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Research Article

A COMPARATIVE ANALYSIS OF TWO SEMI ANALYTIC APPROACHES IN SOLVING SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT: The resolution of systems of first-order ordinary differential equations (ODEs) is a critical endeavor with extensive applications in various scientific and engineering fields. This study presents a rigorous comparative assessment of two semi-analytic methodologies: the Variational Iterative Method (VIM) and the New Iterative Method (NIM). Addressing a significant research gap, our investigation explores the relative merits and demerits of these approaches. We provide a comprehensive examination of VIM, a well-established method, alongside NIM, a relatively less explored approach, to identify their comparative strengths and limitations. Furthermore, the study enriches existing knowledge in numerical methods for ODEs by highlighting essential performance characteristics such as convergence properties, computational efficiency, and accuracy across a diverse array of ODE systems. Through meticulous numerical experimentation, we uncover practical insights into the efficacy of VIM and NIM, bridging a critical knowledge gap in the field of numerical ODE solvers. Our findings demonstrate VIM as the more effective method, thereby enhancing the understanding of semi-analytic approaches for solving ODE systems and providing valuable guidance for practitioners and researchers in selecting the most appropriate method for their specific applications.

Keywords: Semi-analytic approaches, Systems of ODEs, Comparative analysis, Analytical solution, Computational efficiency

1. INTRODUCTION

Ordinary Differential Equations (ODEs) are crucial in various scientific and engineering fields, serving as mathematical models for understanding dynamic behavior in physical, biological, and engineering systems. They are essential for predicting population dynamics, chemical reactions, and fluid flow. Nonlinear nonlinear ODEs are particularly important in representing natural phenomena across different processes. While exact solutions are sought after, the practical need for approximations has led to the development of various numerical and analytical methods tailored for solving ODE systems. Recent studies have explored different approaches: Zada et al. [1] employed a numerical method for solving fractional-order inhomogeneous ODEs, Nasir et al. [2] investigated numerical solutions for systems of coupled fractional-order equations using the New Iterative Method (NIM), and Belal et al. [3] utilized the reduced differential transform method for addressing ODEs like beam and airy equations. Additionally, Shittu et al. [4] focused on analyzing and applying a Variational Iteration-based scheme for approximating solutions of various ODEs, highlighting its effectiveness, especially in handling nonlinear variables.

The Variational Iterative Method (VIM) and the New Iterative Method (NIM) are evaluated quantitatively for solving first-order differential equations. VIM, developed in the 19th century and refined later, minimizes a function called the Lagrange multiplier to solve various differential equations, offering an alternative to traditional numerical methods [5], [6]. It creates an auxiliary function through a variational principle and iteratively improves resolution. NIM, introduced by Daftardar and Jafari in 2006 and later known as the Daftardar-Jafari method, employs an iterative scheme to solve problems [7]-[12]. NIM has shown success in handling differential equations, fractional differential equations, and partial differential equations. Both methods yield precise solutions with reduced computational effort compared to purely numerical methods, but they rely on initial approximations, requiring careful selection and additional effort [13-14]. Consideration of convergence analysis and stability is crucial, especially for complex systems or equations with discontinuities. This research aims to understand and manage first-order ODE systems, recognizing the benefits of these methods and their potential contributions to mathematical modeling and analysis.

Numerous researchers have dedicated significant efforts to studying first-order ordinary differential equation systems. Abdelhakem et al. [15] carried out an investigation that utilizes the first derivatives of Legendre polynomials within a pseudo-Galerkin spectral method, incorporating operational matrices to address systems of ordinary differential-algebraic equations. Msmali et al. [16] presented an innovative application of the differential transform method to solve first-order differential equation systems, establishing fundamental definitions, properties, and theorems in their study. The paper of Al-Ahmad et al.[17] constructed a numerical scheme combining differential transform, Laplace transform, and Pade approximants in resolving first-order ordinary differential equation systems, overcoming challenges and providing accurate approximations, as demonstrated by examples with exact solution correspondences. While many

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researchers have extensively investigated first-order ordinary differential equation systems, more research needs to be done on applying VIM and NIM to such systems Ababneh.[18],Muhammad et al.[19].
The research in the domain of solving systems of first-order differential equations using the VIM and NIM identifies a notable gap in the quantitative evaluation of their performance compared to each other. Both methods have been individually explored; more comprehensive comparative analyses are needed in recent literature. The novelty of this study lies in bridging this gap by conducting a rigorous quantitative assessment of the VIM and NIM, considering their efficiency, accuracy, and computational implications in addressing systems of first-order differential equations. The motivation stems from the significance of enhancing numerical methods for solving such systems and the potential impact on diverse applications. A thorough background review reveals that existing literature has predominantly focused on individual applications of VIM and NIM, emphasizing their advantages and limitations, but a direct quantitative comparison between the two methods is notably absent, prompting the need for this research. Conducting a comprehensive quantitative comparison of the VIM and the NIM for solving systems of first-order differential equations is justified by the lack of direct comparative studies in recent literature, providing valuable insights into their relative effectiveness and guiding their optimal application in diverse fields.

