$\prod_{i=1}^{n} \left\lfloor e^{-\left[\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right) + r_i - l_i\right]^2} \right\rfloor$$

finds the distance traveled on all axes for a given bundle and multiplies (2) by 1 if the terminal point is the desired point, multiplies by 0 if not.

3.1. Arranging the Steps

We systematically assign vectors to the notations of the form a_j. First, we establish another notation for steps. Let $(h_1, h_2, \dots, h_n)$ be a step in V_n. We write d_b to the notation of that step for every $h_b = 1$ with $1 \leq b \leq n$ and we have $u < y$ for $\dots d_u d_y \dots$. For example, the step $(1, 0, 0, 1)$ is given by the notation $d_1 d_4$. This notation tells which axes the steps move on.

Note that the notation $d_1 d_4$ represents $(1, 0, 0, 1)$ only if $n = 4$. For example, $d_1 d_4$ represents $(1, 0, 0, 1, 0)$ for $n = 5$.

The length of a step is the number of axes that step moves on (also the number of 1's in the vector $(h_1, h_2, \ldots, h_n)$ and the number of d_b terms in the notation of that step). We start sorting the steps by their length. The length of the steps ascends from 1 to n.

Lemma 3.1 *The notations a_j with $\sum_{s=1}^{k}\binom{n}{s} \leq j \leq \sum_{s=1}^{k+1}\binom{n}{s}$ represents steps with length of $k+1$.*

Proof It is easy to see that there are $\binom{n}{k}$ steps with length of k. The number of all steps with a length smaller than $k+1$ is $\sum_{s=1}^{k}\binom{n}{s}$. In our system of arranging steps, these steps that have length smaller than $k+1$ comes before (by comes before, we mean $y < u$ for a_y being a step with length less than $k+1$ and a_u being a step with a length of $k+1$) those with a length of $k+1$. □

Now we turn our attention to arranging steps of a fixed length. The arrangement of steps is very similar to an alphabetical arrangement. Assume d_b denotes the b-th letter in the alphabet. For example, d_1 denotes a, d_2 denotes b, d_3 denotes c and so on. We transform the notations consisting of d_b's to words. For example, $d_1d_3d_4d_8$ transforms into acdh. Next we do the classic alphabetical arrangement. The arrangement of steps for $n = 4$ is shown below:

$a_1 = d_1$	$a_5 = d_1d_2$	$a_{11} = d_1d_2d_3$	$a_{15} = d_1d_2d_3d_4$
$a_2 = d_2$	$a_6 = d_1d_3$	$a_{12} = d_1d_2d_4$	
$a_3 = d_3$	$a_7 = d_1d_4$	$a_{13} = d_1d_3d_4$	
$a_4 = d_4$	$a_8 = d_2d_3$	$a_{14} = d_2d_3d_4$	
	$a_9 = d_2d_4$		
	$a_{10} = d_3d_4$		

3.2. Distance Traveled on one Axis

We need to determine the distance traveled on a specific axis for a given bundle of steps. We call this axis the observed axis and represent it with d_i. We want to find all steps that have d_i in its notation. We can show such steps with

$$\underbrace{\overbrace{\ldots\ d_i}^{\text{p+1}}\overbrace{\ldots}^{\text{k-p}}}_{\text{k+1}}. \tag{3}$$

As shown in the notation, $k+1$ is the length of the steps and p is the number of d_b terms that are written before the observed axis. This tells that there are $k-p$ d_b terms written after the observed axis.

Lemma 3.2 *(i) The valid interval for i is $1 \leq i \leq n$.*

(ii) The valid interval for k *is* $0 \leq k \leq n-1$.

(iii) The valid interval for p *is* $\max(0, k+i-n) \leq p \leq \min(k, i-1)$.

Proof

(i) There are n axes in the n-dimensional space.

(ii) The minimum length of a step is 1 and the maximum length of a step is n. Hence, $0 \leq k \leq n-1$.

(iii) There are $i-1$ choices of axes before the observed axis and we choose p of them. Hence $p \leq i-1$. There can be only k more terms other than the observed axis, since the length of a step is $k+1$. Hence, we get $p \leq k$. It is easy to see that it is necessary to choose the smaller one of $i-1$ and k for the maximum valid value of p.

There are $n-i$ choices of axes after the observed axis and we choose $k-p$ of them. Hence, we get $p \leq k+i-n$. On the other hand, we know $p \geq 0$. It is easy to see that it is necessary to choose the greater one of these values for the minimum valid value of p.

