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Parameterized Newton-type inequalities associated with convex functions via quantum calculus

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Abstract. Using the concept of quantum derivatives and integrals, we first develop a new parameterized identity in this work. This parameterized quantum identity is used to demonstrate parameterized quantum Newton-type inequalities related to convex functions. We also demonstrate how setting $q \to 1^-$ allows the newly generated inequalities to be recovered into some existing inequalities. In order to validate the recently discovered inequalities, we conclude by providing mathematical examples of convex functions along with some graphical analysis.

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1. Introduction

Complex analysis, number theory, and many other subjects can benefit from the growing body of knowledge on convexity. With its diverse applications, convexity has a substantial impact on people's lives as well. A function $\Omega : [a_0, a_1] \to \mathbb{R}$ is convex if it satisfies an inequality:

$$\Omega(\zeta \mathrm{x} + (1-\zeta)\mathrm{y}) \leq \zeta\Omega(\mathrm{x}) + (1-\zeta)\Omega(\mathrm{y})$$

where $\mathrm{x}, \mathrm{y} \in [a_0, a_1]$ and $\zeta \in [0, 1]$.

The Simpson and Newton-type inequalities are the most well-known integral inequalities for convex functions. Thomas Simpson (1710-1761) is credited with the invention of Simpson's rules, which are well-known methods for numerical integration and approximations of definite integrals. As a result of Johannes Kepler's use of a comparable estimate nearly 100 years ago, these methods are sometimes referred to as Kepler's rule. Estimates based on a three-step quadratic kernel are frequently referred to as Newton-type inequalities since Simpson's technique is composed of a three-point Newton-Cotes quadrature algorithm.

(1) The Simpson's 1/3 rule is represented by the following way:

$$\int_{a_0}^{a_1} \Omega(\mathrm{x})dx \approx \frac{a_1 - a_0}{6}\left[\Omega(a_0) + 4\Omega\left(\frac{a_0 + a_1}{2}\right) + \Omega(a_1)\right],$$

see [13] for more details.

(2) The Newton-Cotes quadrature formula or Simpson's 3/8 rule is represented in the following way:

$$\int_{a_0}^{a_1} \Omega(\mathrm{x})d\mathrm{x} \approx \frac{a_1 - a_0}{8}\left[\Omega(a_0) + 3\Omega\left(\frac{2a_0 + a_1}{3}\right) + 3\Omega\left(\frac{a_0 + 2a_1}{3}\right) + \Omega(a_1)\right],$$

see [4, 20] for more details.

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Inequalities of the Simpson- and Newton-type are exhibited hereunder:

Theorem 1. *[13] Suppose that $\Omega : [a_0, a_1] \to \mathbb{R}$ is a four times continuously differentiable function on (a_0, a_1) and*

$$\left\|\Omega^{(4)}\right\|_\infty = \sup_{\mathrm{x}\in(a,b)} |\Omega^{(4)}(\mathrm{x})| < \infty,$$

then

$$\left|\frac{1}{6}\left[\Omega(a_0) + 4\Omega\left(\frac{a_0 + a_1}{2}\right) + \Omega(a_1)\right] - \frac{1}{a_1 - a_0}\int_{a_0}^{a_1} \Omega(\mathrm{x})d\mathrm{x}\right| \le \frac{1}{2880}\left\|\Omega^{(4)}\right\|_\infty (a_1 - a_0)^4.$$

Theorem 2. *[20] Suppose that $\Omega : [a_0, a_1] \to \mathbb{R}$ is a four times continuously differentiable function on (a_0, a_1) and*

$$\left\|\Omega^{(4)}\right\|_\infty = \sup_{\mathrm{x}\in(a_0,a_1)} |\Omega^{(4)}(\mathrm{x})| < \infty,$$

then

$$\left|\frac{1}{8}\left[\Omega(a_0) + 3\Omega\left(\frac{2a_0 + a_1}{3}\right) + 3\Omega\left(\frac{a_0 + 2a_1}{3}\right) + \Omega(a_1)\right] - \frac{1}{a_1 - a_0}\int_{a_0}^{a_1} \Omega(\mathrm{x})d\mathrm{x}\right|$$
$$\le \frac{1}{6480}\left\|\Omega^{(4)}\right\|_\infty (a_1 - a_0)^4.$$

Currently, many researchers have focused on the Newton-type inequalities, see [15,16,23,25,29] and the references quoted therein. Particularly, some researchers have studied on the Newton-type inequalities by using quantum calculus, found in [2,8,9,24], see also references quoted in these articles. $q-$calculus, another name for quantum calculus, creates q-analogs of classical mathematics, that can be recovered by setting $q \to 1^-$. Numerous areas of physics and mathematics, including relativity theory, mechanics, quantum theory, orthogonal polynomials, number theory, and hypergeometric functions, make extensive use of the q-calculus (see [14, 19]). The q-parameter was first introduced in Newton's infinite series by the eminent mathematician Euler (1707–1783), who also developed the q-calculus. To define q-integral and q-derivative of continuous functions across the interval $(0, \infty)$, popularly known as calculus without limits, Jackson [17] explored the Euler notion in 1910. The ideas of q-fractional integral inequalities and q-Riemann-Liouville fractional integral inequalities were investigated by Al-Salam [6] in 1966. The q-calculus core fundamental principles were summarized by Kac and Cheung in their book in 2002, [19]. Tariboon and Ntouyas in particular presented the q-integral and q-derivative of continuous functions over finite intervals in [31], 2013. In [1, 3, 5, 7, 11, 12, 18, 21, 22, 26–28, 32–35] and the references cited therein, several novel results can be discovered.

We propose to demonstrate updated versions of the quantum Newton-type inequalities connected to convex functions, which are inspired by the ongoing studies. We further demonstrate that the recently discovered inequalities are the extension of the known Newton-type inequalities.

The remaining paper are arranged as follows: Definitions and the basic ideas of q-calculus are presented in Section 2. Quantum parameterized Newton-type inequalities linked to convex functions are discussed in Section 3. In Section 4, we provide mathematical illustrations to illustrate our key findings. Finally, we wrap up our findings in the concluding Section.

2. Basic Definitions and Preliminary Results

The present section summarises definitions and a basic introduction to the q-calculus. Let q remain constant for the duration of this paper with $0 < q < 1$ and $[a_0, a_1] \subseteq \mathbb{R}$ with $a_0 < a_1$. The quantum analogue of any number n can be given as:

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \quad n \in \mathbb{N},$$

Definition 1. *[31] The q-derivative on $[a_0, a_1]$, for a continuous function $\Omega : [a_0, a_1] \to \mathbb{R}$ is determined by:*

$$_{a_0}D_q\Omega(\mathrm{x}) = \begin{cases} \dfrac{\Omega(\mathrm{x}) - \Omega\left(q\mathrm{x} + (1 - q)a_0\right)}{(1 - q)(\mathrm{x} - a_0)}, & \text{if } \mathrm{x} \ne a_0; \\ \lim\limits_{\mathrm{x}\to a_0} {}_{a_0}D_q\Omega(\mathrm{x}), & \text{if } \mathrm{x} = a_0. \end{cases} \tag{1}$$

Ω is stated to be q_{a_0}-differentiable function provided that ${}_{a_0}D_q\Omega(\mathrm{x})$ exists.

