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Some Classical Operators on Vanishing Weighted Orlicz-Morrey Spaces

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Abstract. In this paper, we investigate the boundedness of the maximal operator in vanishing weighted Orlicz-Morrey spaces. As an application of this result, we show that these spaces are also invariont with respect to some classical operators of harmonic analysis, such as Riesz potential, fractional maximal operator, Hardy operator.

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1. Introduction

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important role in the study of local behaviour and regularity properties of solutions to PDE (see e.g. [10,15] and reference therein). It is well known that the Morrey spaces are non-separable if $\lambda > 0$. The lack of approximation tools for the entire Morrey space has motivated the introduction of appropriate subspaces like vanishing spaces. The definition of the vanishing Morrey spaces involves several vanishing conditions. Each condition generate a closed subspace of $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$. We use the notation of [4] and show these conditions as (V_0), (V_∞) and (V^*). We refer to [1–3, 13, 14] for the discussion of boundedness of some classical operators in vanishing Morrey spaces.

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n),$$

where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one.

The preservation of the vanishing properties (V_0) and (V_∞) of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by maximal operator and Riesz potential was studied in the papers [5, 8, 11]. Boundedness of the maximal operator and Riesz potential in weighted Orlicz-Morrey spaces $\mathcal{M}^{\Phi,\varphi}_w(\mathbb{R}^n)$ was investigated in [6] and [12], respectively.

The purpose of this paper is to introduce vanishing weighted Orlicz-Morrey spaces $V_0\mathcal{M}^{\Phi,\varphi}_w(\mathbb{R}^n)$ and $V_\infty\mathcal{M}^{\Phi,\varphi}_w(\mathbb{R}^n)$ and to show that vanishing properties (V_0) and (V_∞) of $\mathcal{M}^{\Phi,\varphi}_w(\mathbb{R}^n)$ are preserved under the action of maximal operator and Riesz potential.

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. The (Lebesgue) measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and χ_E denotes its characteristic function. $\varphi(B) \equiv \varphi(x,r)$ for a function φ defined on $\mathbb{R}^n \times (0,\infty)$ and $B \in \mathcal{B} := \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$. We use C as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \le CB$ for some independent constant $C > 0$, and $A \approx B$ means $A \lesssim B \lesssim A$.

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2. Preliminaries

Even though, the A_p class is well known, for completeness, we offer the definition of A_p weight functions.

Definition 2.1. For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0, \infty)$ is said to be an A_p weight if

$$\sup_{B\in\mathcal{B}}\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

A locally integrable function $w : \mathbb{R}^n \to [0, \infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|}\int_B w(y)dy \le Cw(x), \qquad a.e.\ \ x \in B$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\ge1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.
We recall the definition of Young functions.

Definition 2.2. A function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \Phi(\infty) = \infty$.

A Young function Φ is said to satisfy the Δ_2-condition, denoted also by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 0$.
A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.
For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \varnothing = \infty).$$

Next we recall an important pair of indices used for Young functions. For any Young function Φ, write

$$h_\Phi(t) = \sup_{s>0}\frac{\Phi(st)}{\Phi(s)}, \quad t > 0.$$

The lower and upper dilation indices of Φ are defined by

$$i_\Phi = \lim_{t\to0^+}\frac{\log h_\Phi(t)}{\log t} \text{ and } I_\Phi = \lim_{t\to\infty}\frac{\log h_\Phi(t)}{\log t},$$

respectively. We refer to [9] for more details about the indices of Orlicz spaces.

Definition 2.3 (Weighted Orlicz Space)**.** For a Young function Φ and $w \in A_\infty$, the set

$$L^\Phi_w(\mathbb{R}^n) \equiv \left\{f - \text{measurable} : \int_{\mathbb{R}^n}\Phi(k|f(x)|)w(x)dx < \infty \text{ for some } k > 0\right\}$$

is called the weighted Orlicz space. The local weighted Orlicz space $L^\Phi_{w,\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{_B} \in L^\Phi_w(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that, $L^\Phi_w(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi_w(\mathbb{R}^n)} \equiv \|f\|_{L^\Phi_w} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\Phi\Big(\frac{|f(x)|}{\lambda}\Big)w(x)dx \le 1\right\}$$

and we have

$$\int_{\mathbb{R}^n}\Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi_w}}\Big)w(x)dx \le 1. \tag{2.1}$$

For $\Omega \subset \mathbb{R}^n$, let

$$\|f\|_{L^\Phi_w(\Omega)} := \|f\chi_\Omega\|_{L^\Phi_w}.$$

In [6], the weighted Orlicz–Morrey space $\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify weighted Orlicz spaces and generalized weighted Morrey spaces. The definition of $\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ is as follows:

Definition 2.4. Let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $w \in A_\infty$ and Φ any Young function. Denote by $\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized weighted Orlicz-Morrey space, the space of all functions $f \in L^{\Phi}_{w,\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) < \infty,$$

where $\mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = \varphi(x,r)^{-1}\,\Phi^{-1}\big(w(B(x,r))^{-1}\big)\,\|f\|_{L^{\Phi}_w(B(x,r))}$.

