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Research Article

Extensions of the operator Bellman and operator Hölder type inequalities

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ABSTRACT. In this paper, we employ the concept of operator means as well as some operator techniques to establish new operator Bellman and operator Hölder type inequalities. Among other results, it is shown that if $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are continuous fields of positive invertible operators in a unital C^*-algebra $\mathscr{A}$ such that $\int_\Omega A_t\, d\mu(t) \leq I_{\mathscr{A}}$ and $\int_\Omega B_t\, d\mu(t) \leq I_{\mathscr{A}}$, and if ω_f is an arbitrary operator mean with the representing function f, then

$$\left(I_{\mathscr{A}} - \int_\Omega (A_t \omega_f B_t)\, d\mu(t)\right)^p \geq \left(I_{\mathscr{A}} - \int_\Omega A_t\, d\mu(t)\right) \omega_{f^p} \left(I_{\mathscr{A}} - \int_\Omega B_t\, d\mu(t)\right)$$

for all $0 < p \leq 1$, which is an extension of the operator Bellman inequality.

Keywords: Bellman inequality, Cauchy-Schwarz inequality, Hölder inequality, operator mean, Hadamard product, continuous field of operators, C^*-algebra

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{L}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity $I_{\mathscr{H}}$. An operator $A \in \mathscr{L}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and in this case we write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. The set of all positive invertible operators is denoted by $\mathscr{L}(\mathscr{H})_+$. For self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. Also, an operator $A \in \mathscr{L}(\mathscr{H})$ is said to be contraction, if $A^*A \leq I_{\mathscr{H}}$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ ($t \in \mathrm{sp}(A)$) implies that $f(A) \geq g(A)$.

Let f be a continuous real valued function defined on an interval J. It is called operator monotone on J if $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$ with spectra in J. It is said to be operator concave on J if $\lambda f(A) + (1-\lambda) f(B) \leq f(\lambda A + (1-\lambda)B)$ for all self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$ with spectra in J and all $\lambda \in [0,1]$, see, e.g., [10]. Every nonnegative continuous function f is operator monotone on $[0, +\infty)$ if and only if f is operator concave on $[0, +\infty)$, see [11, Theorem 8.1]. A map Ψ on $\mathscr{L}(\mathscr{H})$ is called positive if $\Psi(A) \geq 0$ whenever $A \geq 0$ and is said to be unital if $\Psi(I_{\mathscr{H}}) = I_{\mathscr{H}}$. If Ψ is a unital positive linear map and f is an operator concave function on an interval J, then

$$f(\Psi(A)) \geq \Psi(f(A)) \qquad \text{(Davis-Choi-Jensen's inequality)} \tag{1.1}$$

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for every self-adjoint operator A on $\mathscr{H}$, whose spectrum is contained in J, see also [11, 17]. Let A and B be bounded linear operators on a Hilbert space $\mathscr{H}$. The operator $A \otimes B$ on $\mathscr{H} \otimes \mathscr{H}$ is defined by $(A \otimes B)(x \otimes y) = Ax \otimes By$ for every $x, y \in \mathscr{H}$. From this definition, it is clear that the tensor product of positive operators is positive. Furthermore, for operators $A, B, C, D \in \mathscr{L}(\mathscr{H})$, by the definition of the tensor product, we have $(A \otimes B)(C \otimes D) = AC \otimes BD$ and if A and B are positive, then $(A \otimes B)^r = A^r \otimes B^r$ for all $r \geq 0$. For a given orthonormal basis $\{e_j\}$ of a Hilbert space $\mathscr{H}$, the Hadamard product $A \circ B$ of two operators $A, B \in \mathscr{L}(\mathscr{H})$ is defined by $\langle A \circ Be_i, e_j\rangle = \langle Ae_i, e_j\rangle\langle Be_i, e_j\rangle$. It is known that the Hadamard product can be presented by filtering the tensor product $A \otimes B$ through a positive linear map. In fact, $A \circ B = U^*(A \otimes B)U$, where $U : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ue_j = e_j \otimes e_j$, see [3, 4, 9, 23].

The axiomatic theory for operator means of positive invertible operators has been developed by Kubo and Ando [16]. A binary operation ρ on $\mathscr{L}(\mathscr{H})_+$ is called an operator mean, if the following conditions are satisfied:

(i) $A \leq C$ and $B \leq D$ imply $A \,\rho\, B \leq C \,\rho\, D$;
(ii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \rho B_n \downarrow A \,\rho\, B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iii) $T^*(A \,\rho\, B)T \leq (T^*AT)\rho(T^*BT)$ $(T \in \mathscr{L}(\mathscr{H}))$;
(iv) $I_{\mathscr{H}} \,\rho\, I_{\mathscr{H}} = I_{\mathscr{H}}$.

It is easy to see that $T^*(A \,\rho\, B)T = (T^*AT)\,\rho\,(T^*BT)$ for all invertible operators T. In particular, $(\alpha A \,\rho\, \alpha B) = \alpha\,(A \,\rho\, B)\,,\ (\alpha \geq 0)$. There exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0, \infty)$ via $f(t)I_{\mathscr{H}} = I_{\mathscr{H}} \,\rho\, (tI_{\mathscr{H}})$ $(t > 0)$ with $f(1) = 1$. In addition,

$$A \,\rho\, B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

for all $A, B \in \mathscr{L}(\mathscr{H})_+$. The operator monotone function f is called the representing function of ρ. If f and g are the representing functions of the operator means ρ_f and ρ_g, respectively, then $f \leq g$ on $(0, +\infty)$ if and only if $(A \,\rho_f\, B) \leq (A \,\rho_g\, B)$ for all positive invertible operators A and B. The functions $f_{\sharp_\mu}(t) = t^\mu$, $f_{\nabla_\mu}(t) = (1-\mu) + \mu t$, and $f_{!_\mu}(t) = \left(\frac{(1-\mu)+t^{-1}\mu}{2}\right)^{-1}$ on $(0, \infty)$ give the operator weighted geometric mean $A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^\mu A^{\frac{1}{2}}$, the operator weighted arithmetic mean $A\nabla_\mu B = (1-\mu)A + \mu B$, and the operator weighted harmonic mean $A!_\mu B = \left(\frac{(1-\mu)A^{-1}+\mu B^{-1}}{2}\right)^{-1}$, respectively, for all $\mu \in (0, 1)$. An operator mean ρ is symmetric if $A\rho B = B\rho A$ for all $A, B \in \mathscr{L}(\mathscr{H})_+$. For a symmetric operator mean ρ, a parametrized operator mean ρ_t, $0 \leq t \leq 1$, is called an interpolational path for ρ if it satisfies

(1) $A \,\rho_0\, B = A$, $A\rho_{1/2}B = A \,\rho\, B$, and $A \,\rho_1\, B = B$;
(2) $(A \,\rho_p\, B)\,\rho\,(A \,\rho_q\, B) = A \,\rho_{\frac{p+q}{2}}\, B$ for all $p, q \in [0, 1]$;
(3) The map $t \in [0, 1] \mapsto A \,\rho_t\, B$ is norm continuous for each A and B.