Putting $a_0 = 0$ in Definition 1, then (1) takes the form:

$$D_q\Omega(\mathrm{x}) = \frac{\Omega(\mathrm{x}) - \Omega(q\mathrm{x})}{(1-q)(\mathrm{x})},$$

it is a derivation of q-Jackson's work; for more information, see [17].

Definition 2. *[31] Let $\Omega : [a_0, a_1] \to \mathbb{R}$ be a continuous mapping, then q_{a_0}-integral on $[a_0, a_1]$ can be stated as:*

$$\int_{a_0}^{\mathrm{x}} \Omega(\zeta)\ {}_{a_0}D_q\zeta = (1-q)(\mathrm{x}-a_0)\sum_{n=0}^{\infty} q^n\Omega\left(q^n\mathrm{x} + (1-q^n)\,a_0\right) \tag{2}$$

for $\mathrm{x} \in [a_0, a_1]$. Ω is called a q_{a_0}-integrable function provided that $\int_{a_0}^{\mathrm{x}} \Omega(\zeta)\ d_q\zeta$ exists for all $x \in [a_0, a_1]$.

Substituting $a_0 = 0$ in Definition 2, then (2) can be given as:

$$\int_0^{\mathrm{x}} \Omega(\zeta)\ d_q\zeta = (1-q)\mathrm{x}\sum_{n=0}^{\infty} q^n\Omega(q^n\mathrm{x}), \tag{3}$$

it is an integral of q-Jackson's work; for more information, see [17]. Moreover, Jackson [17] introduced q-Jackson integral on the interval $[a_0, a_1]$ is the subsequent form:

$$\int_{a_0}^{a_1} \Omega(\zeta)\ d_q\zeta = \int_0^{a_1} \Omega(\zeta)\ d_q\zeta - \int_0^{a_0} \Omega(\zeta)\ d_q\zeta.$$

Lemma 1. *[30] If $\Omega,\ g : [a_0, a_1] \to \mathbb{R}$ are continuous mappings, then the undermentioned representation is valid:*

$$\begin{aligned}&\int_0^c g(\zeta)\ {}_{a_0}D_q\Omega(\zeta a_1 + (1-\zeta)a_0)\ d_q\zeta \\ &= \left.\frac{g(\zeta)\Omega(\zeta a_1 + (1-\zeta)a_0)}{a_1 - a_0}\right|_0^c - \frac{1}{a_1-a_0}\int_0^c D_qg(\zeta)\Omega(q\zeta a_1 + (1-q\zeta)a_0)\ d_q\zeta.\end{aligned} \tag{4}$$

Lemma 2. *[32] The following expression holds:*

$$\int_{a_0}^{a_1} (\mathrm{x}-a_0)^{\alpha}\ {}_{a_0}D_q\mathrm{x} = \frac{(a_1-a_0)^{\alpha+1}}{[\alpha+1]_q},$$

where $\alpha \in \mathbb{R} - \{-1\}$.

3. Main Results

In order to present new q-parameterized Newton-type inequalities, "we first present the following lemma involving three parameters:

Lemma 3. *Suppose that $\Omega : [a_0, a_1] \to \mathbb{R}$ is a q_{a_0}-differentiable function on (a_0, a_1) such that ${}_{a_0}D_q\Omega$ is continuous and integrable on $[a_0, a_1]$. Then, we have the following identity:*

$$\begin{aligned}&\frac{1}{3}\left[\lambda\Omega(a_0) + (1-\lambda+\mu)\Omega\left(\frac{2a_0+a_1}{3}\right) + (1-\mu+\gamma)\Omega\left(\frac{a_0+2a_1}{3}\right) + (1-\gamma)\Omega(a_1)\right] \\ &- \frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}} \Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x} + \int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}} \Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x} + \int_{\frac{a_0+2a_1}{3}}^{a_1} \Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right] \\ = &\frac{a_1-a_0}{9}\left[\int_0^1 (q\zeta-\lambda)\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3} + (1-\zeta)a_0\right)\ d_q\zeta\right. \\ &+ \int_0^1 (q\zeta-\mu)\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3} + (1-\zeta)\frac{2a_0+a_1}{3}\right)\ d_q\zeta \\ &\left.+ \int_0^1 (q\zeta-\gamma)\ {}_{a_0}D_q\Omega\left(\zeta a_1 + (1-\zeta)\frac{a_0+2a_1}{3}\right)\ d_q\zeta\right].\end{aligned} \tag{5}$$

Proof. Let

$$\begin{aligned}
&\frac{a_1-a_0}{9}\left[\int_0^1 (q\zeta-\lambda)\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a_0\right)\ d_q\zeta\right.\\
&+\int_0^1 (q\zeta-\mu)\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3}+(1-\zeta)\frac{2a_0+a_1}{3}\right)\ d_q\zeta\\
&\left.+\int_0^1 (q\zeta-\gamma)\ {}_{a_0}D_q\Omega\left(\zeta a_1+(1-\zeta)\frac{a_0+2a_1}{3}\right)\ d_q\zeta\right]\\
&=\frac{(a_1-a_0)}{9}\left[I_1+I_2+I_3\right]. \qquad (6)
\end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned}
I_1 =&\ \int_0^1 (q\zeta-\lambda)\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a_0\right)\ d_q\zeta\\
=&\ \frac{3}{a_1-a_0}(q\zeta-\lambda)\,\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a_0\right)\Bigg|_0^1-\frac{3}{a_1-a_0}\int_0^1 q\Omega\left(q\zeta\frac{2a_0+a_1}{3}+(1-q\zeta)a_0\right)\ d_q\zeta\\
=&\ \frac{3}{a_1-a_0}(q-\lambda)\,\Omega\left(\frac{2a_0+a_1}{3}\right)+\frac{3\lambda\Omega(a_0)}{(a_1-a_0)}-\frac{3}{a_1-a_0}(1-q)\sum_{n=0}^{\infty}q^{n+1}\Omega\left(q^{n+1}\frac{2a_0+a_1}{3}+(1-q^{n+1})a_0\right)\\
=&\ \frac{3}{a_1-a_0}(q-\lambda)\,\Omega\left(\frac{2a_0+a_1}{3}\right)+\frac{3\lambda\Omega(a_0)}{(a_1-a_0)}-\frac{3}{a_1-a_0}(1-q)\sum_{n=1}^{\infty}q^{n}\Omega\left(q^{n}\frac{2a_0+a_1}{3}+(1-q^{n})a_0\right)\\
=&\ \frac{3}{a_1-a_0}(q-\lambda)\,\Omega\left(\frac{2a_0+a_1}{3}\right)+\frac{3\lambda\Omega(a_0)}{(a_1-a_0)}\\
&-\frac{3}{a_1-a_0}(1-q)\left[\sum_{n=0}^{\infty}q^{n}\Omega\left(q^{n}\frac{2a_0+a_1}{3}+(1-q^{n})a_0\right)-\Omega\left(\frac{2a_0+a_1}{3}\right)\right]\\
=&\ \frac{3\lambda\Omega(a_0)}{(a_1-a_0)}+\frac{3(1-\lambda)}{(a_1-a_0)}\Omega\left(\frac{2a_0+a_1}{3}\right)-\frac{9}{(a_1-a_0)^2}\int_a^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}, \qquad (7)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
I_2 =&\ \int_0^1 (q\zeta-\mu)\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3}+(1-\zeta)\frac{2a_0+a_1}{3}\right)\ d_q\zeta\\
=&\ \frac{3\mu}{(a_1-a_0)}\Omega\left(\frac{2a_0+a_1}{3}\right)+\frac{3(1-\mu)}{(a_1-a_0)}\Omega\left(\frac{a_0+2a_1}{3}\right)-\frac{9}{(a_1-a_0)^2}\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}, \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
I_3 =&\ \int_0^1 (q\zeta-\gamma)\ {}_{a_0}D_q\Omega\left(tb+(1-\zeta)\frac{a_0+2a_1}{3}\right)\ d_q\zeta\\
=&\ \frac{3\gamma}{(a_1-a_0)}\Omega\left(\frac{a_0+2a_1}{3}\right)+\frac{3(1-\gamma)}{(a_1-a_0)}\Omega\left(b\right)-\frac{9}{(a_1-a_0)^2}\int_{\frac{a_0+2a_1}{3}}^{b}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}. \qquad (9)
\end{aligned}$$

