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On Bivariate Complex Schurer-type Stancu Operators

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Abstract. The study focuses on the approximation features of the bivariate generalization of the complex Schurer form of Stancu-type operators. We have obtained a Voronovskaja type solution that provides quantitative estimates for bivariate complex operators coupled to analytic functions. Furthermore, the exact order of approximation is provided.

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1. Introduction

In the literature, Stancu created a novel category of linear positive operators known as Stancu operators as follows

$$L_{n,r}(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k,r}(x), \quad x\in[0,1], \tag{1.1}$$

where f belongs to the set of continuous functions defined on the interval $[0,1]$, a non-negative integer parameter r, n is a natural number such that $n>2r$. The function $p_{n,k,r}(x)$ is defined as follows:

$$p_{n,k,r}(x)=\begin{cases}(1-x)\,b_{n-r,k}(x); & 0\le k<r\\ (1-x)\,b_{n-k,r}(x)+xb_{n-r,k-r}(x); & r\le k\le n-r\\ xp_{n-r,k-r}(x); & n-r<k<n\end{cases},$$

and

$$b_{n,k}(x)=\begin{cases}\binom{n}{k}x^k(1-x)^{n-k}; & 0\le k\le n\\ 0; & k<0 \text{ or } k>n\end{cases}, \quad x\in[0,1], \tag{1.2}$$

is the well-known Bernstein basis polynomials (see [13, 14]). For the special cases $r=0$ and $r=1$, Stancu operators defined by (1.1) give the classical Bernstein operators. Yang et al. [15], considered multivariate setting of the Stancu operators on a simplex and give that Lipschitz property of the original function is preserved by the these operators. On the other hand, approximation properties of complex forms of the Stancu operators and their some generalizations were investigated in [3–5].

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Stancu utilized the probabilistic technique to create a linear positive operator called $L_{n,r}^{\alpha,\beta}$, which is of the Bernstein type. This operator depends on a non-negative integer r and two real parameters α and β, which satisfies the condition $0 \leq \alpha \leq \beta$,

$$L_{n,r}^{\alpha,\beta}(f,x) = \sum_{k=0}^{n-r} b_{n-r,k}(x)\left[(1-x) f\left(\frac{k+\alpha}{n+\beta}\right) + x f\left(\frac{k+r+\alpha}{n+\beta}\right)\right],$$

where $f \in C[0,1]$ and $x \in [0,1]$ such that r is a non-negative integer, $n \in \mathbb{N}$ such that $n > 2r$ (see in [12]). In addition, the author examined the approximation properties of the $L_{n,r}^{\alpha,\beta}$ operators and derived a formula for the residual of the approximation formula for the operators $L_{n,r} = L_{n,r}^{0,0}$. This was accomplished either through the use of second-order divided differences or by employing an integral representation of the residual. In addition, the author derived an asymptotic estimate for this remainder and utilized the modulus of continuity to examine the order of approximation of the operators $L_{n,r}$.

This equation introduces the $L_{n,p,r}^{\alpha,\beta}$ operator, defined as follows:

$$L_{n,p,r}^{\alpha,\beta}(f,z) = \sum_{k=0}^{n+p-r} b_{n+p-r,k}(z)\left[(1-z) f\left(\frac{k+\alpha}{n+\beta}\right) + z f\left(\frac{k+r+\alpha}{n+\beta}\right)\right],$$

where $f \in C[0,1+p]$ and $x \in [0,1]$ such that α,β are real parameters with $0 \leq \alpha \leq \beta$, r is a non-negative integer, $n \in \mathbb{N}$, $n+p > 2r$, $p \in \mathbb{N} \cup \{0\}$. The approximation properties of this operator for complex variables were studied in depth in [6], providing insights into how this operator behaves in approximating functions within the space $C[0,1+p]$ under complex variables.