The power means $Am_rB = A^{\frac{1}{2}}\left(\frac{I_{\mathscr{H}}+(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^r}{2}\right)^{\frac{1}{r}} A^{\frac{1}{2}}$ are some typical interpolational means for $r \in [-1, 1]$. Their interpolational paths are

$$Am_{r,t}B = A^{\frac{1}{2}}\left((1-t)I_{\mathscr{H}} + t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}} \qquad (t \in [0, 1]).$$

In particular, $Am_{1,t}B = A\nabla_t B = (1-t)A + tB$, $Am_{0,t}B = A\sharp_t B$, and $Am_{-1,t}B = A!_t B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$. If Ψ is a unital positive linear map on $\mathscr{L}(\mathscr{H})$ and ω is an operator mean, then we have

$$\Psi(A\,\omega\,B) \le \Psi(A)\,\omega\Psi(B) \tag{1.2}$$

for all positive invertible operators A and B, see [11, Theorem 5.8]. For more information about operator means, see [11, 16].

The classical Hölder inequality asserts that

$$\left(\sum_{j=1}^{n} x_j\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} y_j\right)^{\frac{1}{q}} \ge \sum_{j=1}^{n} x_j^{\frac{1}{p}} y_j^{\frac{1}{q}}, \tag{1.3}$$

where x_j, y_j $(1 \le j \le n)$ are positive real numbers and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $p = q = 2$ the above inequality states that the celebrated Cauchy-Schwarz inequality.

Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and ω be an operator mean. Then the operator mean ω is concave on pairs of positive invertible operators i.e.,

$$\left(\sum_{j=1}^{n} A_j\right) \omega \left(\sum_{j=1}^{n} B_j\right) \ge \sum_{j=1}^{n} (A_j\,\omega\,B_j), \tag{1.4}$$

where for the weighted operator mean is an extension of the operator Hölder inequality as follows

$$\left(\sum_{j=1}^{n} A_j\right) \sharp_\nu \left(\sum_{j=1}^{n} B_j\right) \ge \sum_{j=1}^{n} (A_j \sharp_\nu B_j) \qquad \text{for all } 0 \le \nu \le 1. \tag{1.5}$$

As a special case of the inequality (1.4), we have

$$(A + B)\,\omega\,(C + D) \ge (A\,\omega\,C) + (B\,\omega\,D) \tag{1.6}$$

for all positive invertible operators A, B, C, D and an operator mean ω, see [11, Theorem 5.7].

Bellman [6] proved that if p is a positive integer and a, b, a_j, b_j $(1 \le j \le n)$ are positive real numbers such that $\sum_{j=1}^{n} a_j^p \le a^p$ and $\sum_{j=1}^{n} b_j^p \le b^p$, then

$$\left((a+b)^p - \sum_{j=1}^{n} (a_j + b_j)^p\right)^{1/p} \ge \left(a^p - \sum_{j=1}^{n} a_j^p\right)^{1/p} + \left(b^p - \sum_{j=1}^{n} b_j^p\right)^{1/p}.$$

A multiplicative analogue of this inequality for $p = 2$ is due to Aczél, see [1] and its operator version in [20]. Popoviciu [22] extended Aczél's inequality for $p \ge 1$. During the last decades, several generalizations, refinements, and applications of the Bellman inequality in various settings have been given and some results related to integral inequalities are presented, see [1, 3, 5, 6, 7, 8, 12, 15, 18, 19, 20, 25].

In [19], the authors showed the following generalization of the operator Bellman inequality

$$\left(I_{\mathscr{H}} - \left(\sum_{j=1}^{n} A_j\,\omega_f\,B_j\right)\right)^p \ge \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \omega_{f^p} \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right), \tag{1.7}$$

where A_j, B_j $(1 \le j \le n)$ are positive invertible operators such that $\sum_{j=1}^{n} A_j \le I_{\mathscr{H}}$, $\sum_{j=1}^{n} B_j \le I_{\mathscr{H}}$, ω_f is a mean with the representing function f and $0 < p \le 1$.

Let $\mathscr{A}$ be a C^*-algebra of operators acting on a Hilbert space, let Ω be a locally compact Hausdorff space, and let $\mu(t)$ be a Radon measure on Ω. A field $(A_t)_{t\in\Omega}$ of operators in $\mathscr{A}$ is called a *continuous field of operators* if the function $t \mapsto A_t$ is norm continuous on Ω and the function $t \mapsto \|A_t\|$ is integrable. One can form the Bochner integral $\int_\Omega A_t\, d\mu(t)$, which is the unique element in $\mathscr{A}$ such that

$$\varphi\left(\int_\Omega A_t\, d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t) \tag{1.8}$$

for every linear functional φ in the norm dual $\mathscr{A}^*$ of $\mathscr{A}$, see [13]. Let $\mathcal{C}(\Omega, \mathscr{A})$ denote the set of bounded continuous functions on Ω with values in $\mathscr{A}$, which is a C^*-algebra under the pointwise operations and the norm $\|(A_t)\| = \sup_{t\in\Omega} \|A_t\|$, see [13].

In this paper, by the concept of operator means, we obtain a refinement of the inequalities (1.2). By using this refinement, we present some refinements of the operator Hölder inequality (1.5) and the operator Bellman inequality (1.7) for positive invertible operators. Furthermore, we generalize and refine some derived results for *continuous fields of operators* in a C^*-algebra $\mathscr{A}$.

2. Refinements of some Generalized operator inequalities

In this section, by the concept of operator means, we present some refinements of the operator Hölder inequality and the operator Bellman inequality. We need the following lemmas to illustrate our result.