Substituting the inequalities (7) - (9) in the inequality (6), we get the required inequality (5). □

Remark 1. *If we set $\lambda=\frac{3}{8}$, $\mu=\frac{1}{2}$ and $\gamma=\frac{5}{8}$ in Lemma 3, we get*

$$\begin{aligned}
&\frac{1}{8}\left[\Omega(a_0)+3\Omega\left(\frac{2a_0+a_1}{3}\right)+3\Omega\left(\frac{a_0+2a_1}{3}\right)+\Omega(a_1)\right]\\
&-\frac{1}{(a_1-a_0)}\left[\int_a^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{b}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\\
=&\ \frac{a_1-a_0}{9}\left[\int_0^1\left(q\zeta-\frac{3}{8}\right)\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a\right)\ d_q\zeta\right.
\end{aligned}$$

$$+\int_0^1\left(q\zeta-\frac{1}{2}\right)\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3}+(1-\zeta)\frac{2a_0+a_1}{3}\right)\ d_q\zeta$$
$$+\int_0^1\left(q\zeta-\frac{5}{8}\right)\ {}_{a_0}D_q\Omega\left(\zeta a_1+(1-\zeta)\frac{a_0+2a_1}{3}\right)\ d_q\zeta\Bigg],$$

which is proved in [23].

Theorem 3. *Under the conditions of Lemma 3, if $|{}_{a_0}D_q f|$ is a convex function on $[a_0, a_1]$, then one can attain the succeeding quantum inequality:*

$$\left|\frac{1}{3}\left[\lambda\Omega(a_0)+(1-\lambda+\mu)\Omega\left(\frac{2a_0+a_1}{3}\right)+(1-\mu+\gamma)\Omega\left(\frac{a_0+2a_1}{3}\right)+(1-\gamma)\Omega(a_1)\right]\right.$$
$$\left.-\frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\right|$$
$$\le\frac{a_1-a_0}{9}\left[(\Lambda_1(q)+\Lambda_3(q)+\Lambda_5(q))\left|\ {}_{a_0}D_q\Omega(a_0)\right|+(\Lambda_2(q)+\Lambda_4(q)+\Lambda_6(q))\left|\ {}_{a_0}D_q\Omega(a_1)\right|\right],\tag{10}$$

where $\Lambda_\jmath(q), \jmath=1,\cdots,6$ are defined by

$$\Lambda_1(q)=\begin{cases}\dfrac{2\lambda+(5\lambda-2)q+(5\lambda-2)q^2+(3\lambda-3)q^3}{[2]_q3[3]_q}, & for\ 0<q<\lambda;\\[2mm] \dfrac{-2\lambda+6\lambda^2-2\lambda^3+(6\lambda^2-5\lambda+2)q+(6\lambda^2-5\lambda+2)q^2+(3-3\lambda)q^3}{[2]_q3[3]_q}, & for\ \lambda\le q<1,\end{cases}$$

$$\Lambda_2(q)=\begin{cases}\dfrac{\lambda+(\lambda-1)[q+q^2]}{3[2]_q[3]_q}, & for\ 0<q<\lambda;\\[2mm] \dfrac{2\lambda^3-\lambda+(1-\lambda)[q+q^2]}{[2]_q3[3]_q}, & for\ \lambda\le q<1,\end{cases}$$

$$\Lambda_3(q)=\begin{cases}\dfrac{\mu+(3\mu-1)[q+q^2]+(2\mu-2)q^3}{[2]_q3[3]_q}, & for\ 0<q<\mu;\\[2mm] \dfrac{4\mu^2-2\mu^3-\mu+(4\mu^2-3\mu+1)q+(4\mu^2-3\mu+1)q^2+(2-2\mu)q^3}{[2]_q3[3]_q}, & for\ \mu\le q<1,\end{cases}$$

$$\Lambda_4(q)=\begin{cases}\dfrac{2\mu+(3\mu-2)q+(3\mu-2)q^2+(\mu-1)q^3}{[2]_q3[3]_q}, & for\ 0<q<\mu;\\[2mm] \dfrac{2\mu^3+2\mu^2-2\mu+(2\mu^2-3\mu+2)q+(2\mu^2-3\mu+2)q^2+(1-\mu)q^3}{[2]_q3[3]_q}, & for\ \mu\le q<1,\end{cases}$$

$$\Lambda_5(q)=\begin{cases}\dfrac{\gamma q+\gamma q^2+(\gamma-1)q^3}{[2]_q3[3]_q}, & for\ 0<q<\gamma;\\[2mm] \dfrac{2\gamma^2-2\gamma^3+(2\gamma^2-\gamma)q+(2\gamma^2-\gamma)q^2+(1-\gamma)q^3}{[2]_q3[3]_q}, & for\ \gamma\le q<1,\end{cases}$$

$$\Lambda_6(q)=\begin{cases}\dfrac{3\gamma+(5\gamma-3)q+(5\gamma-3)q^2+(2\gamma-2)q^3}{[2]_q3[3]_q}, & for\ 0<q<\gamma;\\[2mm] \dfrac{4\gamma^2+2\gamma^3-3\gamma+(4\gamma^2-5\gamma+3)q+(4\gamma^2-5\gamma+3)q^2+(2-2\gamma)q^3}{[2]_q3[3]_q}, & for\ \gamma\le q<1.\end{cases}$$