2. METHODS

2.1. The New Iterative Method (NIM)

The New Iterative Method (NIM) is a computational technique used to find approximate solutions to a wide range of linear and nonlinear differential equations [4, 9]. This method involves generating a sequence of functions that converge to the exact solution through an iterative process. NIM is known for its simplicity and efficiency, often providing accurate solutions with fewer iterations compared to traditional methods. To illustrate the concept of the New Iteration Method, let us examine the system of first-order differential equations represented as:

$$\begin{aligned} j_1' &= j_1, j_2 \ldots j_m, n \\ j_2' &= j_1, j_2 \ldots j_m, n \\ &\vdots \\ j_m' &= j_1, j_2 \ldots j_m, n \end{aligned} \tag{1}$$

Given the initial conditions:

$$\begin{cases} j_1(n_0) = \alpha \\ j_2(n_0) = \beta \\ \vdots \\ j_m(n_0) = \tau \end{cases} \tag{2}$$

Consider the arbitrary constants $\alpha, \beta, \cdots, \tau$ and the range [I, j] applicable to the variable m, Belal. [20]. In the context of the functional equation presented, let us delve into the essence of the NIM.

$$\begin{aligned} j_1(n) &= f_1(n) + \mathrm{N}\left[P_1(n)\right] \\ j_2(n) &= f_2(n) + \mathrm{N}\left[P_2(n)\right] \\ &\vdots \\ j_m(n) &= f_m(n) + \mathrm{N}\left[P_m(n)\right] \end{aligned} \tag{3}$$

The function $f(n)$ is familiar in this scenario where $(\underline{B} \to \underline{B})$, and N is considered a nonlinear nonlinear operator within a Banach space Jasim.[22], Belal.[23]. In the context of interpreting the New Iterative Method, our goal is to discover new iterates (solutions). The representation of the solution to equation (3) can be exhibited in the following series:

$$\begin{cases} J_1(n) = \sum_{i=0}^{\infty} j_i^1(n) \\ J_2(n) = \sum_{i=0}^{\infty} j_i^2(n) \\ \vdots \\ J_m(n) = \sum_{i=0}^{\infty} j_i^m(n) \end{cases} \tag{4}$$

The expression of the recurrence relation can be stated as:

$$\left.\begin{aligned} & J_0^1 = f_1 \\ & J_1^1 = \mathrm{N}\left(J_0^1\right) \\ & J_{i+1}^1 = \mathrm{N}\left(J_0^1 + J_1^1 + \cdots + J_i^1\right) - \mathrm{N}\left(J_0^1 + J_1^1 + \cdots + J_{i-1}^1\right) \\ & J_0^2 = f_2 \\ & J_1^2 = \mathrm{N}\left(J_0^2\right) \\ & J_{i+1}^2 = \mathrm{N}\left(J_0^2 + J_1^2 + \cdots + J_i^2\right) - \mathrm{N}\left(J_0^2 + J_1^2 + \cdots + J_{i-1}^2\right) \\ & \vdots \\ & J_0^m = f_n \\ & j_1^m = \mathrm{N}\left(j_0^m\right) \\ & J_{i+1}^m = \mathrm{N}\left(J_0^m + J_1^m + \cdots + J_i^m\right) - \mathrm{N}\left(J_0^m + J_1^m + \cdots + J_{i-1}^m\right) \\ & i = 1, 2, 3, \cdots \end{aligned}\right\} \tag{5}$$

This leads to the subsequent equation:

$$\begin{aligned} & \left(J_0^1 + J_1^1 + J_2^1 + \cdots + J_{i+1}^1\right) = \mathrm{N}\left(J_0^1 + J_1^1 + J_2^1 + \cdots + J_i^1\right) \\ & \left(J_0^2 + J_1^2 + J_2^2 + \cdots + J_{i+1}^2\right) = \mathrm{N}\left(J_0^2 + J_1^2 + J_2^2 + \cdots + J_i^2\right) \\ & \vdots \\ & \left(J_0^m + J_1^m + J_2^m + \cdots + J_{i+1}^m\right) = \mathrm{N}\left(J_0^m + J_1^m + J_2^m + \cdots + J_i^m\right) \end{aligned} \tag{6}$$

As a result, we obtain

$$\sum_{i=0}^{\infty} J_i = f + \mathrm{N}\left[\sum_{i=0}^{\infty} J_i\right] \tag{7}$$