The study of approximation properties for operators on various domains, especially on movable compact disks, shows a parallel with the generalizations of complex operators. There are also studies in this field in the literature. The study [11], defines and approximates Durrmeyer-type operators on a simplex, preserving affine functions and connecting multidimensional "genuine" Durrmeyer operators to multidimensional Bernstein operators. The research [1], introduces the Bernstein-Stancu-Chlodowsky operator, a new positive linear extension of Bernstein-Stancu operators in two variables. As n increases, it operates on a triangular domain with moving edges. The investigation [2], develops Bernstein-type operators on a simplex with a movable curved side that reproduce exponential functions. In addition, the paper [10] introduces a complex q-Baskakov-Stancu operator and investigates its approximation properties, offering a quantitative estimate of convergence, a Voronovskaja-type result, and the exact order of approximation in compact disks. These studies together showcase advancements in approximation theory across multiple dimensions and domains.

The objective of this study is to analyze the approximation properties of bivariate complex Schurer-type Stancu operators that have a tensor product structure. We extend the approximation findings presented in [6] for the complex Schurer-type of Stancu operators from the case of a single variable to the case of two variables. For this purpose, we obtain a quantitative upper estimate for the complex Schurer-type of Stancu operator and its derivatives on compact disks. We then determine the exact order of approximation for these operators also the qualitative Voronovskaja type result.

At first, we introduce certain ideas in the two-variable situation that are logical expansions of the standard concepts in the one-variable situation. Let $D_{R_1} := \{z \in \mathbb{C} : |z| < R_1\}$, $D_{R_2} := \{w \in \mathbb{C} : |w| < R_2\}$ and $D_{R_1} \times D_{R_2}$ denotes an open polydisk (of center 0 and radius R), where $R = (R_1, R_2)$ and $|z| \leq r_1, |w| \leq r_2, r_1 < R_1$ with $r_2 < R_2$. Let also $\overline{D_R} := \overline{D}_{R_1} \times \overline{D}_{R_2} = \left\{(z,w) \in \mathbb{C}^2 : |z| \leq R_1, |w| \leq R_2\right\}$ denotes the closed polydisk.

The bivariate complex Schurer-type of Stancu operators were defined in the following,

$$L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)(z,w) = \sum_{k=0}^{n+p-r} \sum_{j=0}^{m+q-s} p_{n+p,k,r}(z)\, p_{m+q,j,s}(w)\, f\left(\frac{k+\alpha}{n+\beta}, \frac{j+\gamma}{m+\delta}\right), \tag{1.3}$$

where, in the polydisk $\overline{D_R}$, f is an analytical function of two complex variables (z,w), $p_{n+p,k,r}(z)$, $p_{m+q,j,s}(w)$ are similar to the previous Equation (1.2), such that $\alpha,\beta,\gamma,\delta$ are real parameters with $0 \leq \alpha \leq \beta$ and $0 \leq \gamma \leq \delta$, r,s are non-negative integers, $n,m \in \mathbb{N}$ such that $n+p > 2r, m+q > 2s$, and $p,q \in \mathbb{N} \cup \{0\}$.

2. Approximation Properties of Bivariate Complex Schurer-Type of Stancu Operators

Theorem 2.1. *Let r, s are a non-negative integers, $n, m \in \mathbb{N}$ such that $n+p > 2r$, $m+q > 2s$ and $f : D_R \to \mathbb{C}$ is analytic in D_R with $f(z,w) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} z^k w^j$, and $R_1 > 1+p+\alpha+\beta$, $R_2 > 1+q+\gamma+\delta$, for fixed $p, q \in \mathbb{N}\cup\{0\}$, $0 \le \alpha \le \beta$, $0 \le \gamma \le \delta$. We have,*

i) *Let $1 \le \rho_1 \le \rho_1(1+p)$, $1 \le \rho_2 \le \rho_2(1+q)$ and $\rho_1(1+p+\alpha+\beta) < R_1$, $\rho_2(1+q+\gamma+\delta) < R_2$ be arbitrary fixed. For all $|z| \le \rho_1$, $|w| \le \rho_2$ we have*

$$\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)(z,w) - f(z,w)\right| \le M_{\rho_1,\rho_2 n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f),$$