Proof. Considering Lemma 3 and taking absolute value on both sides of the identity, we have

$$\left|\frac{1}{3}\left[\lambda\Omega(a_0)+(1-\lambda+\mu)\Omega\left(\frac{2a_0+a_1}{3}\right)+(1-\mu+\gamma)\Omega\left(\frac{a_0+2a_1}{3}\right)+(1-\gamma)\Omega(a_1)\right]\right.$$

$$
\begin{aligned}
&-\frac{1}{(a_1-a_0)}\left[\int_a^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\Bigg|\\
&\le\frac{a_1-a_0}{9}\left[\int_0^1|q\zeta-\lambda|\left|\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a_0\right)\right|\ d_q\zeta\right.\\
&+\int_0^1|q\zeta-\mu|\left|\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3}+(1-\zeta)\frac{2a_0+a_1}{3}\right)\right|\ d_q\zeta\\
&\left.+\int_0^1|q\zeta-\gamma|\left|\ {}_{a_0}D_q\Omega\left(\zeta a_1+(1-\zeta)\frac{a_0+2a_1}{3}\right)\right|\ d_q\zeta\right]\\
&\le\frac{a_1-a_0}{9}\left[\int_0^1|q\zeta-\lambda|\left(\frac{3-\zeta}{3}|\ {}_{a_0}D_q\Omega(a_0)|+\frac{\zeta}{3}|\ {}_{a_0}D_q\Omega(a_1)|\right)\ d_q\zeta\right.\\
&+\int_0^1|q\zeta-\mu|\left(\frac{2-\zeta}{3}|\ {}_{a_0}D_q\Omega(a_0)|+\frac{1+\zeta}{3}|\ {}_{a_0}D_q\Omega(a_1)|\right)\ d_q\zeta\\
&\left.+\int_0^1|q\zeta-\gamma|\left(\frac{1-\zeta}{3}|\ {}_{a_0}D_q\Omega(a_0)|+\frac{2+\zeta}{3}|\ {}_{a_0}D_q\Omega(a_1)|\right)\ d_q\zeta\right].
\end{aligned}
$$

Using Lemma 2, the succeeding integrals can be evaluated easily:

$$
\begin{aligned}
\Lambda_1(q)&=\int_0^1|q\zeta-\lambda|\,\frac{3-\zeta}{3}\ d_q\zeta\\
&=\begin{cases}\dfrac{2\lambda+(5\lambda-2)q+(5\lambda-2)q^2+(3\lambda-3)q^3}{[2]_q3[3]_q}, & for\ 0<q<\lambda;\\[2mm] \dfrac{-2\lambda+6\lambda^2-2\lambda^3+(6\lambda^2-5\lambda+2)q+(6\lambda^2-5\lambda+2)q^2+(3-3\lambda)q^3}{[2]_q3[3]_q}, & for\ \lambda\le q<1,\end{cases}\\
\Lambda_2(q)&=\int_0^1|q\zeta-\lambda|\,\frac{\zeta}{3}\ d_q\zeta\\
&=\begin{cases}\dfrac{\lambda+(\lambda-1)q+(\lambda-1)q^2}{[2]_q3[3]_q}, & for\ 0<q<\lambda;\\[2mm] \dfrac{2\lambda^3-\lambda+(1-\lambda)q+(1-\lambda)q^2}{[2]_q3[3]_q}, & for\ \lambda\le q<1,\end{cases}\\
\Lambda_3(q)&=\int_0^1|q\zeta-\mu|\,\frac{2-\zeta}{3}\ d_q\zeta\\
&=\begin{cases}\dfrac{\mu+(3-1)q+(3\mu-1)q^2+(2\mu-2)q^3}{6[2]_q[3]_q}, & for\ 0<q<\mu;\\[2mm] \dfrac{4\mu^2-2\mu^3-\mu+(4\mu^2-3\mu+1)q+(4\mu^2-3\mu+1)q^2+(2-2\mu)q^3}{[2]_q3[3]_q}, & for\ \mu\le q<1,\end{cases}\\
\Lambda_4(q)&=\int_0^1|q\zeta-\mu|\,\frac{1+\zeta}{3}\ d_q\zeta\\
&=\begin{cases}\dfrac{2\mu+(3\mu-2)q+(3\mu-2)q^2+(\mu-1)q^3}{[2]_q3[3]_q}, & for\ 0<q<\mu;\\[2mm] \dfrac{2\mu^3+2\mu^2-2\mu+(2\mu^2-3\mu+2)q+(2\mu^2-3\mu+2)q^2+(1-\mu)q^3}{[2]_q3[3]_q}, & for\ \mu\le q<1,\end{cases}\\
\Lambda_5(q)&=\int_0^1|q\zeta-\gamma|\,\frac{1-\zeta}{3}\ d_q\zeta
\end{aligned}
$$

$$=\begin{cases}\dfrac{\gamma q+\gamma q^2+(\gamma-1)q^3}{[2]_q 3[3]_q}, & for\ 0<q<\gamma;\\ \dfrac{2\gamma^2-2\gamma^3+(2\gamma^2-\gamma)q+(2\gamma^2-\gamma)q^2+(1-\gamma)q^3}{[2]_q 3[3]_q}, & for\ \gamma\le q<1,\end{cases}$$

$$\Lambda_6(q)=\int_0^1 |q\zeta-\gamma|\,\frac{2+\zeta}{3}\,d_q\zeta$$

$$=\begin{cases}\dfrac{3\gamma+(5\gamma-3)q+(5\gamma-3)q^2+(2\gamma-2)q^3}{[2]_q 3[3]_q}, & for\ 0<q<\gamma;\\ \dfrac{4\gamma^2+2\gamma^3-3\gamma+(4\gamma^2-5\gamma+3)q+(4\gamma^2-5\gamma+3)q^2+(2-2\gamma)q^3}{[2]_q 3[3]_q}, & for\ \gamma\le q<1.\end{cases}$$

So, the proof is established. □

Corollary 1. *If we set $\lambda=0$, $\mu=\frac{1}{2}$ and $\gamma=1$ in Theorem 3, then the inequality* (10) *becomes*

$$\Bigg|\frac{\Omega\left(\frac{2a_0+a_1}{3}\right)+\Omega\left(\frac{a_0+2a_1}{3}\right)}{2} \tag{11}$$

$$-\frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\Bigg|$$

$$\le\frac{a_1-a_0}{9}\left[(\Delta_1(q)+\Delta_3(q)+\Delta_5(q))\left|\ {}_{a_0}D_q\Omega(a_0)\right|+(\Delta_2(q)+\Delta_4(q)+\Delta_6(q))\left|\ {}_{a_0}D_q\Omega(a_1)\right|\right], \tag{12}$$

where $\Delta_\jmath(q), \jmath=1,\cdots,6$ are given by

$$\Delta_1(q)=\begin{cases}\dfrac{2q+2q^2+3q^3}{[2]_q 3[3]_q}, & for\ 0\le q<1,\end{cases}$$

$$\Delta_2(q)=\begin{cases}\dfrac{q+q^2}{[2]_q 3[3]_q}, & for\ 0\le q<1,\end{cases}$$

$$\Delta_3(q)=\begin{cases}\dfrac{1+q+q^2-2q^3}{[2]_q 6[3]_q}, & for\ 0<q<1/2;\\ \dfrac{1+2q+2q^2+4q^3}{[2]_q 12[3]_q}, & for\ 1/2\le q<1,\end{cases}$$

$$\Delta_4(q)=\begin{cases}\dfrac{2-q-q^2-q^3}{[2]_q 6[3]_q}, & for\ 0<q<1/2;\\ \dfrac{-1+4q+4q^2+2q^3}{[2]_q 12[3]_q}, & for\ 1/2\le q<1,\end{cases}$$

$$\Delta_5(q)=\begin{cases}\dfrac{q+q^2}{[2]_q 3[3]_q}, & for\ 0<q<1;\end{cases}$$

$$\Delta_6(q)=\begin{cases}\dfrac{3+2q+2q^2}{[2]_q 3[3]_q}, & for\ 0<q<1;\end{cases}$$

Remark 2. *With $\lambda=\frac{3}{8}$, $\mu=\frac{1}{2}$, $\gamma=\frac{5}{8}$ and setting $q\to1^-$ in inequality* (12), *then we get*

$$\left|\frac{1}{8}\left[\Omega(a_0)+3\Omega\left(\frac{2a_0+a_1}{3}\right)+3\Omega\left(\frac{a_0+2a_1}{3}\right)+\Omega(a_1)\right]-\frac{1}{(a_1-a_0)}\int_{a_0}^{a_1}\Omega(x)d\mathrm{x}\right|$$

$$\le\frac{25(a_1-a_0)}{576}\left[|\Omega'(a_0)|+|\Omega'(a_1)|\right],$$

which is proven in [16].