For a Young function Φ and $w \in A_\infty$, we denote by $\mathcal{G}_\Phi^w$ the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that

$$\inf_{B\in\mathcal{B};\, r_B\le r_{B_0}} \varphi(B) \gtrsim \varphi(B_0) \quad \text{for all } B_0 \in \mathcal{B}$$

and

$$\inf_{B\in\mathcal{B};\, r_B\ge r_{B_0}} \frac{\varphi(B)}{\Phi^{-1}\big(w(B)^{-1}\big)} \gtrsim \frac{\varphi(B_0)}{\Phi^{-1}\big(w(B_0)^{-1}\big)} \quad \text{for all } B_0 \in \mathcal{B},$$

where r_B and r_{B_0} denote the radius of the balls B and B_0, respectively.

It will be assumed that the functions φ are of the class $\mathcal{G}_\Phi^w$ in the sequel. We refer to [7, Section 5] for more information about this condition.

We consider the following subspaces of $\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$:

Definition 2.5. The vanishing weighted Orlicz-Morrey space at origin $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0.$$

The vanishing weighted Orlicz-Morrey space at infinity $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to \infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0.$$

The vanishing subspace $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ and $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ are nontrivial if $\mathcal{G}_\Phi^w$ satisfies the additional conditions

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\Phi^{-1}\big(w(B(x,r))^{-1}\big)}{\varphi(x,r)} = 0$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} = 0, \tag{2.2}$$

respectively. Since then, they contain bounded functions with compact support.

Now, we define operators investigated in this paper:

The Hardy-Littlewood maximal operator is one of the most central operators in modern harmonic analysis and theory of partial differential equations. It is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

It is well know that the maximal operator controls various other important operators of harmonic analysis. This is the case of the sharp maximal function

$$M^\sharp f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy,$$

where $f_B = \frac{1}{|B|}\int_B f(z)dz$. We have the obvious estimate by the triangle inequality

$$(M^\sharp f)(x) \le 2(Mf)(x), \quad x \in \mathbb{R}^n. \tag{2.3}$$

The multidimensional Hardy operators H and $\mathcal{H}$, defined by

$$Hf(x) := \frac{1}{|x|^n} \int_{|y|<|x|} f(y)dy \quad \text{and} \quad \mathcal{H}f(x) := \int_{|y|>|x|} \frac{f(y)}{|y|^n}dy.$$

Using that $|x-y|<2|x|$ in the integral defining H, we get the pointwise estimate

$$H(|f|)(x) \le 2^n v_n Mf(x), \quad x \in \mathbb{R}^n, \tag{2.4}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$.

We shall consider more general Hardy type operator H_α and $\mathcal{H}_\alpha$, $0 \le \alpha < n$, defined for appropriate functions f by

$$H_\alpha f(x) := |x|^{\alpha-n}\int_{|y|<|x|} f(y)dy \quad \text{and} \quad \mathcal{H}_\alpha f(x) := \int_{|y|>|x|}\frac{f(y)}{|y|^{n-\alpha}}dy.$$

It can be easily shown that operator H_α is dominated by the fractional maximal operator M_α and Riesz potential operator I_α, for $x \in \mathbb{R}^n$

$$M_\alpha f(x) = \sup_{r>0}\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}|f(y)|dy, \quad I_\alpha f(x) = \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

More precisely, for $0<\alpha<n$ there holds

$$|H_\alpha f(x)| \le v_n 2^{n-\alpha}(M_\alpha f)(x) \le 2^{n-\alpha} I_\alpha(|f|)(x), \quad x \in \mathbb{R}^n. \tag{2.5}$$

The Hardy operator $\mathcal{H}_\alpha$ is also dominated by the Riesz potential operator (cf. [1, Lemma 2.3]) as follows:

$$\mathcal{H}_\alpha(|f|)(x) \le 2^{n-\alpha} I_\alpha(|f|)(x), \quad x \in \mathbb{R}^n. \tag{2.6}$$

3. Auxiliary Estimates

The following estimates play an essential role in the proof of our results.