□

Lemma 3.3 *Let the part before the observed axis be fixed, more specifically* D. *Let* a_{q+1} *be the step* $Dd_i d_{i+1} d_{i+2} \ldots d_{i+k-p-1} d_{i+k-p}$. *Then, all* a_j *steps with* $q+1 \leq j \leq j + \binom{n-i}{k-p}$ *travel on the* i*-th axis.*

Proof For a fixed part before the observed axis, there are $\binom{n-i}{k-p}$ steps. There are $n-i$ possible axes that can be written after the observed axis and we choose $k-p$ of them.

Next, we show that all of these steps are consecutive. Because of the alphabetical arrangement that we made, the observed axis and the part before it does not change until we go through all different $\binom{n-i}{k-p}$ combinations for the part after the observed axis. □

Lemma 3.4 *All different subsets of* $\{1, 2, 3, \ldots, i-2, i-1\}$ *with* $i-p-1$ *elements are the sets* $\{m_1, m_2, m_3, \ldots, m_{z-1}, m_z\}$ *with* $1 \leq z \leq i-p-1$, $m_{z-1} - 1 \geq m_z \geq i-p-z$, $m_0 = i$ *and* $m_z \in \mathbb{N}$.

Proof Consider all subsets of $\{1, 2, 3, \ldots, i-2, i-1\}$ that consists of $i-p-1$ elements. Arrange each set in descending order. Let m_z be the z-th element from left of a subset, we get $1 \leq z \leq i-p-1$ and $m_{z-1} - 1 \geq m_z$. Next we show that $m_z \geq i-p-z$. There are $i-p-z-1$ elements to the right of m_z which are all smaller than m_z. This implies that $m_z \geq i-p-z$. Lastly we show that $m_0 = i$. We know $i-1 \geq m_1$ as m_1 is the greatest number in a subset. Thus, $m_0 = i$. □

We find all steps that travel on i-th axis for fixed values of i, k and p. We denote the function that finds the coefficients of all such steps in n-dimensional space by $f_n(i, k, p)$. $F_n(i, k, p)$ denotes the function that finds all steps with given i, k and p values. Note that if we find all such steps we can find the distance traveled by simply changing each a_j term with r_j. We say that all steps with given i, k and p values whose parts before the observed axis are the same are a section. For example, when observing d_3 in 6-dimensional space, $d_1d_3d_4$, $d_1d_3d_5$ and $d_1d_3d_6$ is a section.

Theorem 3.5

$$F_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} a_{S+v}, & \textit{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} a_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v} \right), & \textit{if } i \neq p+1 \end{cases} \tag{4}$$

with $S = \sum_{s=1}^{k} \binom{n}{s}$, $m_0 = i$, $g_0 = p$ *and* $g_t = g_{t-1} - m_{t-1} + m_t + 1$.

Proof We first proof the case $i = p+1$. Every step with such i and p values can be denoted by $d_1d_2\ldots d_{i-1}d_i\ldots$. Because of the alphabetical arrangement, these steps precedes those of the same length. By Lemma 3.3, these steps are consecutive. All steps with a length smaller than $k+1$ is $S = \sum_{s=1}^{k} \binom{n}{s}$ and there are $\binom{n-i}{k-p}$ steps because we choose $k-p$ axes out of $n-i$ axis for the part after the observed axis.

Next we proof the case $i \neq p+1$. In the case $i = p+1$, we had to use all d_b with $b < i$ in the notation of a step. But for the case $i \neq p+1$ there is some d_b that is not used in the notation of a step. Out of $i-1$ axes we choose not to use $i-p-1$ of them. m_t terms represent these unused axes. For example, if $(m_1, m_2) = (3,1)$, d_3 and d_1 are not used in the notation of a step. Notice that for a fixed set of unused axes, all steps form a section. The number of steps in a section is $\binom{n-i}{k-p}$.

We split all steps with given i, k and p values into sections and for each section, we find the number of steps before that section. $\sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1}$ generates m_t sets that forms the sections.

Consider not using $d_{m_{i-p-1}}$ but using all d_b with $b < m_{i-p-1}$. If $d_{m_{i-p-1}}$ is observed, the part before it will be fixed. Lemma 3.3 implies that if $d_{m_{i-p-1}}$ is not used, then we must have gone through all different combinations for the part after it. There are $m_{i-p-1}-1$ axes before $d_{m_{i-p-1}}$, thus there are $\binom{n-m_{i-p-1}}{k-g_{i-p-1}}$ different combinations for the part after $d_{m_{i-p-1}}$ with $g_{i-p-1} = m_{i-p-1}-1$. Because steps of the form $d_1d_2\ldots d_{m_{ip-1}-1}d_{m_{ip-1}}$ precedes those of the same length, there are $S + \binom{n-m_{m_{i-p-1}}}{k-g_{i-p-1}}$ steps before the ones that do not use $d_{m_{i-p-1}}$. Note that first $m_{i-p-1}-1$ terms are fixed to $d_1d_2\ldots d_{m_{i-p-1}-1}$ and we denote this by D.