Remark 3. *Setting* $\lambda = \frac{3}{8}$, $\mu = \frac{1}{2}$ *and* $\gamma = \frac{5}{8}$ *in inequality* (12) *then one acquires Theorem 3 of [23].*

Theorem 4. *In light of Lemma 3 and* $r > 1$, *if* $|{}_{a_0}D_q\Omega|^r$ *is a convex function on* $[a_0, a_1]$, *then we acquire the undermentioned inequality:*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[\lambda\Omega(a_0) + (1-\lambda+\mu)\Omega\left(\frac{2a_0+a_1}{3}\right) + (1-\mu+\gamma)\Omega\left(\frac{a_0+2a_1}{3}\right) + (1-\gamma)\Omega(a_1)\right] \\
&- \frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}} \Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x} + \int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}} \Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x} + \int_{\frac{a_0+2a_1}{3}}^{a_1} \Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\Bigg| \\
&\le \frac{a_1-a_0}{9}\Big[(\Lambda_7(q))^{1-\frac{1}{r}}\left(\Lambda_1(q)|{}_{a_0}D_q\Omega(a_0)|^r + \Lambda_2(q)|{}_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}} \\
&+ (\Lambda_8(q))^{1-\frac{1}{r}}\left(\Lambda_3(q)|{}_{a_0}D_q\Omega(a_0)|^r + \Lambda_4(q)|{}_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}} \\
&+ (\Lambda_9(q))^{1-\frac{1}{r}}\left(\Lambda_5(q)|{}_{a_0}D_q\Omega(a_0)|^r + \Lambda_6(q)|{}_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}}\Big],
\end{aligned} \tag{13}$$

where $\Lambda_\jmath(q), \jmath = 1, \cdots, 6$ *are defined in Theorem 3, and* $\Lambda_\jmath(q), \jmath = 7, \cdots, 9$ *are defined by*

$$\Lambda_7(q) = \begin{cases} \dfrac{[2]_q\lambda - q}{[2]_q}, & for\ 0 < q < \lambda; \\ \dfrac{2\lambda^2 - \lambda[2]_q + q}{32[2]_q}, & for\ \lambda \le q < 1, \end{cases}$$

$$\Lambda_8(q) = \begin{cases} \dfrac{[2]_q\mu - q}{[2]_q}, & for\ 0 < q < \mu; \\ \dfrac{2\mu^2 - \mu[2]_q + q}{32[2]_q}, & for\ \mu \le q < 1, \end{cases}$$

$$\Lambda_9(q) = \begin{cases} \dfrac{[2]_q\gamma - q}{[2]_q}, & for\ 0 < q < \gamma; \\ \dfrac{2\gamma^2 - \gamma[2]_q + q}{32[2]_q}, & for\ \gamma \le q < 1, \end{cases}$$

Proof. Employing modulus on both sides of identity in Lemma 3 along with power mean inequality, and taking into account the convexity of $|{}_{a_0}D_qf|^r$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[\Omega(a_0) + 3\Omega\left(\frac{2a_0+a_1}{3}\right) + 3\Omega\left(\frac{a_0+2a_1}{3}\right) + \Omega(a_1)\right] - \frac{1}{(a_1-a_0)}\int_{a_0}^{a_1}\Omega(x)_{a_0}d_q\mathrm{x}\right| \\
&\le \frac{a_1-a_0}{9}\left[\int_0^1 |q\zeta-\lambda|\left|{}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3} + (1-\zeta)a_0\right)\right| d_q\zeta\right. \\
&+ \int_0^1 |q\zeta-\mu|\left|{}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3} + (1-\zeta)\frac{2a_0+a_1}{3}\right)\right| d_q\zeta \\
&\left.+ \int_0^1 |q\zeta-\gamma|\left|{}_{a_0}D_q\Omega\left(\zeta a_1 + (1-\zeta)\frac{a_0+2a_1}{3}\right)\right| d_q\zeta\right] \\
&\le \frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\lambda|\ d_q\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1 |q\zeta-\lambda|\left|{}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3} + (1-\zeta)a_0\right)\right|^r d_q\zeta\right)^{\frac{1}{r}}\right] \\
&+ \frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\mu|\ d_q\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1 |q\zeta-\mu|\left|{}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3} + (1-\zeta)\frac{2a_0+a_1}{3}\right)\right|^r d_q\zeta\right)^{\frac{1}{r}}\right] \\
&+ \frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\gamma|\ d_q\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1 |q\zeta-\gamma|\left|{}_{a_0}D_q\Omega\left(\zeta a_1 + (1-\zeta)\frac{a_0+2a_1}{3}\right)\right|^r d_q\zeta\right)^{\frac{1}{r}}\right]
\end{aligned}$$

$$\leq \frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\lambda|\ d_q\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1 |q\zeta-\lambda|\left(\frac{3-\zeta}{3}|\ {}_{a_0}D_q\Omega(a_0)|^r+\frac{\zeta}{3}|\ {}_{a_0}D_q\Omega(a_1)|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]$$
$$+\frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\mu|\ d_q\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1 |q\zeta-\mu|\left(\frac{2-\zeta}{3}|\ {}_{a_0}D_q\Omega(a_0)|^r+\frac{1+\zeta}{3}|\ {}_{a_0}D_q\Omega(a_1)|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]$$
$$+\frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\gamma|\ d_q\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1 |q\zeta-\gamma|\left(\frac{1-\zeta}{3}|\ {}_{a_0}D_q\Omega(a_0)|^r+\frac{2+\zeta}{3}|\ {}_{a_0}D_q\Omega(a_1)|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right].$$

Using Lemma 2, the succeeding integrals can be evaluated easily:

$$\Lambda_7(q)=\int_0^1 |q\,\zeta-\lambda|\ d_q\zeta=\begin{cases}\dfrac{[2]_q\lambda-q}{[2]_q}, & for\ 0<q<\lambda;\\[2mm] \dfrac{2\lambda^2-\lambda[2]_q+q}{32[2]_q}, & for\ \lambda\leq q<1,\end{cases}$$

$$\Lambda_8(q)=\int_0^1 |q\,\zeta-\mu|\ d_q\zeta=\begin{cases}\dfrac{[2]_q\mu-q}{[2]_q}, & for\ 0<q<\mu;\\[2mm] \dfrac{2\mu^2-\mu[2]_q+q}{32[2]_q}, & for\ \mu\leq q<1,\end{cases}$$

$$\Lambda_9(q)=\int_0^1 |q\,\zeta-\gamma|\ d_q\zeta=\begin{cases}\dfrac{[2]_q\gamma-q}{[2]_q}, & for\ 0<q<\gamma;\\[2mm] \dfrac{2\gamma^2-\gamma[2]_q+q}{32[2]_q}, & for\ \gamma\leq q<1.\end{cases}$$