Ultimately, the estimation of the solution in terms of m iterations is provided as:

$$J = \lim_{m\to\infty}\left(J_0 + J_1 + J_2 + \cdots + J_{m-1}\right) \tag{8}$$

2.2. Convergence of the New Iterative Method

Next, we investigate the convergence of the New Iterative Method for solving a broad class of functional equations. Let $\underline{e} = \underline{u}^* - \underline{u}$, where $\underline{u}^*$ is the analytical solution, $\underline{u}$, represents the approximate solution, while $\underline{e}$ denotes the error in the solution, thus satisfies the relation;

$$\underline{e}(\underline{x}) = f(\underline{x}) + N(\underline{e}(\underline{x})) \tag{9}$$

and the recurrence relation becomes;

$$\begin{aligned} \underline{e}_0 &= f, \\ \underline{e}_1 &= N(\underline{e}_0), \\ \underline{e}_{m+1} &= N(\underline{e}_0 + \underline{e}_1 + \cdots + \underline{e}_m) - N(\underline{e}_0 + \underline{e}_1 + \cdots + \underline{e}_{m-1}), \qquad m = 1, 2, \cdots \end{aligned} \tag{10}$$

For a situation where $\|N(x) - N(y)\| \leq k\|x - y\|, 0 < k < 1$, then it can be analyze as thus;

$$\begin{aligned} &\underline{e}_0 = f, \\ &\|\underline{e}_1\| = \|N(\underline{e}_0)\| \leq k\|\underline{e}_0\|, \\ &\|\underline{e}_2\| = \|N(\underline{e}_0 + \underline{e}_1) - N(\underline{e}_0)\| \leq k\|\underline{e}_1\| \leq k^2\|\underline{e}_0\|, \\ &\|\underline{e}_3\| = \|N(\underline{e}_0 + \underline{e}_1 + \underline{e}_2) - N(\underline{e}_0 + \underline{e}_1)\| \leq k\|\underline{e}_2\| \leq k^3\|\underline{e}_0\|, \\ &\quad\vdots \\ &\|\underline{e}_{m+1}\| = \|N(\underline{e}_0 + \cdots + \underline{e}_m) - N(\underline{e}_0 + \cdots + \underline{e}_{m-1})\| \leq k\|\underline{e}_m\| \leq k^{m+1}\|\underline{e}_0\|, \qquad m = 0, 1, 2, \cdots \end{aligned} \tag{11}$$

Thus $\underline{e}_{m+1} \to 0$ as $m \to \infty$, signifying that the NIM is convergent for solving the general functional equation.

2.3. The Variational Iterative Method (VIM)

Based on various work by authors [25-26]. Take into account the following ordinary differential equations (system):

$$\begin{aligned} &L_1(J_1, J_2, \ldots, J_m) + N_1(J_1, J_2, \cdots, J_m) = g_1 \\ &L_2(J_1, J_2, \ldots, J_m) + N_2(J_1, J_2, \cdots, J_m) = g_2 \\ &\vdots \\ &L_m(J_1, J_2, \ldots, J_m) + N_m(J_1, J_2, \cdots, J_m) = g_m \end{aligned} \tag{12}$$

In this context $L_1, L_2, \cdots, L_m$ represents linear operators while $N_1, N_2, \cdots, N_m$ denoting nonlinear operators. $J_m = J_m(n, k, t)$ and $g_m = g_m(n, k, t)$. One may establish a correction functional in the following manner;

$$J_{1,n+1} = J_{1,n} + \int_0^s \lambda_1 [L_1(j_{1,n}, j_{2,n}, \cdots j_{m,n}) + N_1(\tilde{j}_{1,n}, \tilde{j}_{2,n}, \cdots, \tilde{j}_{m,n}) - g_1] dn$$
$$J_{2,n+1} = J_{2,n} + \int_0^s \lambda_2 [L_2(j_{1,n}, j_{2,n}, \cdots j_{m,n}) + N_2(\tilde{j}_{1,n}, \tilde{j}_{2,n}, \cdots, \tilde{j}_{m,n}) - g_2] dn \quad (13)$$
$$J_{m,n+1} = J_{m,n} + \int_0^s \lambda_m [L_m(j_{1,n}, j_{2,n}, \cdots j_{m,n}) + N_m(\tilde{j}_{1,n}, \tilde{j}_{2,n}, \cdots, \tilde{j}_{m,n}) - g_m] dn$$

Lagrange multipliers $\lambda_1, \lambda_2, \cdots, \lambda_m$ are generally identified optimally in this scenario using variational theory and stationary conditions. The subscript m denotes the mth order approximation, where j_m is regarded as a constrained variation such as $\partial j_m = 0$ Mohammad [24].

$$L_i(j_i) + N_i(j_1, \cdots, j_m) = g_i(n), \quad i = 1, 2, \cdots, m \quad (14)$$
$$j_{i,n+1} = j_{i,n} + \int_0^s \lambda_i [L_i(j_{i,n}(n) + N_i(j_{1,n}(n), \cdots, j_{m,n}(n)) - g_i(s)) dn \quad (15)$$