where

$$\begin{aligned}
M_{\rho_1,\rho_2 n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f) = {} & \frac{6\rho_2^2(1+q)^2}{m+q+\gamma}\sum_{k=0}^{\infty}\sum_{j=2}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k 2\rho_2^{j-2}(q+1)^{j-2} j(j-1) \\
& + \frac{12(\gamma+s)\rho_2(1+q)}{m+q+\gamma}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k j\left[\rho_2(1+q)\right]^{j-1} \\
& + \frac{3\rho_2}{m+\delta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k\left[(1+q+\gamma+\delta)\right]^j - \frac{3}{m+\delta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k\rho_2^j \\
& + \frac{12\left[\rho_2(1+q+\gamma+\delta)\right]^2}{m+q-s}\sum_{k=0}^{\infty}\sum_{j=2}^{\infty}\left|c_{k,j}\right|\left[\rho_2(1+q+\gamma+\delta)\right]^{j-2} j(j-1) \\
& + \frac{2\rho_1^2(1+p)^2}{n+p+\alpha}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^{k-2}\rho_2^j(1+p)^{k-2}k(k-1) \\
& + \frac{4(r+\alpha)\left[\rho_1(1+p)\right]}{n+p+\alpha}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|(r+\alpha)\left[\rho_1(p+1)\right]^{k-1}\rho_2^j k \\
& + \frac{1}{n+\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left[\rho_1(1+p+\alpha+\beta)\right]^k\rho_2^j - \frac{1}{n+\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^k\rho_2^j \\
& + \frac{4\left[\rho_1(1+p+\alpha+\beta)\right]^2}{n+p-r}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left[\rho_1(1+p+\alpha+\beta)\right]^{k-2}\rho_2^j k(k-1) < \infty.
\end{aligned}$$

ii) *Let $1 \le \rho_1 < \rho_1^* \le \rho_1(p+1)$, $1 \le \rho_2 < \rho_2^* \le \rho_2(q+1)$ and $\rho_1(1+p+\alpha+\beta) < R_1$, $\rho_2(1+q+\gamma+\delta) < R_2$ then for all $|z| \le \rho_1$, $|w| \le \rho_2$ and $n, m, j_1, j_2 \in \mathbb{N}\cup\{0\}$ be with $j_1 + j_2 \ge 1$, we have*

$$\left|\frac{\partial^{j_1+j_2} L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)}{\partial z^{j_1}\partial w^{j_2}}(z,w) - \frac{\partial^{j_1+j_2} f}{\partial z^{j_1}\partial w^{j_2}}(z,w)\right| \le M_{\rho_1^*,\rho_2^* n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)\,\frac{(j_1)!\rho_1^*}{\left(\rho_1^*-\rho_1\right)^{j_1+1}}\,\frac{(j_2)!\rho_2^*}{\left(\rho_2^*-\rho_2\right)^{j_2+1}},$$

where $M_{\rho_1^*,\rho_2^* n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)$ is provided as stated above.

Proof. (i) Denoting $e_{k,j}(z,w) = e_k(z)e_j(w)$ where $e_j(z) = z^j$. By taking into account the Lemma 2.6 from [6] with definition of (1.3) and $f(z,w) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} z^k w^j$ we get,

$$\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)(z,w) - f(z,w)\right| \le \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(e_{k,j})(z,w) - e_{k,j}(z,w)\right|,$$

and here from [8] by following the steps in the proof of Theorem 2.2.1 (i) we can write,

$$\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(e_{k,j})(z,w) - e_{k,j}(z,w)\right| = \left|L_{n,p,r}^{\alpha,\beta}(e_k,z)\,L_{m,q,s}^{\gamma,\delta}\left(e_j,w\right) - z^k w^j\right|. \tag{2.1}$$

By considering the estimates for two cases $0 \leq k \leq n+p-r$ and $k > n+p-r$ from [6](proof of Theorem 3.1) and $0 \leq j \leq m+q-s$ and $j > m+q-s$ for second variable, we write

$$\begin{aligned}\left|L_{n,p,r}^{\alpha,\beta}(e_k,z)-e_k(z)\right| &\leq 2\rho_1^k(1+p)^k\frac{k(k-1)+2k(\alpha+r)}{(n+p+\alpha)}+\frac{\rho_1^k}{n+\beta}\left[(1+p+\alpha+\beta)^k-1\right]\\&+4\frac{k(k-1)}{n+p-r}\left[\rho_1(1+p+\alpha+\beta)\right]^k,\\ \left|L_{n,p,r}^{\alpha,\beta}(e_k)(z)\right| &\leq 3\rho_1^{k+1}(1+p+\alpha+\beta)^k,\end{aligned}$$