So, the proof is established. □

Corollary 2. *Setting $\lambda=0$, $\mu=\frac{1}{2}$ and $\gamma=1$ in Theorem 4, then the inequality (13) becomes*

$$\left|\frac{\Omega\left(\frac{2a_0+a_1}{3}\right)+\Omega\left(\frac{a_0+2a_1}{3}\right)}{2}\right.$$
$$\left.-\frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_a d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\right|$$
$$\leq\frac{a_1-a_0}{9}\left[(\Delta_7(q))^{1-\frac{1}{r}}\left(\Delta_1(q)|_{a_0}D_q\Omega(a_0)|^r+\Delta_2(q)|_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}}\right.$$
$$+(\Delta_8(q))^{1-\frac{1}{r}}\left(\Delta_3(q)|_{a_0}D_q\Omega(a_0)|^r+\Delta_4(q)|_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}}$$
$$\left.+(\Delta_9(q))^{1-\frac{1}{r}}\left(\Delta_5(q)|_{a_0}D_q\Omega(a_0)|^r+\Delta_6(q)|_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}}\right]$$

where $\Delta_j(q), j=1,\cdots,6$ are defined in Corollary 1, and $\Delta_j(q), j=7,\cdots,9$ are defined by

$$\Delta_7(q)=\frac{q}{[2]_q32}$$
$$\Delta_8(q)=\begin{cases}\dfrac{1-q}{2[2]_q}, & for\ 0<q<\frac{1}{2};\\[2mm] \dfrac{q}{64[2]_q}, & for\ \frac{1}{2}\leq q<1,\end{cases}$$
$$\Delta_9(q)=\frac{1}{[2]_q}.$$

Remark 4. *Setting $\lambda=\frac{3}{8}$, $\mu=\frac{1}{2}$, $\gamma=\frac{5}{8}$ and $q\to 1^-$ in inequality (13), it becomes*

$$\left|\frac{1}{8}\left[\Omega(a_0)+3\Omega\left(\frac{2a_0+a_1}{3}\right)+3\Omega\left(\frac{a_0+2a_1}{3}\right)+\Omega(a_1)\right]-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}\Omega(x)\ d\mathrm{x}\right|$$

$$\leq \frac{a_1-a_0}{36}\left\{\left(\frac{17}{16}\right)^{1-1/r}\left(\frac{251\left|\Omega'(a_0)\right|^r+937\left|\Omega'(a_1)\right|^r}{1152}\right)^{1/r}+\left(\frac{\left|\Omega'(a_0)\right|^r+\left|\Omega'(a_1)\right|^r}{2}\right)^{1/r}\right.$$
$$\left.+\left(\frac{17}{16}\right)^{1-1/r}\left(\frac{937\left|\Omega'(a_0)\right|^r+251\left|\Omega'(a_1)\right|^r}{1152}\right)^{1/r}\right\}.$$

which is given in [29].

Remark 5. *Setting $\lambda=\frac{3}{8}$, $\mu=\frac{1}{2}$ and $\gamma=\frac{5}{8}$ in Theorem 4, then one can recapture Theorem 4 proved in [23].*

Theorem 5. *Under the assumptions of Lemma 3 for $r>1$ such that $s^{-1}+r^{-1}=1$, if $|_{a_0}D_q\Omega|^r$ is a convex function on $[a_0,a_1]$, then one can attain the succeeding quantum inequality:*

$$\left|\frac{1}{3}\left[\lambda\Omega(a_0)+(1-\lambda+\mu)\Omega\left(\frac{2a_0+a_1}{3}\right)+(1-\mu+\gamma)\Omega\left(\frac{a_0+2a_1}{3}\right)+(1-\gamma)\Omega(a_1)\right]\right.$$
$$\left.-\frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\right|$$
$$\leq\frac{a_1-a_0}{9}\left[(1-\lambda)\left(\frac{(3q+2)\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right.$$
$$+(1-\mu)\left(\frac{(2q+1)\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+(q+2)\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}$$
$$\left.+(1-\gamma)\left(\frac{q\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+(2q+3)\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right]. \tag{14}$$

Proof. Taking modulus and applying the Hölder's inequality in Lemma 3, and using the convexity of $|_{a_0}D_qf|^r$, we have

$$\left|\frac{1}{3}\left[\lambda\Omega(a_0)+(1-\lambda+\mu)\Omega\left(\frac{2a_0+a_1}{3}\right)+(1-\mu+\gamma)\Omega\left(\frac{a_0+2a_1}{3}\right)+(1-\gamma)\Omega(a_1)\right]\right.$$
$$\left.-\frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\right|$$
$$\leq\frac{a_1-a_0}{9}\left[\int_0^1|q\zeta-\lambda|\left|\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a_0\right)\right|\ d_q\zeta\right.$$
$$+\int_0^1|q\zeta-\mu|\left|\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3}+(1-\zeta)\frac{2a_0+a_1}{3}\right)\right|\ d_q\zeta$$
$$\left.+\int_0^1|q\zeta-\gamma|\left|\ {}_{a_0}D_q\Omega\left(\zeta a_1+(1-\zeta)\frac{a_0+2a_1}{3}\right)\right|\ d_q\zeta\right]$$
$$\leq\frac{a_1-a_0}{9}\left[\left(\int_0^1|q\zeta-\lambda|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left|\ {}_{a_0}D_q\Omega\left(\zeta\frac{2a_0+a_1}{3}+(1-\zeta)a_0\right)\right|^r\ d_q\zeta\right)^{\frac{1}{r}}\right]$$
$$+\frac{a_1-a_0}{9}\left[\left(\int_0^1|q\zeta-\mu|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left|\ {}_{a_0}D_q\Omega\left(\zeta\frac{a_0+2a_1}{3}+(1-\zeta)\frac{2a_0+a_1}{3}\right)\right|^r\ d_q\zeta\right)^{\frac{1}{r}}\right]$$
$$+\frac{a_1-a_0}{9}\left[\left(\int_0^1|q\zeta-\gamma|^s\right)^{\frac{1}{s}}\left(\int_0^1\left|\ {}_{a_0}D_q\Omega\left(\zeta a_1+(1-\zeta)\frac{a_0+2a_1}{3}\right)\right|^r\ d_q\zeta\right)^{\frac{1}{r}}\right]$$
$$\leq\frac{a_1-a_0}{9}\left[\left(\int_0^1|q\zeta-\lambda|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{3-\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\frac{\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]$$
$$+\frac{a_1-a_0}{9}\left[\left(\int_0^1|q\zeta-\mu|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{2-\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\frac{1+\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]$$

$$+\frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\gamma|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{1-\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\frac{2+\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right].$$

By using inequality (3), we have

$$\begin{aligned}\int_0^1 |q\,\zeta-\lambda|^s\ d_q\zeta &= (1-q)\sum_{n=0}^{\infty} q^n\left|q^{n+1}-\lambda\right|^s\\ &\leq (1-q)\sum_{n=0}^{\infty} q^n\left|1-\lambda\right|^s\\ &= (1-q)(1-\lambda)^s\frac{1}{(1-q)}\\ &= (1-\lambda)^s.\end{aligned}$$