λ_i represents Lagrange multipliers in a general sense, $\partial j_{i,n+1} = 0, \quad i = 1, 2, \cdots, m$.. The value of the Lagrange multiplier, represented by λ, is computed using the formula $\lambda = (-1)^m \frac{1}{(m-1)!}(s-\xi)^{m-1}$, m represents the count of occurrences of differentials. In the framework of the VIM, Lagrange multipliers are seamlessly incorporated into the auxiliary functional formulation Tang [25]. These additional terms enforce constraints or boundary conditions pertinent to the addressed problem. Their inclusion leads to a modification of the auxiliary function, which becomes dependent on both the unknown function and these multipliers Shoaib [26]. Optimal values for Lagrange multipliers are determined most effectively through variation theory. The subsequent conditions are established by fixing the functions defined by equation (12).

$$\lambda_i'(s)\big|_{s=j} = 0$$
$$1 + \lambda_i(s)\big|_{s=j} = 0, \quad i = 1, 2, \cdots, m$$
$$p_{i,n+1} = p_{i,n} + \int_0^s \lambda_i [L_i(p_{i,n}(n)) + N_i(p_{1,n+1}(n), \ldots, p_{i-1,n+1}(n), p_{i,n}(n), \ldots, p_{m,n}(n)) - g_i(n)] dn \quad (16)$$

For $i = 1, 2, \ldots, m$. Indeed, the revised values $j_{1,n+1,} j_{2,n+1}, \ldots, j_{i-1,n+1}$ are employed to locate $j_{i,n+1}$. This technique expedites the convergence of the sequence system

3. NUMERICAL SIMULATIONS

We utilized the VIM and NIM methodologies on various sets of ordinary differential equations (ODEs), employing them on benchmark problems with predefined solutions, which were conducted to evaluate their precision, convergence characteristics, and computational efficacy. Additionally, the outcomes obtained via Maple 2021 software were juxtaposed with those of established methods to discern their strengths and weaknesses. The computed data is organized into tables and visually depicted through graphical illustrations.

Problem 1: We examine the subsequent system of first-order ODE utilizing both VIM and NIM approaches.

$$\frac{dj}{dn} = k(n), \quad \frac{dk}{dn} = 2j(n) - k(n)$$, conditions: $j(0) = 1,\ k(0) = -1.$ Analytical: $$j(n) = \frac{e^n}{3} + \frac{2e^{-2n}}{3}, \quad k(n) = \frac{e^n}{3} - \frac{4e^{-2n}}{3}$$

NIM solution;

$$P(n)=\sum p_1(n)=p_0(n)+p_1(n)+p_2(n)+p_3(n)+...+p_n(n)$$

Re-arrange the equation to get

$$j'(n)=k(n)$$
$$k'(n)=2j(n)-k(n)$$

Take the integral of both sides

$$\int_0^n j'(n)=\int_0^n k(n)dn$$
$$\int_0^n k'(n)=\int_0^n (2j(n)-k(n))dn$$

It is resolved in its form.
The first term for NIM;

$$j_1=-n$$
$$k_1=3n$$

Second term for NIM;

$$j_2=-n+\frac{3}{2}n^2$$
$$k_2=3n-\frac{5}{2}n^2$$

Third term for NIM;

$$j_3=-n+3n^2-\frac{5}{6}n^3$$
$$k_3=3n-5n^2+\frac{11}{6}n^3$$

VIM Method; Re-write the system of equations into;

$$\frac{dj}{dn}-k(n)=0$$
$$\frac{dk}{dn}-2j(n)-k(n)=0$$

And substitute it into the general VIM formula to get

$$J_{n+1}(n) = J_n(n) + \int_0^n \lambda(n)[j'(n) - k(n)]dn$$

$$K_{n+1}(n) = K_n(n) + \int_0^n \lambda(n)[k'(n) - 2j(n) - k(n)]dn$$

Find the value of λ and substitute

$$L = \frac{d}{dn} \rightarrow \lambda(n) = (-1)'\frac{(1)}{1} = -1$$

$$\lambda(n) = -1$$

To arrive at

$$J_{n+1}(n) = J_n(n) - \int_0^n [j'(n) - k_n(n)]dn$$

$$K_{n+1}(n) = K_n(n) - \int_0^n [k'(n) - 2j_n(n) - k_n(n)]dn$$

We then use a Maple 2017 to resolved it.
The first term for VIM;

$$j_1 = 1 - n$$

$$k_1 = 1 + 3n$$

Second term for VIM;

$$j_2 = 1 - n + \frac{3}{2}n^2$$

$$k_2 = 1 + 3n - \frac{5}{2}n^2$$

Third term for VIM;

$$j_3 = 1 - n + \frac{3}{2}n^2 - \frac{5}{6}n^3$$

$$k_3 = 1 + 3n - \frac{5}{2}n^2 + \frac{11}{6}n^3$$

Problem 2: We employ the New iterative and Variational iterative methods to compute the solution for the system of first-order ordinary differential equation below;

$$\begin{aligned}\frac{dj}{dn} &= 2j(n) - 3k(n)\\ \frac{dk}{dn} &= -2j(n) + k(n)\end{aligned}, \quad \text{conditions; } j(0) = 8,\ k(0) = 3. \qquad \text{Analytical: } \begin{aligned} j(n) &= 3e^{4n} + 5e^{-n}\\ k(n) &= -2e^{4n} + 5e^{-n}\end{aligned}$$