and

$$\begin{aligned}\left|L_{m,q,s}^{\gamma,\delta}\left(e_j,w\right)-e_j(w)\right| &\leq 2\rho_2^j(1+q)^j\frac{j(j-1)+2j(\gamma+s)}{(m+q+\gamma)}+\frac{\rho_2^j}{m+\delta}\left[(1+q+\gamma+\delta)^j-1\right]\\&+4\frac{j(j-1)}{m+q-s}\left[\rho_2(1+q+\gamma+\delta)\right]^j,\end{aligned}$$

for $k,j \in \mathbb{N}$, $\rho_1,\rho_2 \geq 1$, and for all $|z| \leq \rho_1$ and $|w| \leq \rho_2$ and by using equation (2.1), we obtain

$$\begin{aligned}\left|L_{n,p,r}^{\alpha,\beta}(e_k,z).L_{m,q,s}^{\gamma,\delta}\left(e_j,w\right)-z^kw^j\right| &\leq \left|L_{n,p,r}^{\alpha,\beta}(e_k,z).L_{m,q,s}^{\gamma,\delta}\left(e_j,w\right)-L_{n,p,r}^{\alpha,\beta}(e_k,z).w^j\right|+\left|L_{n,p,r}^{\alpha,\beta}(e_k,z).w^j-z^kw^j\right|\\&\leq \left|L_{n,p,r}^{\alpha,\beta}(e_k,z)\right|.\left|L_{m,q,s}^{\gamma,\delta}\left(e_j,w\right)-w^j\right|+\left|w^j\right|.\left|L_{n,p,r}^{\alpha,\beta}(e_k,z)-z^k\right|\\&\leq 3\rho_1^k(1+p+\alpha+\beta)^k\left[\frac{2\rho_2^j(q+1)^j}{m+q+\gamma}j(j-1)+\frac{4j(\gamma+s)\left[\rho_2(q+1)\right]^j}{m+q+\gamma}\right.\\&\left.+\frac{1}{m+\delta}\left[\rho_2(1+q+\gamma+\delta)\right]^j-\frac{1}{m+\delta}\rho_2^j+\frac{4\left[\rho_2(1+q+\gamma+\delta)\right]^j}{m+q-s}j(j-1)\right]\\&+\rho_2^j\left[\frac{2\rho_1^k(p+1)^k}{n+p+\alpha}k(k-1)+\frac{4(r+\alpha)\left[\rho_1(p+1)\right]^k}{n+p+\alpha}k+\frac{1}{n+\beta}\left[\rho_1(1+p+\alpha+\beta)\right]^k\right.\\&\left.-\frac{1}{n+\beta}\rho_1^k+\frac{4\left[\rho_1(1+p+\alpha+\beta)\right]^k}{n+p-r}k(k-1)\right],\end{aligned}$$

which the coeffients' criteria imply according to $c_{k,j}$

$$\begin{aligned}\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)(z,w)-f(z,w)\right| &\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(e_{k,j})(z,w)-e_{k,j}(z,w)\right|\\&\leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left\{3\rho_1^k(1+p+\alpha+\beta)^k\left[\frac{2\rho_2^j(q+1)^j}{m+q+\gamma}j(j-1)+\frac{4j(\gamma+s)\left[\rho_2(q+1)\right]^j}{m+q+\gamma}\right.\right.\\&\left.+\frac{1}{m+\delta}\left[\rho_2(1+q+\gamma+\delta)\right]^j-\frac{1}{m+\delta}\rho_2^j+\frac{4\left[\rho_2(1+q+\gamma+\delta)\right]^j}{m+q-s}j(j-1)\right]\\&+\rho_2^j\left[\frac{2\rho_1^k(p+1)^k}{n+p+\alpha}k(k-1)+\frac{4(r+\alpha)\left[\rho_1(p+1)\right]^k}{n+p+\alpha}k\right.\\&\left.\left.+\frac{1}{n+\beta}\left[\rho_1(1+p+\alpha+\beta)\right]^k-\frac{1}{n+\beta}\rho_1^k+\frac{4\left[\rho_1(1+p+\alpha+\beta)\right]^k}{n+p-r}k(k-1)\right]\right\}\\&\leq \frac{6\rho_2^2(1+q)^2}{m+q+\gamma}\sum_{k=0}^{\infty}\sum_{j=2}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k\,2\rho_2^{j-2}(q+1)^{j-2}j(j-1)\end{aligned}$$