After that, consider not using $d_{m_{i-p-2}}$ but using all d_b with $m_{i-p-1} < b < m_{i-p-2}$. $m_{i-p-2} - m_{i-p-1} - 1$ axes gets fixed after D. If $d_{m_{i-p-2}}$ is observed, there are $\binom{n-m_{m_{i-p-2}}}{k-g_{i-p-2}}$ with $g_{i-p-2} = g_{i-p-1} + m_{i-p-2} - m_{i-p-1} - 1$ different combinations for the part after it. There are $S + \binom{n-m_{i-p-1}}{k-g_{i-p-1}} +$

$\binom{n-m_{i-p-2}}{k-g_{i-p-2}}$ steps before the ones that do not use $d_{m_{i-p-1}}$ and $d_{m_{i-p-2}}$. The same idea applies to all other m_t terms and we get $g_t = g_{t-1} - m_{t-1} + m_t + 1$. If this equation is summed up for $1 \le t \le i-p-1$, we get $g_{i-p-1} = g_0 - m_0 + m_{i-p-1} + i - p - 1$ which implies $g_0 = p$. □

Example 3.6 *Let $n = 4$. The arrangement of these steps is made in Section 3. We can see that $F_4(3,1,1) = (a_6, a_8)$ since $F_4(3,1,1)$ is the steps with a length of 2 and 1 d_b term before d_3 in 4-dimensional space. Plugging the values into ((4)) we get*

$$F_4(3,1,1) = \sum_{z=1}^{1} \sum_{m_z=2-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{1}{0}} a_{S+\left[\sum_{t=1}^{1} \binom{4-m_t}{1-g_t}\right]+v} \right)$$

with $S = 4$, $m_0 = 3$ and $g_0 = 1$. We further simplify,

$$F_4(3,1,1) = \sum_{m_1=1}^{2} a_{\left(4+\binom{4-m_1}{1-g_1}+1\right)}.$$

1. $m_1 = 1$, $g_1 = 0$. We get a_8. 2. $m_1 = 2$, $g_1 = 1$. We get a_6.

Example 3.7 *Let $n = 5$. The arrangement of these steps is*

$a_1 = d_1$	$a_6 = d_1 d_2$	$a_{16} = d_1 d_2 d_3$	$a_{26} = d_1 d_2 d_3 d_4$	$a_{31} = d_1 d_2 d_3 d_4 d_5$
$a_2 = d_2$	$a_7 = d_1 d_3$	$a_{17} = d_1 d_2 d_4$	$a_{27} = d_1 d_2 d_3 d_5$	
$a_3 = d_3$	$a_8 = d_1 d_4$	$a_{18} = d_1 d_2 d_5$	$a_{28} = d_1 d_2 d_4 d_5$	
$a_4 = d_4$	$a_9 = d_1 d_5$	$a_{19} = d_1 d_3 d_4$	$a_{29} = d_1 d_3 d_4 d_5$	
$a_5 = d_5$	$a_{10} = d_2 d_3$	$a_{20} = d_1 d_3 d_5$	$a_{30} = d_2 d_3 d_4 d_5$	
	$a_{11} = d_2 d_4$	$a_{21} = d_1 d_4 d_5$		
	$a_{12} = d_2 d_5$	$a_{22} = d_2 d_3 d_4$		
	$a_{13} = d_3 d_4$	$a_{23} = d_2 d_3 d_5$		
	$a_{14} = d_3 d_5$	$a_{24} = d_2 d_4 d_5$		
	$a_{15} = d_4 d_5$	$a_{25} = d_3 d_4 d_5$		

Now, we show that $F_5(3,2,1) = (a_{21}, a_{24}, a_{25})$. Notice that even though a_{24} and a_{25} are consecutive, they do not form a section. Simplifying (4) gives

$$F_5(4,2,1) = \sum_{z=1}^{2} \sum_{m_z=3-z}^{m_{z-1}-1} a_{15+\left[\sum_{t=1}^{2} \binom{5-m_t}{2-g_t}\right]+1} = \sum_{m_1=2}^{3} \sum_{m_2=1}^{m_1-1} a_{15+\left[\sum_{t=1}^{2} \binom{5-m_t}{2-g_t}\right]+1}.$$

1. $m_1 = 2$, $m_2 = 1$, $g_1 = 0$, $g_2 = 0$. $a_{\left(15+\binom{3}{2}+\binom{4}{2}+1\right)} = a_{25}$.