Lemma 2.1 ([18])**.** *Let $A, B \in \mathscr{L}(\mathscr{H})_+$ be such that A is contraction, let h be a nonnegative operator monotone function on $[0, +\infty)$, and let ω_f be an operator mean with the representing function f. Then*

$$A\, \omega_{hof} B \le h(A\, \omega_f B).$$

In the following lemma, we present an operator inequality for three arbitrary operator means.

Lemma 2.2. *Let σ, τ, ρ be three arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$. Then*

$$A \le (A\, \sigma\, B)\, \rho\, (A\, \tau\, B) \le B \tag{2.9}$$

for all positive invertible operators A and B such that $A \le B$.

Proof. Assume that A and B are positive invertible operators such that $A \le B$. Applying the properties of operator means, we have

$$A = A\, \sigma\, A \le A\, \sigma\, B \le B\, \sigma\, B = B \quad \text{and} \quad A = A\, \tau\, A \le A\, \tau\, B \le B\, \tau\, B = B.$$

Moreover, if $\sigma \le \tau$, i.e., $A\, \sigma\, B \le A\, \tau\, B$, then

$$(A \le) \qquad A\, \sigma\, B \le (A\, \sigma\, B)\rho(A\, \tau\, B) \le A\, \tau\, B \qquad (\le B) \tag{2.10}$$

and if $\tau \le \sigma$, i.e., $A\, \tau\, B \le A\, \sigma\, B$, then

$$(A \le) \qquad A\, \tau\, B \le (A\, \sigma\, B)\, \rho\, (A\, \tau\, B) \le A\, \sigma\, B \qquad (\le B). \tag{2.11}$$

Combining inequalities (2.10) and (2.11), we get

$$A \le (A\, \sigma\, B)\, \rho\, (A\, \tau\, B) \le B,$$

as required. ∎

Remark 2.1. *Assume that $\sigma, \tau, \rho_1, \rho_2$ are arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$ and A, B are positive invertible operators such that $A \le B$. Then, applying Lemma 2.1, we get*

$$A \le (A\, \sigma\, B)\, \rho_1\, (A\, \tau\, B) \le (A\, \sigma\, B)\, \rho_2\, (A\, \tau\, B) \le B,$$

where $\rho_1 \le \rho_2$. To see this, note that, if $\rho_1 \le \rho_2$, then for the positive invertible operators $A\, \sigma\, B$ and $A\, \tau\, B$, we have

$$(A\, \sigma\, B)\, \rho_1\, (A\, \tau\, B) \le (A\, \sigma\, B)\, \rho_2\, (A\, \tau\, B).$$

Moreover, by Lemma 2.1, we have

$$A \leq (A\,\sigma\, B)\,\rho_1\,(A\,\tau\, B) \qquad \text{and} \qquad (A\,\sigma\, B)\,\rho_2\,(A\,\tau\, B) \leq B$$

for arbitrary operator means σ, τ with $\sigma \leq \tau$ or $\tau \leq \sigma$. Combining the above inequalities, we get desired result.

Remark 2.2. *Assume that σ_f and σ_g are arbitrary operator means with the representing functions f and g, respectively, with $f \leq g$ or $g \leq f$. As a special case of Lemma 2.1 for $\rho = \nabla_\lambda$, $(0 \leq \lambda \leq 1)$, we have*

$$A \leq A\sigma_{(1-\lambda)f+\lambda g}B \leq B \tag{2.12}$$

for all positive invertible operators A and B such that $A \leq B$. To see this, note that

$$\begin{aligned}(A\,\sigma_f\, B)\nabla_\lambda(A\,\sigma_g\, B) &= (1-\lambda)A^{\frac{1}{2}}f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} + \lambda A^{\frac{1}{2}}g(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}\Big((1-\lambda)f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) + \lambda g(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})\Big)A^{\frac{1}{2}} \\ &= A\,\sigma_{(1-\lambda)f+\lambda g}\, B.\end{aligned}$$

Hence, by Lemma 2.1, we get

$$A \leq (A\,\sigma_f\, B)\nabla_\lambda(A\,\sigma_g\, B) = A\,\sigma_{(1-\lambda)f+\lambda g}\, B \leq B,$$

as required.

As an application of the above result, we have the next lemma, which is a refinement of the inequality (1.2).

Lemma 2.3. *Let $\sigma, \tau, \rho, \omega$ be arbitrary operator means such that $\sigma \leq \tau$ or $\tau \leq \sigma$, and let Ψ be a unital positive linear map on $\mathscr{L}(\mathscr{H})$. Then*

$$\begin{aligned}\big(\Psi(A)\,\omega\,\Psi(B)\big)^p &\geq \Big(\Psi^p(A\,\omega\, B)\,\sigma\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big)\,\rho\,\Big(\Psi^p(A\,\omega\, B)\,\tau\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big) \\ &\geq \Psi^p(A\,\omega\, B)\end{aligned} \tag{2.13}$$

for all positive invertible operators A, B and $0 < p \leq 1$.

Proof. Applying the inequality (1.2) and the operator monotonicity of $g(t) = t^p, (0 < p \leq 1)$, we have

$$\Psi^p(A\,\omega\, B) \leq (\Psi(A)\,\omega\Psi(B))^p.$$

Replacing A by $\Psi^p(A\,\omega\, B)$ and B by $(\Psi(A)\,\omega\Psi(B))^p$, respectively, in the inequality (2.9), we have

$$\begin{aligned}\Psi^p(A\,\omega\, B) &\leq \Big(\Psi^p(A\,\omega\, B)\,\sigma\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big)\,\rho\,\Big(\Psi^p(A\,\omega\, B)\,\tau\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big) \\ &\leq \big(\Psi(A)\,\omega\,\Psi(B)\big)^p\end{aligned}$$

for all operator means $\sigma, \tau, \rho, \omega$ such that $\sigma \leq \tau$ or $\tau \leq \sigma$, as required. □

In the first result of this section, we present a refinement of the operator Hölder inequality (1.4) as follows.