$$+\frac{12(\gamma+s)\rho_2(q+1)}{m+q+\gamma}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k j\left[\rho_2(q+1)\right]^{j-1}$$
$$+\frac{3\rho_2}{m+\delta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k\left[(1+q+\gamma+\delta)\right]^j-\frac{3}{m+\delta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^k(1+p+\alpha+\beta)^k\rho_2^j$$
$$+\frac{12\left[\rho_2(1+q+\gamma+\delta)\right]^2}{m+q-s}\sum_{k=0}^{\infty}\sum_{j=2}^{\infty}\left|c_{k,j}\right|\left[\rho_2(1+q+\gamma+\delta)\right]^{j-2}j(j-1)$$
$$+\frac{2\rho_1^2(p+1)^2}{n+p+\alpha}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^{k-2}\rho_2^j(p+1)^{k-2}k(k-1)+\frac{4(r+\alpha)\left[\rho_1(p+1)\right]}{n+p+\alpha}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|(r+\alpha)\left[\rho_1(p+1)\right]^{k-1}\rho_2^j k$$
$$+\frac{1}{n+\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left[\rho_1(1+p+\alpha+\beta)\right]^k\rho_2^j-\frac{1}{n+\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\rho_1^k\rho_2^j$$
$$+\frac{4\left[\rho_1(1+p+\alpha+\beta)\right]^2}{n+p-r}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left[\rho_1(1+p+\alpha+\beta)\right]^{k-2}\rho_2^j k(k-1)$$

this demonstrates (i).

(ii) The simultaneous approximation convergence rate is now available. Let $1\le\rho_1<\rho_1^*\le\rho_1(p+1)$, $1\le\rho_2<\rho_2^*\le\rho_2(q+1)$ and $\rho_1(1+p+\alpha+\beta)<R_1$, $\rho_2(1+q+\gamma+\delta)<R_2$ and $|v_1-z|=\rho_1^*$, $|v_2-w|=\rho_2^*$. Using Cauchy's formula, we obtain

$$\frac{\partial^{j_1+j_2}L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)}{\partial z^{j_1}\partial w^{j_2}}(z,w)-\frac{\partial^{j_1+j_2}f}{\partial z^{j_1}\partial w^{j_2}}(z,w)=\frac{j_1!j_2!}{(2\pi i)^2}\int_{\rho_2^*}\int_{\rho_1^*}\frac{L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)(v_1,v_2)-f(v_1,v_2)}{(v_1-z)^{j_1+1}(v_2-w)^{j_2+1}}dv_1dv_2$$

the use of passing to absolute value $|z|\le\rho_1$,$|w|\le\rho_2$ and taking into consideration that $|v_1-z|\ge\rho_1^*-\rho_1$, $|v_2-w|\ge\rho_2^*-\rho_2$, by using the estimate from (i), we are able to derive

$$\left|\frac{\partial^{j_1+j_2}L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)}{\partial z^{j_1}\partial w^{j_2}}(z,w)-\frac{\partial^{j_1+j_2}f}{\partial z^{j_1}\partial w^{j_2}}(z,w)\right|\le\frac{j_1!j_2!}{(2\pi i)^2}\int_{\rho_2^*}\int_{\rho_1^*}\frac{\left|L_{n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)(v_1,v_2)-f(v_1,v_2)\right|}{|v_1-z|^{j_1+1}|v_2-w|^{j_2+1}}|dv_1dv_2|$$
$$\le\frac{(j_1)!}{2\pi}\frac{(j_2)!}{2\pi}M_{\rho_1^*,\rho_2^*n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)\frac{2\pi\rho_1^*}{\left(\rho_1^*-\rho_1\right)^{j_1+1}}\frac{2\pi\rho_2^*}{\left(\rho_2^*-\rho_2\right)^{j_2+1}}$$
$$\le M_{\rho_1^*,\rho_2^*n,m,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)\frac{(j_1)!\rho_1^*}{\left(\rho_1^*-\rho_1\right)^{j_1+1}}\frac{(j_2)!\rho_2^*}{\left(\rho_2^*-\rho_2\right)^{j_2+1}}$$

of which the theorem is proved. □

The second discovery relates to the Voronovskaja-type theorem concerning operator (1.3). For this, first of all, let's give the theorem in the form of two variables in real spaces which will be used in the proof.