2. $m_1 = 3$,

   (a) $m_2 = 1$, $g_1 = 1$, $g_2 = 0$. $a_{\left(15+\binom{2}{1}+\binom{4}{2}+1\right)} = a_{24}$.

   (b) $m_2 = 2$, $g_1 = 1$, $g_2 = 1$. $a_{\left(15+\binom{2}{1}+\binom{3}{1}+1\right)} = a_{22}$.

Corollary 3.8 *The distance traveled on i-th axis is*

$$\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p). \tag{5}$$

4. The Results

Let

$$K(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ 0, & \text{if } \alpha \neq 0 \end{cases}. \tag{6}$$

We combine this function above with our results. $\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p) - l_i$ is the difference between the distance traveled and the distance wanted to travel on the i-th axis. We write $\alpha = \sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p) - l_i$ in (6). For a given bundle, we multiply all results for $1 \leq i \leq n$. If the terminal point for that bundle is the desired point, the result will be 1.

Corollary 4.1 *In n-dimensional space, the number of paths from origin to $(l_1, l_2, \ldots, l_n)$ using only vectors in $V_n := \{(h_1, h_2, \ldots, h_n) : h_i \in \{0,1\}, 1 \leq i \leq n\}$ is*

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!} \prod_{i=1}^{n} K\left(\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right) + r_i - l_i\right)$$

with

$$f_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} r_{S+v}, & \text{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v}\right), & \text{if } i \neq p+1 \end{cases},$$

$S = \sum_{s=1}^{k} \binom{n}{s}$, $m_0 = i$, $g_0 = p$, $g_t = g_{t-1} - m_{t-1} + m_t + 1$ *and* $x > 1$.

This formula can be generalized to counting the number of paths between any two lattice points in n-dimensional space.

Corollary 4.2 *In n-dimensional space, the number of paths from $(e_1, e_2, \ldots, e_n)$ to $(l_1, l_2, \ldots, l_n)$ using only vectors in $V_n := \{(h_1, h_2, \ldots, h_n) : h_i \in \{0,1\}, 1 \leq i \leq n\}$ is*

$$\sum_{j=1}^{2^n-1} \sum_{r_j=0}^{\infty} \frac{\left(\sum_{b=1}^{2^n-1} r_b\right)!}{\prod_{c=1}^{2^n-1} r_c!} \prod_{i=1}^{n} \left\lfloor x^{-\left[\left(\sum_{k=1}^{n-1} \sum_{p=\max(0,k+i-n)}^{\min(k,i-1)} f_n(i,k,p)\right) + r_i - (l_i - e_i)\right]^2} \right\rfloor$$

with

$$f_n(i,k,p) = \begin{cases} \sum_{v=1}^{\binom{n-i}{k-p}} r_{S+v}, & \textit{if } i = p+1 \\ \sum_{z=1}^{i-p-1} \sum_{m_z=i-p-z}^{m_{z-1}-1} \left(\sum_{v=1}^{\binom{n-i}{k-p}} r_{S+\left[\sum_{t=1}^{i-p-1} \binom{n-m_t}{k-g_t}\right]+v} \right), & \textit{if } i \neq p+1 \end{cases},$$

$S = \sum_{s=1}^{k} \binom{n}{s}, \; m_0 = i, \; g_0 = p, \; g_t = g_{t-1} - m_{t-1} + m_t + 1$ *and* $x > 1$.

We calculate the number of paths from origin to $(l_1, l_2, \ldots, l_n) = (1, 1, \ldots, 1)$ for $1 \leq n \leq 6$ and we get the sequence $1, 3, 13, 75, 541, 4683$. This sequence is OEIS sequence A000670, also called Fubini numbers. The formula for the n-th number in this sequence is $a_n = \sum_{i=1}^{n} \binom{n}{i} a_{n-i}$.

Corollary 4.3 *Let $L(n)$ be the number of lattice paths from origin to $(l_1, l_2, \ldots, l_n) = (1, 1, \ldots, 1)$ using steps in V_n. Then,*

$$L(n) = \sum_{i=1}^{n} \binom{n}{i} a_{n-i}. \tag{7}$$

Thus, we calculate the number of paths from origin to $(l_1, l_2, \ldots, l_n) = (2, 2, \ldots, 2)$ for $1 \leq n \leq 5$. We get the numbers 1, 13, 409, 23917 and 2244361 which appears as OEIS sequence A055203.

l_2 \ l_1	0	1	2	3	4	5	6	7
7	575							
6	377	6287						
5	231	3417	25695					
4	129	1671	11049	50191				
3	63	705	4047	16081	50191			
2	25	239	1177	4047	11049	25695		
1	7	57	239	705	1671	3417	6287	
0	1	7	25	63	129	231	377	575

Figure 2: The number of paths from origin to $(l_1, l_2, 3)$ using steps in V_3 for $l_1 + l_2 \leq 7$

The numbers in Figure 2 was generated using Python.