NIM solution;

$$P(n) = \sum p_1(n) = p_0(n) + p_1(n) + p_2(n) + p_3(n) + \ldots + p_n(n)$$

Re-arrange the equation

$$\begin{aligned} j'(n) &= 2j(n) - 3k(n)\\ k'(n) &= -2j(n) + k(n)\end{aligned}$$

Take the integral of both sides

$$\begin{aligned}\int_0^n j'(n) &= \int_0^n (2j(n) - 3k(n))dn\\ \int_0^n k'(n) &= \int_0^n (-2j(n) + k(n))dn\end{aligned}$$

Maple 2017 is then utilized to solve the remaining part of the equation.
The first term for NIM;

$$\begin{aligned} j_1 &= 7n\\ k_1 &= -13n\end{aligned}$$

Second term for NIM;

$$\begin{aligned} j_2 &= 7n + \frac{53}{2}n^2\\ k_2 &= -13n - \frac{27}{2}n^2\end{aligned}$$

Third term for NIM;

$$\begin{aligned} j_3 &= 7n + 53n^2 - \frac{187}{6}n^3\\ k_3 &= -13n - 27n^2 - \frac{133}{6}n^3\end{aligned}$$

VIM solution; Re-write the system of equations into;

$$\frac{dj}{dn}-2j(n)+3k(n)=0$$
$$\frac{dk}{dn}+2j(n)-k(n)=0$$

And substitute it into the general VIM formula to get

$$J_{n+1}(n)=J_n(n)+\int_0^n \lambda(n)[k'(n)-2j_n(n)-k_n(n)]dn$$
$$K_{n+1}(n)=K_n(n)+\int_0^n \lambda(n)[k'(n)+2j_n(n)-k_n(n)]dn$$

Find the value of λ and substitute

$$L=\frac{d}{dn}\rightarrow\lambda(n)=(-1)'\frac{(1)}{1}=-1$$
$$\lambda(n)=-1$$

Thus the correctional function is obtained

$$J_{n+1}(n)=J_n(n)-\int_0^n [j'(n)-2j_n(n)+3k_n(n)]dn$$
$$K_{n+1}(n)=K_n(n)-\int_0^n [k'(n)+2j_n(n)-k_n(n)]dn$$

Next, Maple 2017 is employed to compute the solution.
The first term for VIM;

$$j_1=8+7n$$
$$k_1=3-13n$$

Second term for VIM;

$$j_2=8+7n+\frac{53}{2}n^2$$
$$k_2=3-13n-\frac{27}{2}n^2$$

Third term for VIM;

$$j_2=8+7n+\frac{53}{2}n^2+\frac{187}{6}n^3$$
$$k_2=3-13n-\frac{27}{2}n^2-\frac{133}{6}n^3$$

Problem 3: Solve the following system of first-order ODE by NIM and VIM approach

$$\begin{aligned}\frac{dj}{dn} &= j(n)+k(n)\\ \frac{dk}{dn} &= -j(n)+k(n)\end{aligned}$$, conditions are; $j(0)=0,\ k(0)=1.$ Analytical: $$\begin{aligned}j(n) &= -\frac{1}{2}+\frac{e^{2n}}{2}\\ k(n) &= \frac{e^{2n}}{2}+\frac{1}{2}\end{aligned}$$

NIM solution;

$$P(n)=\sum p_1(n)=p_0(n)+p_1(n)+p_2(n)+p_3(n)+\ldots+p_n(n)$$

Re-arrange the equation

$$\begin{aligned}j'(n)-j(n)-k(n)&=0\\ k'(n)+j(n)-k(n)&=0\end{aligned}$$

Take the integral of both sides

$$\begin{aligned}\int_0^n j'(n) &= \int_0^n (j(n)+k(n))dn\\ \int_0^n k'(n) &= \int_0^n (-j(n)+k(n))dn\end{aligned}$$

Next, the solution is computed using Maple 2017.
The first term for NIM;

$$\begin{aligned}j_1 &= n\\ k_1 &= n\end{aligned}$$

Second term for NIM;

$$\begin{aligned}j_2 &= n+n^2\\ k_2 &= n+n^2\end{aligned}$$

Third term for NIM;

$$\begin{aligned}j_3 &= n+n^2+\frac{2}{3}n^3\\ k_3 &= n+n^2+\frac{2}{3}n^3\end{aligned}$$

VIM solution; Re-write the system of equations into;

$$\begin{aligned}\frac{dj}{dn}-j(n)-k(n)&=0\\ \frac{dk}{dn}+j(n)-k(n)&=0\end{aligned}$$