Theorem 2.2 ([9], Theorem 8)**.** *Let $L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f;x,y)$ be real form of the operators given with the definition (1.3), $f\in C^{2,2}([0,1+p]\times[0,1+q])$. Then, for all $(x,y)\in C^{2,2}([0,1+p]\times[0,1+q])$ we have*

$$\lim_{n\to\infty}n\left(L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f;x,y)-f(x,y)\right)=px\frac{\partial f}{\partial x}(x,y)+qy\frac{\partial f}{\partial y}(x,y)+\frac{1}{2}\left\{x(1-x)\frac{\partial^2 f}{\partial x^2}(x,y)+y(1-y)\frac{\partial^2 f}{\partial y^2}(x,y)\right\}.$$

Now, we present the following qualitative asymptotic formula by applying the asymptotic Voronovskaja-type theorem for the bivariate complex Schurer-type of Stancu operator. An explanation of qualitative Voronovskaja theorem is provided below.

Theorem 2.3. *For fixed $p,q\in\mathbb{N}\cup\{0\}$, and $R_1>p+1$, $R_2>q+1$ suppose that in D_R, $f:D_R\to\mathbb{C}$ is analytical with $f(z,w)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_{k,j}z^kw^j$. Then, for all $n\in\mathbb{N}$ and $|z|\le\rho_1$, $|w|\le\rho_2$ where $1\le\rho_1\le\rho_1(1+p)$, $1\le\rho_2\le\rho_2(1+q)$*

and $\rho_1(1+p+\alpha+\beta) < R_1$, $\rho_2(1+q+\gamma+\delta) < R_2$ *we get*

$$\lim_{n\to\infty} n\left(L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f;z,w) - f(z,w)\right) = pz\frac{\partial f}{\partial z}(z,w) + qw\frac{\partial f}{\partial w}(z,w) + \frac{1}{2}\left\{z(1-z)\frac{\partial^2 f}{\partial z^2}(z,w) + w(1-w)\frac{\partial^2 f}{\partial w^2}(z,w)\right\}$$

uniformly.

Proof.

$$\left\{n\left(L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f;z,w) - f(z,w)\right)\right\}_{n\in\mathbb{N}},$$

is the sequence of analytic functions that holds uniformly in any compact disk $\overline{D_\rho} = \overline{D_{\rho_1}} \times \overline{D_{\rho_2}}$ with $1 \le \rho_1(p+1) < R_1$, $1 \le \rho_2(q+1) < R_2$. In fact, the following derives from (i) of Theorem 2.1

$$\left|n\left(L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f;z,w) - f(z,w)\right)\right| \le nM_{\rho_1^*,\rho_2^*}{}^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f) := \mathcal{M}_{\rho_1^*,\rho_2^*}{}^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f)$$

for all $n \in \mathbb{N}$ and $(z,w) \in \overline{D_\rho}$ with $1 \le \rho_1(p+1) < R_1$, $1 \le \rho_2(q+1) < R_2$, here where

$$\mathcal{M}_{\rho_1^*,\rho_2^*}{}^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f) < \infty.$$

Then, in accordance with the Vitali Theorem (eg. [7, p.1]) $\left\{n\left(L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f;z,w) - f(z,w)\right)\right\}_{n\in\mathbb{N}}$ is uniformly convergent in any $\overline{D_\rho}$, this proves the theorem. □

To determine the precise order of approximation by $L^{\alpha,\beta,\gamma,\delta}_{n,m,p,q,r,s}(f)$, the following theorem will be helpful.