So, we find that

$$\begin{aligned}&\frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\lambda|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{3-\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\frac{\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]\\ &=\frac{a_1-a_0}{9}\left[(1-\lambda)\left(\frac{(3q+2)\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right].\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\mu|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{2-\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\frac{1+\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]\\ &=\frac{a_1-a_0}{9}\left[(1-\mu)\left(\frac{(2q+1)\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+(q+2)\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right],\end{aligned}$$

and

$$\begin{aligned}&\frac{a_1-a_0}{9}\left[\left(\int_0^1 |q\zeta-\gamma|^s\ d_q\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{1-\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\frac{2+\zeta}{3}\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r\right)\ d_q\zeta\right)^{\frac{1}{r}}\right]\\ &\frac{a_1-a_0}{9}\left[(1-\gamma)\left(\frac{q\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+(2q+3)\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right].\end{aligned}$$

So, the proof is established. □

Corollary 3. *Setting $\lambda=0$, $\mu=\frac{1}{2}$ and $\gamma=1$ in Theorem 4, then the inequality* (14) *becomes*

$$\begin{aligned}&\left|\frac{\Omega\left(\frac{2a_0+a_1}{3}\right)+\Omega\left(\frac{a_0+2a_1}{3}\right)}{2}-\frac{1}{(a_1-a_0)}\left[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a}d_q\mathrm{x}+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}\right.\right.\\ &\left.\left.+\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\right]\right|\leq\frac{a_1-a_0}{9}\left[\left(\frac{(3q+2)\left|\ {}_{a_0}D_q\Omega(a_0)\right|^r+\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right.\\ &\left.+\frac{1}{2}\left(\frac{(2q+1)\left|\ {}_{a}D_q\Omega(a_0)\right|^r+(q+2)\left|\ {}_{a_0}D_q\Omega(a_1)\right|^r}{3[2]_q}\right)^{\frac{1}{r}}\right].\end{aligned}$$

Remark 6. *Setting $\lambda=\frac{3}{8}$, $\mu=\frac{1}{2}$, $\gamma=\frac{5}{8}$ and taking $q\to 1^-$ in inequality* (14), *it becomes*

$$\begin{aligned}&\left|\frac{1}{8}\left[\Omega(a_0)+3\Omega\left(\frac{2a_0+a_1}{3}\right)+3\Omega\left(\frac{a_0+2a_1}{3}\right)+\Omega(a_1)\right]-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}\Omega(x)\ d\mathrm{x}\right|\\ &\leq\frac{a_1-a_0}{9}\left[\frac{5}{8}\left(\frac{5\left|\Omega'(a_0)\right|^r+\left|\Omega'(a_1)\right|^r}{6}\right)^{\frac{1}{r}}+\frac{1}{2}\left(\frac{\left|\Omega'(a_0)\right|^r+\left|\Omega'(a_1)\right|^r}{2}\right)^{\frac{1}{r}}\right.\end{aligned}$$

$$+\frac{3}{8}\left(\frac{|\Omega'(a_0)|^r+5|\Omega'(a_1)|^r}{6}\right)^{\frac{1}{r}}\Bigg].$$

Remark 7. *If we set $\lambda=\frac{3}{8}$, $\mu=\frac{1}{2}$ and $\gamma=\frac{5}{8}$ in Theorem 5, then we get Theorem 5 of [23].*

4. An Example with Graphs

Example 1. *Let consider the function $\Omega:[3,6]\to\mathbb{R}$, $\Omega(\mathrm{x})=\mathrm{x}^2$ with $q\in\left[\frac{1}{2},1\right)$. By Definition 2, we have*

$$\begin{aligned}\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_q\mathrm{x} &= \int_3^4 \mathrm{x}^2\ {}_3d_q\mathrm{x}\\ &= (1-q)\sum_{n=0}^{\infty}q^n\left(3q^n+(1-q^n)4\right)^2\\ &= (1-q)\sum_{n=0}^{\infty}q^n\left(9+6q^n+q^{2n}\right)\\ &= 9+\frac{6}{[2]_q}+\frac{1}{[3]_q}.\end{aligned}$$

Similarly,

$$\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}=\int_4^5\mathrm{x}^2\ {}_4d_q\mathrm{x}=16+\frac{8}{[2]_q}+\frac{1}{[3]_q}$$

and

$$\int_{\frac{a_0+2a_1}{3}}^{a_1}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_qx=\int_5^6\mathrm{x}^2\ {}_5d_q\mathrm{x}=25+\frac{10}{[2]_q}+\frac{1}{[3]_q}.$$

Therefore, we can write

$$\begin{aligned}&\frac{\Omega\left(\frac{2a_0+a_1}{3}\right)+\Omega\left(\frac{a_0+2a_1}{3}\right)}{2}-\frac{1}{(a_1-a_0)}\Bigg[\int_{a_0}^{\frac{2a_0+a_1}{3}}\Omega(\mathrm{x})\ {}_{a_0}d_qx \\ &+\int_{\frac{2a_0+a_1}{3}}^{\frac{a_0+2a_1}{3}}\Omega(\mathrm{x})\ {}_{\frac{2a_0+a_1}{3}}d_q\mathrm{x}+\int_{\frac{a_0+2a_1}{3}}^{b}\Omega(\mathrm{x})\ {}_{\frac{a_0+2a_1}{3}}d_q\mathrm{x}\Bigg]=\frac{41}{2}-\frac{1}{3}\left[50+\frac{24}{[2]_q}+\frac{3}{[3]_q}\right].\end{aligned} \tag{15}$$

On the other hand, by Definition 1, we have

$${}_{a_0}D_q\Omega(\mathrm{x})={}_3D_q\mathrm{x}^2=\frac{\mathrm{x}^2-(q\mathrm{x}+3(1-q))^2}{(\mathrm{x}-3)(1-q)}=3(1-q)+[2]_q\mathrm{x}.$$

It is clear that, $|\ {}_{a_0}D_qf(a_0)|$ is convex on $[3,6]$. Thus, we get

$$\begin{aligned}&\frac{a_1-a_0}{9}\left[(\Delta_1(q)+\Delta_3(q)+\Delta_5(q))\,|\ {}_{a_0}D_q\Omega(a_0)|+(\Delta_2(q)+\Delta_4(q)+\Delta_6(q))\,|\ {}_{a_0}D_q\Omega(a_1)|\right] \\ =\ &\frac{1}{3}\left[6\left(\Delta_1(q)+\Delta_3(q)+\Delta_5(q)\right)+(9+3q)\left(\Delta_2(q)+\Delta_4(q)+\Delta_6(q)\right)\right].\end{aligned} \tag{16}$$

By using the equalities (16) *and* (16) *in Corollary 1, we get the inequality*

$$\begin{aligned}&\left|\frac{41}{2}-\frac{1}{3}\left[50+\frac{24}{[2]_q}+\frac{3}{[3]_q}\right]\right| \\ \leq\ &\frac{1}{3}\left[6\left(\Delta_1(q)+\Delta_3(q)+\Delta_5(q)\right)+(9+3q)\left(\Delta_2(q)+\Delta_4(q)+\Delta_6(q)\right)\right].\end{aligned} \tag{17}$$