Then substitute into the general VIM formula to obtain

$$J_{n+1}(n) = J_n(n) + \int_0^n \lambda(n)[j'(n) - j_n(n) - k_n(n)]dn$$
$$K_{n+1}(n) = K_n(n) + \int_0^n \lambda(n)[k'(n) + j_n(n) - k_n(n)]dn$$

Find the value of λ

$$L = \frac{d}{dn} \to \lambda(n) = (-1)'\frac{(1)}{1} = -1$$
$$\lambda(n) = -1$$

and substitute to obtain

$$J_{n+1}(n) = J_n(n) - \int_0^n [j'(n) - j_n(n) - k_n(n)]dn$$
$$K_{n+1}(n) = K_n(n) - \int_0^n [k'(n) + j_n(n) - k_n(n)]dn$$

The solution is then computed using Maple 2017.
The first term for VIM;

$$j_1 = -n$$
$$k_1 = 1 - n$$

Second term for VIM;

$$j_2 = -n + n^2$$
$$k_2 = 1 - n + n^2$$

Third term for VIM;

$$j_2 = -n + n^2 - \frac{2}{3}n^3$$
$$k_2 = 1 - n + n^2 \frac{2}{3}n^3$$

Table 1. NIM Computational Outcome Problem 1

n	NIMj Estimates	NIMk Estimates	NIMj Analytical Values	NIMk Analytical Values
0.00	1.000000000	-1.000000000	1.0000000000	-1.0000000000
0.10	0.7441666667	-0.1731666667	0.9142108081	-0.7232506980
0.20	0.5733333333	0.5146666667	0.8540142833	-0.4866258085
0.30	0.4825000000	1.074500000	0.8158273601	-0.2817959120
0.40	0.4666666667	1.517333333	0.7968275420	-0.1018303860
0.50	0.5208333333	1.854166667	0.7948267177	0.0590678354
0.60	0.6400000000	2.096000000	0.8081690746	0.2057806509
0.70	0.8191666667	2.253833333	0.8356488782	0.3424549505
0.80	1.0533333330	2.338666667	0.8764446546	0.4726516186
0.90	1.3375000000	2.361500000	0.9300669624	0.5994691861
1.00	1.6666666670	2.333333333	0.9963174647	0.7256468984

Table 2. VIM Computational Outcome Problem 1

n	VIMj Estimates	VIMk Estimates	VIMj Analytical Values	VIMk Analytical Values
0.00	1.0000000000	-1.0000000000	1.0000000000	-1.0000000000
0.10	0.9142125000	-0.7232541667	0.9142108081	-0.7232506980
0.20	0.8540666667	-0.4867333333	0.8540142833	-0.4866258085
0.30	0.8162125000	-0.2825875000	0.8158273601	-0.2817959120
0.40	0.7984000000	-0.1050666667	0.7968275420	-0.1018303860
0.50	0.7994791667	0.4947916667	0.7948267177	0.0590678354
0.60	0.8194000000	0.1826000000	0.8081690746	0.2057806509
0.70	0.8592125000	0.2937458333	0.8356488782	0.3424549505
0.80	0.9210666667	0.3802666667	0.8764446546	0.4726516186
0.90	1.0082125000	0.4374125000	0.9300669624	0.5994691861
1.00	1.1250000000	0.4583333333	0.9963174647	0.7256468984

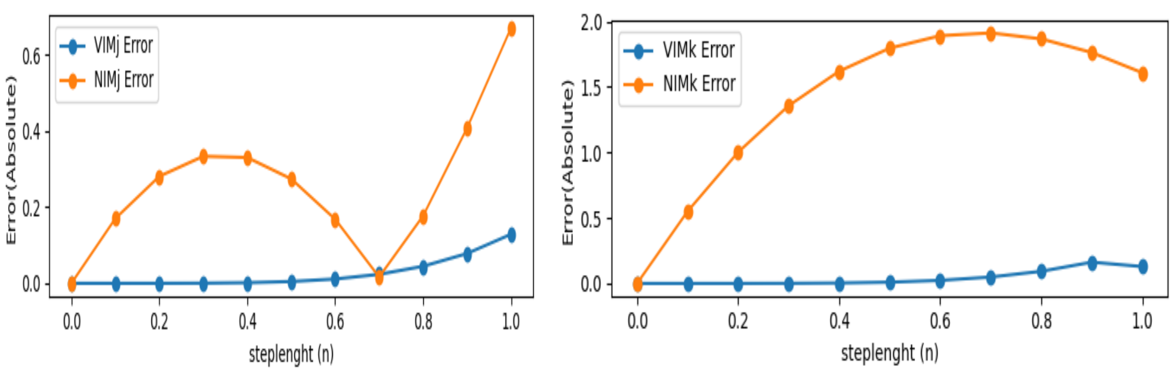

Fig 1. Plots illustrating Error Comaprison Problem 1; (a) VIMj and NIMj (b) VIMk and NIMka