Theorem 2.4. *Given a constant value for* $p, q \in \mathbb{N}\cup\{0\}$, *and* $R_1 > p+1$, $R_2 > q+1$ *suppose that* $f : D_R \to \mathbb{C}$ *is analytic in* D_R *with* $f(z,w) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} z^k w^j$ *for all* $(z,w) \in \overline{D_\rho}$. *Denoting* $\|f\|_{\rho_1,\rho_2} = \sup\{|f(z,w)|; |z| \le \rho_1; |w| \le \rho_2\}$ *and the following complex partial differential equation has no solution for* f.

$$pz\frac{\partial f}{\partial z}(z,w) + \frac{z(1-z)}{2}\frac{\partial^2 f}{\partial z^2}(z,w) + qw\frac{\partial f}{\partial w}(z,w) + \frac{w(1-w)}{2}\frac{\partial^2 f}{\partial w^2}(z,w) = 0, |z| \le \rho_1, |w| \le \rho_2,$$

then we have

$$\left\|L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f) - f\right\|_{\rho_1,\rho_2} \ge \frac{\mathcal{K}_{\rho_1,\rho_{2,f}}}{n},$$

where $\mathcal{K}_{\rho_1,\rho_{2,f}}$ *solely relies on* f, r, s, ρ_1, ρ_2.

Proof. For all $|z| \le \rho_1$, $|w| \le \rho_2$ and $n \in \mathbb{N}$, we are able to write

$$\begin{aligned} L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f)(z,w) - f(z,w) = \frac{1}{n}\Bigg\{&\frac{pz}{2}\frac{\partial f}{\partial z}(z,w) + \frac{z(1-z)}{4}\frac{\partial^2 f}{\partial z^2}(z,w) + \frac{pw}{2}\frac{\partial f}{\partial w}(z,w) + \frac{w(1-w)}{4}\frac{\partial^2 f}{\partial w^2}(z,w) \\ &+n\Bigg[L^{\alpha,\beta,\gamma,\delta}_{n,n,p,q,r,s}(f;z,w) - f(z,w) - pz\frac{\partial f}{\partial z}(z,w) - qw\frac{\partial f}{\partial w}(z,w) \\ &-\frac{1}{2}\left\{z(1-z)\frac{\partial^2 f}{\partial z^2}(z,w) - w(1-w)\frac{\partial^2 f}{\partial w^2}(z,w)\right\}\Bigg]\Bigg\} \end{aligned}$$

taking into the inequalities

$$\|N+M\|_{\rho_1,\rho_2} \ge \left|\|N\|_{\rho_1,\rho_2} - \|M\|_{\rho_1,\rho_2}\right| \ge \|N\|_{\rho_1,\rho_2} - \|M\|_{\rho_1,\rho_2}$$

it follows

$$\begin{aligned}\left\|L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)-f\right\|_{\rho_1,\rho_2} &= \Bigg\|\frac{1}{n}\left\{\frac{pz}{2}\frac{\partial f}{\partial z}(z,w)+\frac{z(1-z)}{4}\frac{\partial^2 f}{\partial z^2}(z,w)+\frac{pw}{2}\frac{\partial f}{\partial w}(z,w)+\frac{w(1-w)}{4}\frac{\partial^2 f}{\partial w^2}(z,w)\right. \\ &+n\left[L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f;z,w)-f(z,w)-pz\frac{\partial f}{\partial z}(z,w)-qw\frac{\partial f}{\partial w}(z,w)\right. \\ &\left.\left.-\frac{1}{2}\left\{z(1-z)\frac{\partial^2 f}{\partial z^2}(z,w)-w(1-w)\frac{\partial^2 f}{\partial w^2}(z,w)\right\}\right]\right\}\Bigg\|_{\rho_1,\rho_2} \\ &\geq \frac{1}{n}\left\{\left\|\frac{pz}{2}\frac{\partial f}{\partial z}(z,w)+\frac{z(1-z)}{4}\frac{\partial^2 f}{\partial z^2}(z,w)+\frac{pw}{2}\frac{\partial f}{\partial w}(z,w)+\frac{w(1-w)}{4}\frac{\partial^2 f}{\partial w^2}(z,w)\right\|_{\rho_1,\rho_2}\right. \\ &-n\Bigg\|\left[L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f;z,w)-f(z,w)-pz\frac{\partial f}{\partial z}(z,w)-qw\frac{\partial f}{\partial w}(z,w)\right. \\ &\left.\left.-\frac{1}{2}\left\{z(1-z)\frac{\partial^2 f}{\partial z^2}(z,w)-w(1-w)\frac{\partial^2 f}{\partial w^2}(z,w)\right\}\right]\Bigg\|_{\rho_1,\rho_2}\right\}.\end{aligned}$$