Theorem 2.1. *Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and $\sigma, \tau, \rho, \omega$ be arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$. Then*

$$\left(\left(\sum_{j=1}^n A_j\right) \omega \left(\sum_{j=1}^n B_j\right)\right)^p$$
$$\ge \left[\left(\sum_{j=1}^n (A_j \omega B_j)\right)^p \sigma \left[\left(\sum_{j=1}^n A_j\right) \omega \left(\sum_{j=1}^n B_j\right)\right]^p\right] \rho \left[\left(\sum_{j=1}^n (A_j \omega B_j)\right)^p \tau \left[\left(\sum_{j=1}^n A_j\right) \omega \left(\sum_{j=1}^n B_j\right)\right]^p\right]$$
$$\ge \left(\sum_{j=1}^n (A_j \omega B_j)\right)^p$$

for $0 < p \le 1$.

Proof. Assume that $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and $\sigma, \tau, \rho, \omega$ are arbitrary operator means with $\sigma \le \tau$ or $\tau \le \sigma$. Note that if $A_1 \oplus \cdots \oplus A_n$ and $B_1 \oplus \cdots \oplus B_n$ are two diagonal operator matrices, then by the definition of operator means, for the operator mean ω, we have

$$(A_1 \oplus \cdots \oplus A_n)\omega(B_1 \oplus \cdots \oplus B_n) = (A_1 \omega B_1) \oplus \cdots \oplus (A_n \omega B_n).$$

Replacing A by $A_1 \oplus \cdots \oplus A_n$ and B by $B_1 \oplus \cdots \oplus B_n$ in the inequality (2.13) and taking Ψ in the inequality (2.13) to be the unital positive linear map defined on the diagonal blocks of operators by $\Psi(A_1 \oplus \cdots \oplus A_n) = \frac{1}{n}\sum_{j=1}^n A_j$, we have the desired result. □

As a consequence of Theorem 2.1, we have a refinement of the operator Hölder inequality involving the weighted geometric mean.

Corollary 2.1. *Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and σ, τ, ρ be arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$. Then*

$$\left(\sum_{j=1}^n (A_j \sharp_\nu B_j)\right)^p$$
$$\le \left[\left(\sum_{j=1}^n (A_j \sharp_\nu B_j)\right)^p \sigma \left[\left(\sum_{j=1}^n A_j\right) \sharp_\nu \left(\sum_{j=1}^n B_j\right)\right]^p\right]$$
$$\rho \left[\left(\sum_{j=1}^n (A_j \sharp_\nu B_j)\right)^p \tau \left[\left(\sum_{j=1}^n A_j\right)^p \sharp_\nu \left(\sum_{j=1}^n B_j\right)\right]^p\right]$$
$$\le \left(\left(\sum_{j=1}^n A_j\right) \sharp_\nu \left(\sum_{j=1}^n B_j\right)\right)^p \tag{2.14}$$

for all $\nu \in [0,1]$ and $0 < p \le 1$. In particular, for $\tau = \sigma$, we have

$$\left(\sum_{j=1}^n (A_j \sharp_\nu B_j)\right)^p \le \left(\sum_{j=1}^n (A_j \sharp_\nu B_j)\right)^p \sigma \left[\left(\sum_{j=1}^n A_j\right) \sharp_\nu \left(\sum_{j=1}^n B_j\right)\right]^p$$
$$\le \left(\left(\sum_{j=1}^n A_j\right) \sharp_\nu \left(\sum_{j=1}^n B_j\right)\right)^p$$

for all $\nu \in [0,1]$ and $0 < p \leq 1$.

Remark 2.3. *Note that if $0 \leq s \leq t \leq 1$, then $A\sharp_s B \leq A\sharp_t B$ for positive invertible operators A and B such that $A \leq B$. Therefore, for positive invertible operators A_j, B_j $(1 \leq j \leq n)$ with $A_jB_j = B_jA_j$ $(1 \leq j \leq n)$ and $\sigma = \sharp_s$, $\rho = \nabla$, and $\tau = \sharp_t$ in Corollary 2.1, we have*

$$\begin{aligned}
&\sum_{j=1}^{n} A_j^{1-\nu} B_j^{\nu} \\
\leq &\frac{1}{2}\left[\left(\sum_{j=1}^{n} A_j^{1-\nu} B_j^{\nu}\right)^{1-s}\left(\sum_{j=1}^{n} A_j\right)^{(1-\nu)s}\left(\sum_{j=1}^{n} B_j\right)^{\nu s}\right] \\
+ &\left[\left(\sum_{j=1}^{n} A_j^{1-\nu} B_j^{\nu}\right)^{1-t}\left(\sum_{j=1}^{n} A_j\right)^{(1-\nu)t}\left(\sum_{j=1}^{n} B_j\right)^{\nu t}\right] \\
\leq &\left(\sum_{j=1}^{n} A_j\right)^{(1-\nu)}\left(\sum_{j=1}^{n} B_j\right)^{\nu}
\end{aligned}$$

for all $0 \leq s \leq t \leq 1$, which is an extension and a refinement of the classical Hölder inequality.

In the following result, we obtain a refinement of the generalized operator Bellman inequality (1.7).

Theorem 2.2. *Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \leq j \leq n)$ be such that $\sum_{j=1}^{n} A_j \leq I_{\mathscr{H}}$, $\sum_{j=1}^{n} B_j \leq I_{\mathscr{H}}$, and let ω_f be an operator mean with the representing function f and $0 < p \leq 1$. Then*

$$\begin{aligned}
&\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \omega_f^p \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right) \\
\leq &\Bigg(\left(\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\mu} \\
&\rho\left(\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\nu} - \sum_{j=1}^{n}(A_j \omega_f B_j)\Bigg)^{p} \\
\leq &\left(I_{\mathscr{H}} - \sum_{j=1}^{n}(A_j \omega_f B_j)\right)^{p}
\end{aligned}$$

for all arbitrary means ρ and $0 \leq \mu \leq \nu \leq 1$.