Table 3. NIM Computational Outcome Problem 2

n	NIMj Estimates	NIMk Estimates	NIMj Analytical Values	NIMk Analytical Values
0.00	8.000000000	3.000000000	8.000000000	3.000000000
0.10	10.92616667	-1.327166667	8.999661184	1.540537694
0.20	15.62933333	-6.597333333	10.77027655	-0.357428090
0.30	22.29650000	-12.94350000	13.66444187	-2.936142742
0.40	31.11466667	-20.49866667	18.21069750	-6.554464618
0.50	42.27083333	-29.39583333	25.19982160	-11.74545890
0.60	55.95200000	-39.76800000	35.81358732	-19.30229458
0.70	72.34516667	-51.74816667	51.81686683	-30.40636702
0.80	91.63733333	-65.46933333	75.84423542	-46.81841558
0.90	114.0155000	-81.06450000	111.8275516	-71.16362058
1.00	139.6666667	-98.66666667	165.6338473	-107.3569029

Table 4. VIM Computational Outcome Problem 2

n	VIMj Estimates	VIMk Estimates	VIMj Analytical Values	VIMk Analytical Values
0.00	8.000000000	3.000000000	8.000000000	3.0000000000
0.10	8.999387500	1.540720833	8.999661184	1.540537694
0.20	10.76086667	-0.351133333	10.77027655	-0.357428090
0.30	13.58738750	-2.884612500	13.66444187	-2.936142742
0.40	17.85920000	-6.319466667	18.21069750	-6.554464618
0.50	24.03385417	-10.96614583	25.19982160	-11.74545890
0.60	32.64620000	-17.18580000	35.81358732	-19.30229458
0.70	44.30838750	-25.39027917	51.81686683	-30.40636702
0.80	59.70986667	-36.04213333	75.84423542	-46.81841558
0.90	79.61738750	-49.65461250	111.8275516	-71.16362058
1.00	104.8750000	-66.79166667	165.6338473	-107.3569029

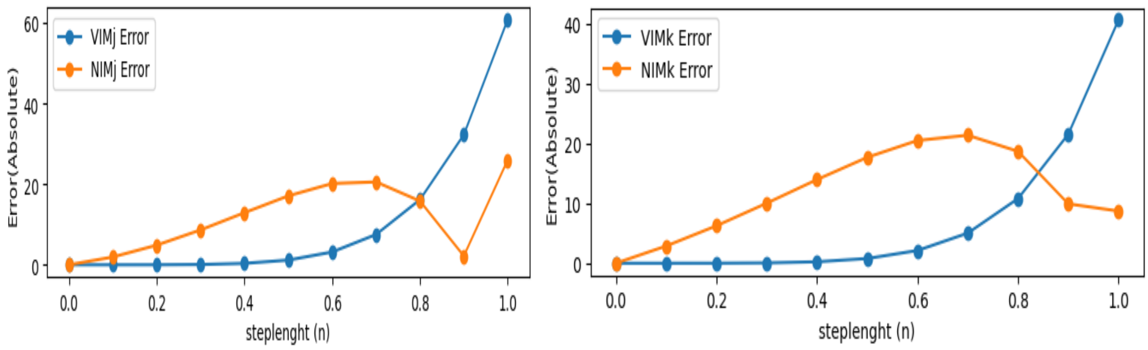

Fig 2. Plots illustrating Error Comaprison Problem 2; (a) VIMj and NIMj (b) VIMk and NIMk

Table 5. NIM Computational Outcome Problem 3

n	NIMj Estimates	NIMk Estimates	NIMj Analytical Values	NIMk Analytical Values
0.00	0.000000000	1.000000000	0.000000000	1.000000000
0.10	0.3306666667	1.330666667	0.1107013790	1.110701379
0.20	0.7253333333	1.725333333	0.2459123490	1.245912349
0.30	1.188000000	2.188000000	0.4110594000	1.411059400
0.40	1.722666667	2.722666667	0.6127704640	1.612770464
0.50	2.333333333	3.333333333	0.8591409140	1.859140914
0.60	3.024000000	4.024000000	1.160058462	2.160058462
0.70	3.798666667	4.798666667	1.527599984	2.527599984
0.80	4.661333333	5.661333333	1.976516212	2.976516212
0.90	5.616000000	6.616000000	2.524823732	3.524823732
1.00	6.666666667	7.666666667	3.194528050	4.194528050

Table 6. VIM Computational Outcome Problem 3

n	VIMj Estimates	VIMk Estimates	VIMj Analytical Values	VIMk Analytical Values
0.00	0.000000000	1.000000000	0.000000000	1.000000000
0.10	-0.9063333333	0.9093666667	0.1107013790	1.110701379
0.20	-0.1648000000	0.8352000000	0.2459123490	1.245912349
0.30	-0.2253000000	0.7747000000	0.4110594000	1.411059400
0.40	-0.2741333333	0.7258666667	0.6127704640	1.612770464
0.50	-0.3125000000	0.6875000000	0.8591409140	1.859140914
0.60	-0.3408000000	0.6592000000	1.1600584620	2.160058462
0.70	-0.3586333333	0.6413666667	1.5275999840	2.527599984
0.80	-0.3648000000	0.6352000000	1.9765162120	2.976516212
0.90	-0.3573000000	0.6427000000	2.5248237320	3.524823732
1.00	-0.3333333333	0.6666666667	3.1945280500	4.194528050