5. Recursive Relation

Theorem 5.1 *Let $L(p)$ be the number of lattice paths from origin to p using steps in V_n. The recursive relation in this sequence is*

$$L(l_1, l_2, \dots, l_n) = \sum_{m=2}^{n-1} \sum_{b=2}^{n} \sum_{v_b=0}^{1} \sum_{v_1=1}^{1} \sum_{b=2}^{n} L(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n). \quad (8)$$

Proof From $(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n)$ with $v_i \in \{0,1\}$ for all i, there is only 1 way of directly (without touching any other lattice points besides the one we want to reach) reaching $(l_1, l_2, \dots, l_n)$. If any $v_i \notin \{0,1\}$, there will be no way of directly reaching $(l_1, l_2, \dots, l_n)$. Summing up the values of $L(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n)$ gives us the number of path to $(l_1, l_2, \dots, l_n)$. But the case where $v_i = 0$ for all i is the point at which we are finding the recursive relation. Every time we fix one of v_i to 1 to solve this. □

In fact, the recursive relation in Theorem 5.1 can be generalized for any set of vectors.

Corollary 5.2 *Let K be a set of vectors and $L(l_1, l_2, \dots, l_n)$ be the number the number of lattice paths from origin to $(l_1, l_2, \dots, l_n)$. Then the recursive relation in this sequence is*

$$\sum L(l_1 - v_1, l_2 - v_2, \dots, l_n - v_n)$$

for $v_i \in v$ and $v \in K$.

6. An Algorithm for Lattice Paths with Restricted Points

We developed an algorithm that finds the number of paths from origin to any point without touching the points in $\mathbb{R}$.

Lemma 6.1 *Consider a set of lattice points. There is either one or no arrangements of these points such that i-th coordinate of a point is greater than or equal to the i-th coordinates of previous points for all i.*

Proof Consider two different lattice points $p_1 = (b_1, b_2, \dots, b_m)$ and $p_2 = (c_1, c_2, \dots, c_m)$. It is easy to see that there is either 1 or 0 arrangement such that $b_j \le c_j$ for $1 \le j \le m$ or $c_j \le b_j$ for $1 \le j \le m$. This means that 2 different points are not interchangable. If there is such an arrangement for a set of lattice points, it will be the only such arrangement. □

Corollary 6.2 *Let $L(p, p')$ denote the number of lattice paths from p to p' (without restrictions). The number of paths from p to p' that do not touch the lattice points in $\mathbb{R}$ can be computed by the following algorithm.*

1. *Let* r_m *be an* m*-element subset of* $\mathbb{R}$*.*

2. *For* $p_i \in r_m$*, calculate the quantity* $(-1)^n L(p,p_1)L(p_1,p_2)\dots L(p_m,p')$ *for all permutations of* r_m*. There can not be more than one nonzero value and if there is one, note it down.*

3. *Do this for all subsets of* $\mathbb{R}$*.*

4. *The sum of the results is the number of paths from* p *to* p' *that do not touch the points in* $\mathbb{R}$*.*

Proof $L(p,p_1)L(p_1,p_2)\dots L(p_m,p')$ gives the number of paths from p to p' that touch the points $(p_1,p_2,\dots,p_m)$ in the given order. By Lemma 6, if there is such permutation, there will be only one. We multiply by $(-1)^n$ because of the inclusion exclusion principle. □

The efficiency of our algorithm lies on the fact that it doesn't compute all paths and check whether each path is using one of the restricted points or not. It utilizes the inclusion exclusion principle to avoid computing all paths.

7. Conclusion

A formula counting the number of paths from origin to the point $(l_1,l_2,\dots,l_n)$ using steps in V_n has been found. The recursive relation between these numbers has been found and it has been observed that the technique used to find this recursive relation applies to general sets of vectors. The formula found can be generalized to find the number of paths between two lattice points. It has been observed that the numbers $L(1,1,\dots,1)$ correspond to Fubini numbers which are the number of arrangements of n competitors. Lastly, an algorithm for lattice paths with restricted lattice points has been given.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Alper Vural]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Cemil Karaçam]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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