Lemma 3.1. *[6, Lemma 4.2] Let Φ be a Young function with $\Phi \in \nabla_2$, $f \in L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^n)$ and $B = B(x,r)$. Assume in addition $w \in A_{i_\Phi}$. Then,*

$$\|Mf\|_{L_w^\Phi(B)} \lesssim \frac{1}{\Phi^{-1}(w(B)^{-1})}\sup_{t>r}\Phi^{-1}\big(w(B(x,t))^{-1}\big)\,\|f\|_{L_w^\Phi(B(x,t))}. \tag{3.1}$$

Lemma 3.2. *[12, Lemma 4.1] Let $0<\alpha<n$, Φ be a Young function, $w \in A_{i_\Phi}$ and $\varphi(x,t)$ satisfies the condition*

$$t^\alpha\varphi(x,t) + \int_t^\infty r^\alpha\,\varphi(x,r)\frac{dr}{r} \le C\varphi(x,t)^\beta \tag{3.2}$$

for some $\beta \in (0,1)$ and for every $x \in \mathbb{R}^n$ and $t>0$. Then, for the operator I_α we have the following pointwise estimate

$$|I_\alpha f(x)| \lesssim (Mf(x))^\beta\,\|f\|_{\mathcal{M}_w^{\Phi,\varphi}}^{1-\beta}. \tag{3.3}$$

4. Main Results

In this section, we show that the subspaces $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ and $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ are invariant with respect to operators M and I_α. Moreover, we also give some corollaries of those results for various operators.

Theorem 4.1. *Let Φ be a Young function with $\Phi \in \nabla_2$, $w \in A_{i_\Phi}$ and $\varphi \in \mathcal{G}_\Phi^w$. Then, the maximal operator M is bounded on $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$.*

Proof. Since M is bounded in $\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ (cf. [6, Theorem 1.1]) we only have to show that it preserves the vanishing property (V_∞):

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0 \quad \Longrightarrow \quad \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) = 0.$$

If $f \in V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$, then for any $\epsilon>0$ there exists $R = R(\epsilon) > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(f;x,t) < \epsilon \qquad \text{for every } t \ge R.$$

Using inequality (3.1), we get

$$\mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) \lesssim \frac{1}{\varphi(x,r)}\sup_{r<t<\infty}\Phi^{-1}\big(w(B(x,t))^{-1}\big)\,\|f\|_{L_w^\Phi(B(x,t))} \lesssim \epsilon$$

for any $x \in \mathbb{R}^n$ and every $r \ge R$ (with the implicit constants independent of x and r). Therefore,

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) = 0$$

and hence $Mf \in V_\infty \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$. □

The operators $M^\sharp$ and H are pointwise dominated by the maximal function in view of the inequalities (2.3) and (2.4). Consequently, the results for the formers could be derived from the results for the latter. Therefore, we get the following corollary:

Corollary 4.2. *Under the same assumptions of Theorem 4.1, the operators $M^\sharp$ and H are bounded in $V_\infty \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$.*

Now, we show that the vanishing property (V_∞) is also preserved by the operators T_α (where T_α stands for any of the operators I_α, M_α, H_α and $\mathcal{H}_\alpha$).

Theorem 4.3. *Let Φ be a Young function with $\Phi \in \nabla_2$, $w \in A_{i_\Phi}$ and $\varphi \in \mathcal{G}_\Phi^w$. Let $\beta \in (0,1)$ and define $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If condition (3.2) holds, then T_α is bounded from $V_\infty \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $V_\infty \mathcal{M}_w^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. Since the operators M_α, H_α and $\mathcal{H}_\alpha$ are pointwise dominated by the Riesz potential operator I_α (cf. (2.5) and (2.6)), it suffices to show the result for the latter operator. The boundedness of the operator I_α in weighted Orlicz-Morrey spaces follows from [12, Theorem 4.3]. To show the preservation of vanishing property, we make use of the pointwise estimate (3.3).