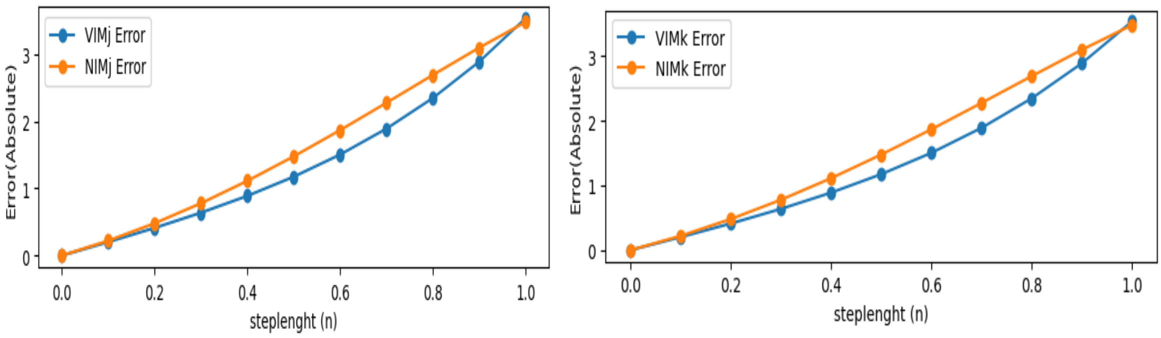

Fig 3. Plots illustrating Error Comaprison Problem 3; (a) VIMj and NIMj (b) VIMk and NIMk

4. DISCUSSIONS OF RESULTS

Following the criteria outlined by Poornima et al. [27], a numerical solution is considered convergent if the absolute disparity between the numerical solution and the analytical solution tends toward zero as 'N' ranges from 1 to N. This evaluation accounts for the error, defined as the absolute difference between the numerical and analytical solutions . A thorough analysis of the results is disclosed. Examining Tables 1 to 6 compares numerical solutions from VIM and NIM methods with analytical solutions for Problems 1-3, revealing their efficacy. Figures 1-3 further analyze associated errors, facilitating the selection of effective numerical techniques for dynamic system simulation.

Numerical Solution Comparison (Tables 1 to 6):
i. Concerning the comparison of numerical solutions (Tables 1 to 6), these tables offer a comprehensive view of the evolving solutions provided by NIM and VIM for Problems 1 to 3.
ii. The numerical results presented herein showcase the models' capability to capture the dynamic behavior exhibited by the examined systems.
iii. In each instance, the solutions display variations that mirror the increasing complexity of the problems, providing valuable insights into how each model responds to diverse conditions.
Comparison of Error Computations (Figures 1-3):
i. Shifting focus to error comparison (Figures 1 to 3), these figures quantify the disparities between the numerical solutions derived from NIM and VIM and the corresponding analytical benchmarks.
ii. As problems become more complex, errors increase, posing challenges in capturing precise system dynamics. Figures (1-3) show VIM's markedly lower errors than NIM.
iii. Figures 1-3 visually compare errors between NIMj, VIMj (subplot a), and NIMk, VIMk (subplot b), aiding in understanding performance differences between NIM and VIM across various problem conditions.

Findings:
i. The outcomes underscore the trade-offs between NIM and VIM, emphasizing careful consideration when choosing numerical methods tailored to specific problem types.
ii. It is evident from the analysis that the error associated with the VIM approach is consistently lower compared to the errors calculated by the NIM approach.
iii. These observations provide valuable insights for future model refinements, pinpointing areas that could be refined and improved to enhance accuracy.

5. CONCLUSION

We have successfully addressed several first-order linear differential equations systems utilizing the New Iterative Method (NIM) and the Variational Iterative Method (VIM). Our numerical investigations demonstrate that both methods yield highly accurate solutions for a given first-order linear ordinary differential equation. Moreover, comparative analysis reveals that both the NIM and VIM offer reduced computational complexity, enhancing simplicity and usability. However, our findings, illustrated through tables and absolute error figures (Figures 1-3), unequivocally establish the Variational Iterative Method as the superior approach among the two methodologies studied. Consequently, our research showcases an effective method characterized by reliability, adequacy, and simplicity for solving first-order linear differential equations systems. In future studies, we aim to extend our analysis to systems of second-order ODEs or even systems of third-order ODEs. This expansion will enable us to further explore the applicability and efficiency of the NIM and VIM in solving higher-order systems. By doing so, we anticipate gaining deeper insights into the performance of these methods across a broader range of differential equation systems, thereby contributing to advancements in numerical analysis and computational mathematics.

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