One can see the validity of the inequality (17) *in Figure 1.*

For $r=2$, it is clear that $|\ {}_{a_0}D_q\Omega(a_0)|^r=\left([2]_q\mathrm{x}+3(1-q)\right)^2$ is convex on $[3,6]$. By the right hand side of Corollary 2, we have

$$\begin{aligned}&\frac{a_1-a_0}{9}\Big[(\Delta_7(q))^{1-\frac{1}{r}}\left(\Delta_1(q)|_{a_0}D_q\Omega(a)|^r+\Delta_2(q)|_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}} \\ &+(\Delta_8(q))^{1-\frac{1}{r}}\left(\Delta_3(q)|_{a_0}D_q\Omega(a_0)|^r+\Delta_4(q)|_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}}\end{aligned} \tag{18}$$

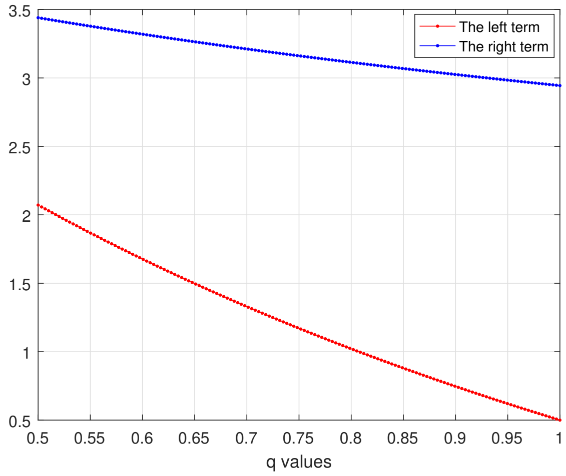

FIGURE 1. An example to Corollary 1, depending on $q \in \left[\frac{1}{2}, 1\right)$, computed and plotted by MATLAB.

$$+(\Delta_9(q))^{1-\frac{1}{r}} \left(\Delta_5(q)|_{a_0}D_q\Omega(a_0)|^r + \Delta_6(q)|_{a_0}D_q\Omega(a_1)|^r\right)^{\frac{1}{r}}\Big]$$
$$= \frac{1}{3}\left[(\Delta_7(q))^{\frac{1}{2}} \left(36\Delta_1(q) + (9+3q)^2 \Delta_2(q)\right)^{\frac{1}{2}}\right.$$
$$+ (\Delta_8(q))^{\frac{1}{2}} \left(36\Delta_3(q) + (9+3q)^2 \Delta_4(q)\right)^{\frac{1}{2}}$$
$$\left.+(\Delta_9(q))^{\frac{1}{2}} \left(36\Delta_5(q) + (9+3q)^2 \Delta_6(q)\right)^{\frac{1}{2}}\right].$$

By using the equalities (16) *and* (18) *in Corollary 2, we get the inequality*

$$\left|\frac{41}{2} - \frac{1}{3}\left[50 + \frac{24}{[2]_q} + \frac{3}{[3]_q}\right]\right| \tag{19}$$
$$\leq \frac{1}{3}\left[(\Delta_7(q))^{\frac{1}{2}} \left(36\Delta_1(q) + (9+3q)^2 \Delta_2(q)\right)^{\frac{1}{2}}\right.$$
$$+ (\Delta_8(q))^{\frac{1}{2}} \left(36\Delta_3(q) + (9+3q)^2 \Delta_4(q)\right)^{\frac{1}{2}}$$
$$\left.+(\Delta_9(q))^{\frac{1}{2}} \left(36\Delta_5(q) + (9+3q)^2 \Delta_6(q)\right)^{\frac{1}{2}}\right].$$

Figure 2 illustrates the validity of inequality (19).

By the right hand side of Corollary 3, we have

$$\frac{a_1 - a_0}{9}\left[\left(\frac{(3q+2)\,|\,_{a_0}D_q\Omega(a_0)|^r + |\,_{a_0}D_q\Omega(a_1)|^r}{3[2]_q}\right)^{\frac{1}{r}}\right.$$
$$\left.+\frac{1}{2}\left(\frac{(2q+1)\,|\,_{a_0}D_q\Omega(a_0)|^r + (q+2)\,|\,_{a_0}D_q\Omega(a_1)|^r}{3[2]_q}\right)^{\frac{1}{r}}\right]$$
$$= \frac{1}{3}\left[\left(\frac{36(3q+2) + (9+3q)^2}{3[2]_q}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{36(2q+1) + (q+2)(9+3q)^2}{3[2]_q}\right)^{\frac{1}{2}}\right].$$

By Corollary 3, we get the inequality

$$\left|\frac{41}{2} - \frac{1}{3}\left[50 + \frac{24}{[2]_q} + \frac{3}{[3]_q}\right]\right| \tag{20}$$

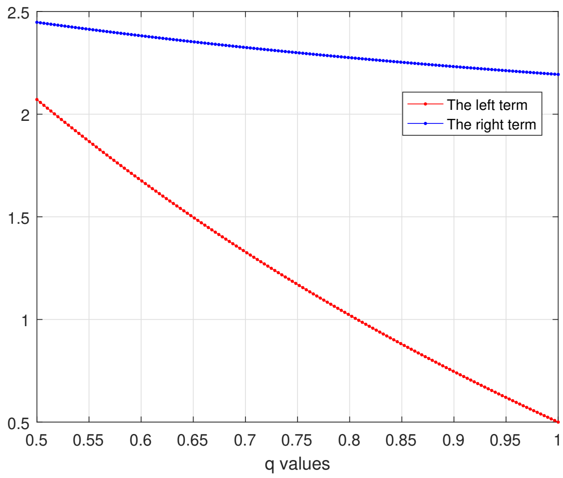

FIGURE 2. An example to Corollary 2, depending on $q \in \left[\frac{1}{2}, 1\right)$, computed and plotted by MATLAB.

$$\leq \frac{1}{3}\left[\left(\frac{36(3q+2)+(9+3q)^2}{3[2]_q}\right)^{\frac{1}{2}}+\frac{1}{2}\left(\frac{36(2q+1)+(q+2)(9+3q)^2}{3[2]_q}\right)^{\frac{1}{2}}\right].$$

One can see the validity of the inequality (20) *in Figure 3.*

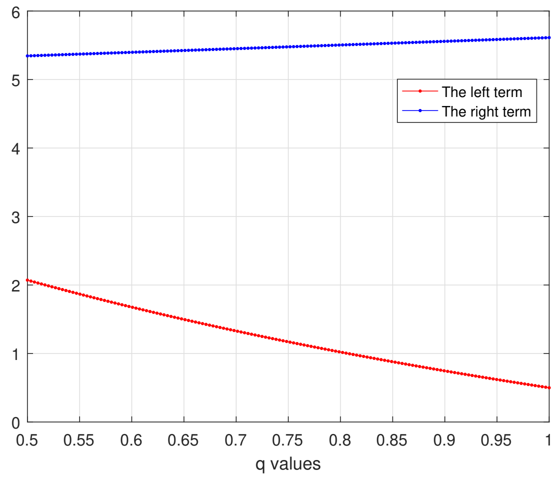

FIGURE 3. An example to Corollary 3, depending on $q \in \left[\frac{1}{2}, 1\right)$, computed and plotted by MATLAB.

5. Concluding Remarks

In this work, we demonstrate new formulations of convex function-related parameterized Newton-type inequalities using quantum calculus. We also showed that by adopting the restriction $q \to 1^-$, the recently discovered inequalities may be reconstructed into classical Newton-type inequalities. Examples in mathematics were provided to demonstrate the newly discovered inequalities. Researchers can use post quantum calculus to obtain inequalities of the Newton type linked to convex functions in upcoming publications.

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