Proof. Applying Theorem 2.1 to $X_j, Y_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \leq j \leq n+1)$ and to two arbitrary operator means ρ, ω_f, and to the weighted geometric means $\sharp_\mu$, and $\sharp_\nu$ such that $0 \leq \mu \leq \nu \leq 1$, we get

$$\begin{aligned}
&\sum_{j=1}^{n+1}(X_j \omega Y_j) \\
&\leq \left[\left(\sum_{j=1}^{n+1}(X_j \omega_f Y_j)\right) \sharp_\mu \left[\left(\sum_{j=1}^{n+1} X_j\right) \omega_f \left(\sum_{j=1}^{n+1} Y_j\right)\right]\right] \\
&\quad \rho \left[\left(\sum_{j=1}^{n+1}(X_j \omega_f Y_j)\right) \sharp_\nu \left[\left(\sum_{j=1}^{n+1} X_j\right) \omega_f \left(\sum_{j=1}^{n+1} Y_j\right)\right]\right] \\
&\leq \left(\sum_{j=1}^{n+1} X_j\right) \omega_f \left(\sum_{j=1}^{n+1} Y_j\right). \qquad (2.15)
\end{aligned}$$

By putting $X_j = A_j, Y_j = B_j$ $(1 \leq j \leq n)$ $X_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n A_j$, and $Y_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n B_j$, and taking $\sigma = \sharp_\mu$ and $\sigma = \sharp_\nu$ in the inequalities (2.15), we get

$$\begin{aligned}
&\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right) \\
&\leq \left[\left(\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right) \sharp_\mu \left(I_{\mathscr{H}} \omega_f I_{\mathscr{H}}\right)\right] \\
&\quad \rho \left[\left(\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right) \sharp_\nu \left(I_{\mathscr{H}} \omega_f I_{\mathscr{H}}\right)\right] \\
&\leq \left(I_{\mathscr{H}} \omega_f I_{\mathscr{H}}\right)
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right) \\
&\leq \left[\left(\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right) \sharp_\mu I_{\mathscr{H}}\right] \\
&\quad \rho \left[\left(\sum_{j=1}^{n}(A_j \omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right) \omega_f \left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right) \sharp_\nu I_{\mathscr{H}}\right] \\
&\leq I_{\mathscr{H}}, \qquad \text{for } 0 \leq \mu \leq \nu \leq 1.
\end{aligned}$$

Using the definition of the operator means $\sharp_\mu$ and $\sharp_\nu$, we have

$$\begin{aligned}
&\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\\
&\le \left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\mu}\\
&\quad \rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\nu}\\
&\le I_{\mathscr{H}}
\end{aligned}$$

for all arbitrary means ρ and $0 \le \mu \le \nu \le 1$. Hence,

$$\begin{aligned}
&\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\\
&\le \left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\mu}\\
&\quad \rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I - \sum_{j=1}^{n} A_j\right)\omega_f\left(I - \sum_{j=1}^{n} B_j\right)\right)^{\nu} - \sum_{j=1}^{n}(A_j\omega_f B_j)\\
&\le I_{\mathscr{H}} - \sum_{j=1}^{n}(A_j\omega_f B_j), \qquad \text{for } 0 \le \mu \le \nu \le 1.
\end{aligned}$$

It follows from the operator monotonicity of $g(t) = t^p$ $(0 < p \le 1)$, the above inequalities, and Lemma 2.1 that

$$\begin{aligned}
&\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_{f^p}\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\\
&\le \Bigg(\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\mu}\\
&\quad \rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I - \sum_{j=1}^{n} A_j\right)\omega_f\left(I - \sum_{j=1}^{n} B_j\right)\right)^{\nu} - \sum_{j=1}^{n}(A_j\omega_f B_j)\Bigg)^{p}\\
&\le \left(I_{\mathscr{H}} - \sum_{j=1}^{n}(A_j\omega_f B_j)\right)^{p} \qquad \text{for } 0 \le \mu \le \nu \le 1,
\end{aligned}$$

as required. □

3. Some extensions for continuous fields of operators

Let $\mathscr{A}$ be a C^*-algebra of operators acting on a Hilbert space, let Ω be a compact Hausdorff space, and let $(A_t)_{t\in\Omega}$ be a continuous field of operators in $\mathscr{A}$. In this section, by using the concept of the *continuous fields of operators*, we present some results involving the operator Hölder type inequalities and the operator Bellman type inequalities.

We need following lemma to illustrate our results.

Lemma 3.4. *Let $\mathscr{A}$ be a C^*-algebra, Ω be a compact Hausdorff space equipped with a Radon measure μ, and let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators. Then*

$$\int_\Omega \int_\Omega (A_t \circ B_s)\, d\mu(t)\, d\mu(s) = \int_\Omega A_t\, d\mu(t) \circ \int_\Omega B_s\, d\mu(s) \qquad (A_t, B_s \in \mathscr{A}). \tag{3.16}$$

Proof. Assume that $\mathscr{A}$ is a C^*-algebra of operators acting on a Hilbert space, Ω is a compact Hausdorff space, and $(A_t)_{t\in\Omega}$ is a continuous field of operators in $\mathscr{A}$. Using [21, Page 78], since $\mathbf{A} : t \mapsto A_t$ is a continuous function from Ω to $\mathscr{A}$, for every operator $A_t \in \mathscr{A}$ and for every $\varepsilon > 0$, we can consider an element of the form

$$I_\lambda(A_t) = \Sigma_{k=1}^n \mathbf{A}(t_k)\mu(E_k) = \Sigma_{k=1}^n A_{t_k}\mu(E_k),$$

where the $E_k{}'s$ form a partition of Ω into disjoint Borel subsets, and

$$t_k \in E_k \subseteq \{t \in \Omega : \|A_t - A_{t_k}\| \le \varepsilon\} \qquad (1 \le k \le n),$$

with $\lambda = \{E_1, \cdots, E_n, \varepsilon\}$. Then $(I_\lambda(A_t))_{\lambda\in\Lambda}$ is a uniformly convergent net to $\int_\Omega A_t d\mu(t)$. It follows from the norm continuity of the tensor product of two operators that for any operator $B \in \mathscr{A}$, we have

$$\int_\Omega (A_t \otimes B)\, d\mu(t) = \left(\int_\Omega A_t\, d\mu(t)\right) \otimes B. \tag{3.17}$$