By denoting $N(z,w):=\frac{pz}{2}\frac{\partial f}{\partial z}(z,w)+\frac{z(1-z)}{4}\frac{\partial^2 f}{\partial z^2}(z,w)+\frac{pw}{2}\frac{\partial f}{\partial w}(z,w)+\frac{w(1-w)}{4}\frac{\partial^2 f}{\partial w^2}(z,w)$ and if we prove that $\|N\|_\rho>0$, by Theorem 2.2, there is an index n_0 that depends on f,ρ_1,ρ_2 such that for all values of n greater than or equal to n_0, the following condition holds:

$$\left\|L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)-f\right\|_{\rho_1,\rho_2}\geq\frac{1}{n}\frac{\|N\|_{\rho_1,\rho_2}}{2}.$$

Also, for $n\in\{1,2,...,n_0-1\}$ we obviously write $\left\|L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)-f\right\|_\rho\geq\frac{H_{n,\rho_1,\rho_2}}{n}$, with $H_{n,\rho_1,\rho_2}(f):=n\left\|L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)-f\right\|_\rho>0$. Finally it follows that

$$\left\|L_{n,n,p,q,r,s}^{\alpha,\beta,\gamma,\delta}(f)-f\right\|_\rho\geq\frac{\mathcal{K}_{\rho_1,\rho_2}(f)}{n},$$

for all $n\in\mathbb{N}$, where

$$\mathcal{K}_{\rho_1,\rho_2}(f):=\min\left\{H_{1,\rho_1,\rho_2},H_{2,\rho_1,\rho_2},...,H_{n_0-1,\rho_1,\rho_2},\frac{\|N\|_{\rho_1,\rho_2}}{2}\right\}$$

and the proof is now completed.

□

The exact order is obtained by combining Theorem 2.1 with Theorem 2.4.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author confirms sole responsibility for the study.

References

[1] Acar, T., Aral, A., *Approximation properties of two dimensional Bernstein-Stancu-Chlodowsky operators*, Le Matematiche **68**(2)(2013), 15–31.

[2] Bozkurt, K., Özsaraç, F., Aral, A., *Bivariate Bernstein polynomials that reproduce exponential functions*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.,**70**(1)(2021), 541–554.

[3] Çetin, N., *A new generalization of complex Stancu operators*, Mathematical Methods in the Applied Sciences, **42**(2019), 5582–5594.

[4] Çetin, N., Başcanbaz-Tunca,G., *Approximation by a new complex generalized Bernstein operator*, An. Univ. Oradea Fasc. Mat. **26**(2)(2019), 129–141.

[5] Çetin, N., *A new complex generalized Bernstein-Schurer operator*, Carpathian J. Math., **37**(1)(2021), 81–89.

[6] Çetin, N., Manav Mutlu, N., *Complex generalized Stancu-Schurer operators,* Mathematica Slovaca, **74**(5)(2024), 1215–1232.

[7] Gal, S.G., Approximation by Complex Bernstein and Convoluation Type Operators, World Scientific, Sigapore, 2009.

[8] Gal, S.G., Approximation by Complex Bernstein and Convolution Type Operators, Series on Concrete and Applicable Mathematics, 8. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, USA, 2009.
[9] Manav Mutlu, N., *On new bivariate Schurer-Stancu-type operators,* (submitted).
[10] Özden, D.S., Arı, D.A., *Approximation by a complex q-Baskakov-Stancu operator in compact disks,* Journal of Inequalities and Applications, (2014), 1–15.
[11] Paltanea, R., *Durrmeyer type operators on a simplex*, Constructive Mathematical Analysis, **4**(2)(2021), 215–228.
[12] Stancu, D.D., *Approximation of functions by means of a new generalized Bernstein operator*, Calcolo, **20**(1983), 211–229.
[13] Stancu, D.D., *Quadrature formulas constructed by using certain linear positive operators*, Numerical Integration (Proc. Conf., Oberwolfach, 1981), **57**(1982), 241–251.
[14] Stancu, D.D., *Approximation of functions by means of a new generalized Bernstein operator*, Calcolo, **20**(1983) 211–229.
[15] Yang, R., Xiong, J., Cao, F., *Multivariate Stancu operators defined on a simplex*, Appl. Math. Comput., **138**(2003), 189–198.