Note that, from (2.1) we get

$$\int_{B(x,r)} \Psi\left(\frac{(Mf(z))^\beta}{\|Mf\|^\beta_{L^\Phi_w(B(x,r))}}\right) w(z)dz = \int_{B(x,r)} \Phi\left(\frac{Mf(z)}{\|Mf\|_{L^\Phi_w(B(x,r))}}\right) w(z)dz \le 1.$$

Thus,

$$\|(Mf)^\beta\|_{L^\Psi_w(B(x,r))} \le \|Mf\|^\beta_{L^\Phi_w(B(x,r))}. \tag{4.1}$$

By (3.3) and (4.1), we get

$$\mathfrak{A}_{\Psi,\eta,w}(I_\alpha f;x,r) \lesssim (\mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r))^\beta \|f\|^{1-\beta}_{\mathcal{M}_w^{\Phi,\varphi}} \tag{4.2}$$

for all $r > 0$ and $x \in \mathbb{R}^n$. If $f \in V_\infty \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$, then $Mf \in V_\infty \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ by Theorem 4.1. Consequently, we have $I_\alpha f \in V_\infty \mathcal{M}_w^{\Psi,\eta}(\mathbb{R}^n)$ taking into account estimate (4.2). □

Theorem 4.4. *Let Φ be a Young function with $\Phi \in \nabla_2$, $w \in A_{i_\Phi}$ and $\varphi \in \mathcal{G}_\Phi^w$. Let also*

$$m_\delta := \sup_{x\in\mathbb{R}^n} \varphi(x,\delta) < \infty \tag{4.3}$$

for every $\delta > 0$, Then, the maximal operator M is bounded on $V_0 \mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$.

Proof. The norm inequalities follow from [6, Theorem 1.1], so we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) = 0. \tag{4.4}$$

In this estimation, we follow some ideas of [14] in such passage to the limit in the case $\Phi(r) = r^p$ and $w \equiv 1$, but base ourselves on Lemma 3.1.

We rewrite the inequality (3.1) in the form

$$\mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) \le C \frac{\sup_{t>r} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \|f\|_{L^\Phi(B(x,t))}}{\varphi(x,r)}. \tag{4.5}$$

To show that $\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) < \varepsilon$ for small r, we split the right-hand side of (4.5):

$$\mathfrak{A}_{\Phi,\varphi,w}(Mf;x,r) \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{4.6}$$

where $\delta_0 > 0$ will be chosen as shown below (we may take $\delta_0 < 1$) and

$$I_{\delta_0}(x,r) := \frac{\sup_{r<t<\delta_0} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \|f\|_{L^\Phi(B(x,t))}}{\varphi(x,r)},$$

$$J_{\delta_0}(x,r) := \frac{\sup_{t>\delta_0} \Phi^{-1}\big(w(B(x,t))^{-1}\big) \|f\|_{L^\Phi(B(x,t))}}{\varphi(x,r)}$$

and it is supposed that $r < \delta_0$. Now, we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,t) < \frac{\varepsilon}{2C}, \quad \text{for all} \quad 0 < t < \delta_0,$$

where C is the constant from (4.6), which is possible since $f \in V\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$. Then, $\Phi^{-1}(w(B(x,t))^{-1})\|f\|_{L^\Phi(B(x,t))} < \frac{\varepsilon}{2C}\varphi(x,t)$ and we obtain the estimate of the first term uniform in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.2). We have

$$J_{\delta_0}(x,r) \le \frac{m_{\delta_0}\|f\|_{\mathcal{M}^{\Phi,\varphi}}}{\varphi(x,r)},$$

where m_{δ_0} is the constant from (4.3) with $\delta = \delta_0$. Then, by (2.2) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} \le \frac{\varepsilon}{2Cm_{\delta_0}\|f\|_{\mathcal{M}^{\Phi,\varphi}}},$$

which completes the proof of (4.4). □

In view of the inequalities (2.3) and (2.4), we get the following corollary:

Corollary 4.5. *Under the same assumptions of Theorem 4.4, the operators* $M^\sharp$ *and* H *are bounded in* $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$.

Similar to the proof of Theorem 4.3 we can show that the vanishing property (V_0) is also preserved by the operators T_α (where T_α stands for any of the operators I_α, M_α, H_α and $\mathcal{H}_\alpha$).

Theorem 4.6. *Let* Φ *be a Young function with* $\Phi \in \nabla_2$, $w \in A_{i_\Phi}$ *and* $\varphi \in \mathcal{G}_\Phi^w$. *Let* $\beta \in (0,1)$ *and define* $\eta(x,t) \equiv \varphi(x,t)^\beta$ *and* $\Psi(t) \equiv \Phi(t^{1/\beta})$. *If conditions (3.2) and (4.3) hold, then* T_α *is bounded from* $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $V_0\mathcal{M}_w^{\Psi,\eta}(\mathbb{R}^n)$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author have read and agreed to the published version of the manuscript.

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