Also, by using the definition of the Bochner integral for any operator $X \in \mathscr{A}$, we have $\int_\Omega (X^* A_t X) d\mu(t)$ $X^* \left(\int_\Omega A_t d\mu(t)\right) X$. Therefore, for an arbitrary operator $B \in \mathscr{A}$, we get

$$\begin{aligned}\int_\Omega (A_t \circ B)\, d\mu(t) &= \int_\Omega V^*(A_t \otimes B) V\, d\mu(t) = V^* \int_\Omega (A_t \otimes B)\, d\mu(t) V\\ &= V^* \left(\int_\Omega A_t\, d\mu(t) \otimes B\right) V = \int_\Omega A_t\, d\mu(t) \circ B \qquad (A_t, B \in \mathscr{A}),\end{aligned} \tag{3.18}$$

where $V : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ve_j = e_j \otimes e_j$, for a given orthonormal basis $\{e_j\}$ of the Hilbert space $\mathscr{H}$. Hence, we have

$$\begin{aligned}
\int_\Omega \int_\Omega (A_t \circ B_s)\, d\mu(t)\, d\mu(s) &= \int_\Omega \int_\Omega V^*(A_t \otimes B_s) V\, d\mu(t)\, d\mu(s) \\
&= \int_\Omega V^* \left(\int_\Omega (A_t \otimes B_s)\, d\mu(t) \right) V\, d\mu(s) \quad \text{(by (3.18))} \\
&= \int_\Omega V^* \left(\left(\int_\Omega A_t\, d\mu(t) \right) \otimes B_s \right) V\, d\mu(s) \quad \text{(by (3.17))} \\
&= \int_\Omega \left(\int_\Omega A_t\, d\mu(t) \right) \circ B_s\, d\mu(s) \\
&= \int_\Omega U^* \left(\left(\int_\Omega A_t\, d\mu(t) \right) \otimes B_s \right) U\, d\mu(s) \\
&= U^* \left(\int_\Omega \left(\int_\Omega A_t\, d\mu(t) \right) \otimes B_s\, d\mu(s) \right) U \quad \text{(by (3.18))} \\
&= U^* \left(\left(\int_\Omega A_t\, d\mu(t) \right) \otimes \left(\int_\Omega B_s\, d\mu(s) \right) \right) U \quad \text{(by (3.17))} \\
&= \int_\Omega A_t\, d\mu(t) \circ \int_\Omega B_s\, d\mu(s) \qquad \text{for } A_t, B_s \in \mathscr{A}.
\end{aligned}$$

□

The first result of this section is the Hölder inequality for *continuous fields of operators* involving an arbitrary operator mean. The main ideas of the following result are stimulated by [4].

Theorem 3.3. *Let $\mathscr{A}$ be a C^*-algebra, Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $C(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators, and let ω_f be an operator mean with the representing function f. Then*

$$\left(\int_\Omega A_s\, d\mu(s) \right) \omega_f \left(\int_\Omega B_s\, d\mu(s) \right) \geq \int_\Omega (A_s\, \omega_f\, B_s)\, d\mu(s). \tag{3.19}$$

Proof. For the *continuous fields of positive invertible operators* $\mathbf{A} = (A_t)_{t\in\Omega} \in \mathscr{A}$ and $\mathbf{B} = (B_t)_{t\in\Omega} \in \mathscr{A}$, we put the positive unital linear map

$$\Psi(S) = \int_\Omega Z^* S Z d\mu(t) \ \ (S \in \mathscr{A}),$$

where $Z = B_t^{\frac{1}{2}} \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}}$. Thus, we have

$$\begin{aligned}
&\left(\int_\Omega A_t\, d\mu(t) \right) \omega_f \left(\int_\Omega B_s\, d\mu(s) \right) \\
&= \left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} f \left(\left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} \int_\Omega A_t\, d\mu(t) \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}} \right) \left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} \\
&= \left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} f \left(\int_\Omega \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}} B_t^{\frac{1}{2}} (B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}) B_t^{\frac{1}{2}} \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}} d\mu(t) \right) \\
&\times \left(\int_\Omega B_s\, d\mu(t) \right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} f\left(\int_\Omega Z^* B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}} Z d\mu(t)\right)\left(\int_\Omega B_s\, d\mu(t)\right)^{\frac{1}{2}} \\
&= \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} f\left(\Phi\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)\right)\left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \\
&\geq \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \Phi\left(f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)\right)\left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \qquad \text{(by (1.1))} \\
&= \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \left(\int_\Omega Z^* f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) Z d\mu(t)\right)\left(\int_\Omega B_s\, d\mu(t)\right)^{\frac{1}{2}} \\
&= \int_\Omega B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}}\, d\mu(t) \\
&= \int_\Omega (A_t\, \omega_f\, B_t)\, d\mu(t),
\end{aligned}
$$

as required. □

Remark 3.4. *In the discrete case $\Omega = \{1, \cdots, n\}$, for positive invertible operators $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$, Theorem 3.4 enforces the inequality (1.4).*

Remark 3.5. *Assume that $\Omega = [0, 1]$ is with the Lebesgue measure and $\mathscr{A} = \mathbb{R}$ is the real numbers. Then $\mathcal{C}([0, 1], \mathbb{R})$ is the C^*-algebra involving all continuous real-valued functions over $[0, 1]$. As a special case of Theorem 3.3, we have the integral version of the Hölder inequality as follows*

$$
\left(\int_a^b f(x)\, dx\right) \omega \left(\int_a^b g(x)\, dx\right) \geq \int_a^b (f(x)\, \omega\, g(x))\, dx,
$$

where $f, g \in \mathcal{C}([0, 1], \mathbb{R})$ are positive functions and ω is an operator mean.

Using the inequality (2.9), we obtain a refinement of the Hölder inequality for *continuous fields of operators* (3.19) as follows.

Theorem 3.4. *Let $\mathscr{A}$ be a C^*-algebra, Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators, and let $\sigma, \tau, \rho, \omega$ be arbitrary operator means such that $\sigma \leq \tau$ or $\tau \leq \sigma$. Then*

$$
\begin{aligned}
\left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right) &\geq \left[\left(\int_\Omega (A_t\, \omega\, B_t)\, d\mu(t)\right) \sigma \left[\left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right)\right]\right] \\
&\quad \rho \left[\left(\int_\Omega (A_t\, \omega\, B_t)\, d\mu(t)\right) \tau \left[\left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right)\right]\right] \\
&\geq \int_\Omega (A_t\, \omega\, B_t)\, d\mu(t).
\end{aligned}
$$

Proof. Using the inequality (3.19) and replacing

$$
A \text{ by } \int_\Omega (A_t\, \omega\, B_t)\, d\mu(t) \qquad \text{and} \qquad B \text{ by } \left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right)
$$

in the inequality (2.9), respectively, we get the desired result. □

In the next result, we obtain an inequality for *continuous fields of operators*.

Theorem 3.5. *Let $\mathscr{A}$ be a C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le A$ and $\int_\Omega B_t\, d\mu(t) \le B$ for some positive invertible operators $A, B \in \mathscr{A}$, and let ω_f be an arbitrary operator mean with the representing function f. Then*

$$\left((A\,\omega_f\,B) - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p \ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(B - \int_\Omega B_t\,d\mu(t)\right) \tag{3.20}$$

for all $0 < p \le 1$.

Proof. Assume $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are *continuous fields* of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le A$ and $\int_\Omega B_t\, d\mu(t) \le B$ for some positive invertible operators $A, B \in \mathscr{A}$. Then we have

$$\begin{aligned} A\,\omega_f\,B &= \left(A - \int_\Omega A_t\,d\mu(t) + \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t) + \int_\Omega B_t\,d\mu(t)\right) \\ &\ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t)\right) + \left(\int_\Omega A_t\,d\mu(t)\,\omega_f\int_\Omega B_t\,d\mu(t)\right) \\ &\qquad \text{(by the inequality (1.6))} \\ &\ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t)\right) + \left(\int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right). \end{aligned}$$

Hence, by the above inequality, the operator monotonicity of $f(t) = t^p$ $(0 < p \le 1)$ and Lemma 2.1, we get

$$\begin{aligned} \left((A\,\omega_f\,B) - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p &\ge \left(\left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t)\right)\right)^p \\ &\ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(B - \int_\Omega B_t\,d\mu(t)\right), \end{aligned}$$

as required. □

In the next result, by using Theorem 3.5, we have the operator Bellman inequality for *continuous fields* in a unital C^*-algebra.

Corollary 3.2. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le I_\mathscr{A}$ and $\int_\Omega B_t\, d\mu(t) \le I_\mathscr{A}$, and let ω_f be an operator mean with the representing function f. Then*

$$\left(I_\mathscr{A} - \int_\Omega (A_t\omega_f B_t)\,d\mu(t)\right)^p \ge \left(I_\mathscr{A} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_\mathscr{A} - \int_\Omega B_t\,d\mu(t)\right) \tag{3.21}$$

for all $0 < p \le 1$.

Remark 3.6. *Assume that $\mathcal{C}([0,1],\mathbb{R})$ is the C^*-algebra involving all continuous real-valued functions over $[0,1]$. As a special case of the inequality (3.21), we have the integral version of the Bellman inequality as follows*

$$\left(1 - \int_a^b (g(x)\omega_f h(x))\,dx\right)^p \ge \left(1 - \int_a^b g(x)\,dx\right)\omega_{f^p}\left(1 - \int_a^b h(x)\,dx\right) \qquad (0 < p \le 1),$$

where $f, g \in \mathcal{C}([0,1],\mathbb{R})$ are positive functions such that $\int_a^b g(x)\,dx \le 1$ and $\int_a^b h(x)\,dx \le 1$, and ω_f is an operator mean with the representing function f. In particular, for $\omega_f = \sharp_{\frac{1}{2}}$, we have

$$1 - \int_a^b \sqrt{g(x)h(x)}\,dx \ge \sqrt{1 - \int_a^b g(x)\,dx}\sqrt{1 - \int_a^b h(x)\,dx}.$$

These two above inequalities are the integral version of the Bellman inequality (1.7).

In the next theorem, we present a refinement of the operator Bellman inequality (3.21) for *continuous fields of operators*.

Theorem 3.6. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega,\mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\,d\mu(t) \le I_{\mathscr{A}}$, $\int_\Omega B_t\,d\mu(t) \le I_{\mathscr{A}}$, and let ω_f be an arbitrary operator mean with the representing function f. Then*

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega (A_t\omega_f B_t)\,d\mu(t)\right)^p\\
&\ge \left(\left(I_{\mathscr{A}} - \int_{\Omega_1} A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_{\Omega_1} B_t\,d\mu(t)\right) - \int_{\Omega_2}(A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p\\
&\ge \left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)
\end{aligned}$$

for all $0 < p \le 1$ and for two disjoint sets $\Omega_1, \Omega_2 \subseteq \Omega$ such that $\Omega = \Omega_1 \cup \Omega_2$.

Proof. Assume that $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are *continuous fields* of positive invertible operators in a unital C^*-algebra $\mathscr{A}$ with $\int_\Omega A_t\,d\mu(t) \le I_{\mathscr{A}}$ and $\int_\Omega B_t\,d\mu(t) \le I_{\mathscr{A}}$. We have

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)\\
&= \left(I_{\mathscr{A}} - \int_{\Omega_1} A_t\,d\mu(t) - \int_{\Omega_2} A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_{\Omega_1} B_t\,d\mu(t) - \int_{\Omega_2} B_t\,d\mu(t)\right)\\
&\le \left(I_{\mathscr{A}} - \int_{\Omega_1} A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_{\Omega_1} B_t\,d\mu(t)\right) - \int_{\Omega_2}(A_t\,\omega_f\,B_t)\,d\mu(t)\\
&\qquad\qquad\text{(by the inequality (3.20))}\\
&\le (I_{\mathscr{A}}\omega_f I_{\mathscr{A}}) - \int_{\Omega_1}(A_t\,\omega_f\,B_t)\,d\mu(t) - \int_{\Omega_2}(A_t\,\omega_f\,B_t)\,d\mu(t)\\
&\qquad\qquad\text{(by the inequality (3.20))}\\
&= I_{\mathscr{A}} - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t).
\end{aligned}$$

Hence, by the above inequalities, the operator monotonicity of $f(t) = t^p\,(0 < p \le 1)$ and Lemma 2.1, we have

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)\\
&\qquad\le \left(\left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)\right)^p\\
&\qquad\le \left(I_{\mathscr{A}} - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p,
\end{aligned}$$

as required. $\square$

In the next theorem, we present another refinement of the operator Bellman inequality (3.21) involving *continuous fields of operators*.

Theorem 3.7. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le I_{\mathscr{A}}$ and $\int_\Omega B_t\, d\mu(t) \le I_{\mathscr{A}}$, let ω_f be an arbitrary operator mean with the representing function f, and let $\lambda : s \in \Omega \longmapsto \lambda_s \in [0,1]$ be a measurable function. Then*

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega (A_s\omega_f B_s)\, d\mu(s)\right)^p \\
&\ge \left(\left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right)\right) - \int_\Omega (1-\lambda_s)\big(A_s\omega_f B_s\big) d\mu(s)\right)^p \\
&\ge \left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s)\right)
\end{aligned}$$

for $0 < p \le 1$.

Proof. Assume that $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are *continuous fields* of positive invertible operators in a unital C^*-algebra $\mathscr{A}$ such that $\int_\Omega A_t\, d\mu(t) \le I_{\mathscr{A}}$ and $\int_\Omega B_t\, d\mu(t) \le I_{\mathscr{A}}$ and $\lambda_s \in [0,1]$ $(s \in \Omega)$. First note that

$$\int_\Omega A_s\, d\mu(s) = \int_\Omega (A_s\nabla_{\lambda_s} A_s)\, d\mu(s) = \int_\Omega \lambda_s A_s\, d\mu(s) + \int_\Omega (1-\lambda_s)A_s\, d\mu(s),$$

$$\int_\Omega B_s\, d\mu(s) = \int_\Omega (B_s\nabla_{\lambda_s} B_s)\, d\mu(s) = \int_\Omega \lambda_s B_s\, d\mu(s) + \int_\Omega (1-\lambda_s)B_s\, d\mu(s),$$

$$I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s) \ge \int_\Omega (1-\lambda_s)A_s\, d\mu(s) \ge 0,$$

and

$$I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s) \ge \int_\Omega (1-\lambda_s)B_s\, d\mu(s) \ge 0.$$

Then,

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s)\right) \\
&= \left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s) - \int_\Omega (1-\lambda_s)A_s\, d\mu(s)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s) - \int_\Omega (1-\lambda_s)B_s\, d\mu(s)\right) \\
&\le \left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right)\right) - \int_\Omega \big[((1-\lambda_s)A_s)\omega_f((1-\lambda_s)B_s)\big] d\mu(s) \\
&\qquad\qquad\text{(by the inequality (3.20))} \\
&\le \left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right)\right) - \int_\Omega (1-\lambda_s)\big(A_s\omega_f B_s\big) d\mu(s) \\
&\qquad\qquad\text{(by the properties of means)}
\end{aligned}$$

$$\leq \left((I_{\mathscr{A}} \omega_f I_{\mathscr{A}}) - \int_\Omega (\lambda_s A_s \omega_f \lambda_s B_s)\, d\mu(s) \right) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s)$$
(by the inequality (3.20))
$$\leq I_{\mathscr{A}} - \int_\Omega \lambda_s (A_s \omega_f B_s)\, d\mu(s) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s)$$
(by the properties of means)
$$= I_{\mathscr{A}} - \int_\Omega (A_s \omega_f B_s)\, d\mu(s).$$

Hence, the operator monotonicity of $f(t) = t^p$ $(0 < p \leq 1)$ Lemma 2.1, and the above inequalities imply that

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s) \right) \omega_{f^p} \left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s) \right) \\
&\leq \left(\left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s) \right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s) \right) \right)^p \\
&\leq \left(\left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s) \right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s) \right) \right) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s) \right)^p \\
&\leq \left(I_{\mathscr{A}} - \int_\Omega (A_s \omega_f B_s)\, d\mu(s) \right)^p
\end{aligned}$$

for $0 < p \leq 1$. This completes the proof. □

In the following result, we obtain the operator Hölder inequality involving the Hadamard product of operators. The main ideas of the next result are stimulated by [2, 24].

Theorem 3.8. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, and let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators. Then*

$$\int_\Omega A_t\, d\mu(t) \circ \int_\Omega B_s\, d\mu(s) \geq \int_\Omega (A_t \sharp_\alpha B_t)\, d\mu(t) \circ \int_\Omega (A_s \sharp_{1-\alpha} B_s)\, d\mu(s)$$

for $0 \leq \alpha \leq 1$.

Proof. Assume that $a, b > 0$ and $X_t, X_s \in \mathscr{A}$ are positive invertible operators. The Heinz inequality [14] asserts that

$$a^{1-\nu} b^{\nu} + a^{\nu} b^{1-\nu} \leq a + b \quad \text{for } 0 \leq \nu \leq 1. \tag{3.22}$$

If we replace b by a^{-1} and take $\mu = 2\nu - 1$ (3.22), then we get

$$a^{\mu} + a^{-\mu} \leq a + a^{-1} \quad \text{for } 0 \leq \mu \leq 1.$$

Replacing a by the positive invertible operator $X_t \otimes X_s^{-1}$ in the above inequality, we get

$$X_t^{\mu} \otimes X_s^{-\mu} + X_t^{-\mu} \otimes X_s^{\mu} \leq X_t \otimes X_s^{-1} + X_t^{-1} \otimes X_s. \tag{3.23}$$

Multiplying both sides of (3.23) by the positive invertible operator $X_t^{\frac{1}{2}} \otimes X_s^{\frac{1}{2}}$, we have

$$X_t^{1+\mu} \otimes X_s^{1-\mu} + X_t^{1-\mu} \otimes X_s^{1+\mu} \leq X_t^2 \otimes I_{\mathscr{A}} + I_{\mathscr{A}} \otimes X_s^2.$$

Now, replacing μ by $2\alpha-1$, X_t by $X_t^{\frac{1}{2}}$, and X_s by $X_s^{\frac{1}{2}}$, respectively, in the above inequality, we get

$$X_t^{\alpha}\otimes X_s^{1-\alpha}+X_t^{1-\alpha}\otimes X_s^{\alpha}\leq X_t\otimes I_{\mathscr{A}}+I_{\mathscr{A}}\otimes X_s \quad \text{for } 0\leq\alpha\leq 1.$$

Now, setting $X_t=A_t^{-\frac{1}{2}}B_tA_t^{-\frac{1}{2}}$ and $X_s=A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}$, and then, multiplying by $A_t^{\frac{1}{2}}\otimes A_s^{\frac{1}{2}}$, in the above inequality, we get

$$(A_t\sharp_\alpha B_t)\otimes(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\otimes(A_s\sharp_\alpha B_s)\leq A_t\otimes B_s+B_s\otimes A_t.$$

Therefore, for the Hadamard product, we have

$$(A_t\sharp_\alpha B_t)\circ(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\circ(A_s\sharp_\alpha B_s)\leq A_t\circ B_s+B_s\circ A_t.$$

Taking the double integral over the above inequality, we get

$$\begin{aligned}&\int_\Omega\int_\Omega\left((A_t\sharp_\alpha B_t)\circ(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\circ(A_s\sharp_\alpha B_s)\right)d\mu(t)\,d\mu(s)\\ &\leq\int_\Omega\int_\Omega\left(A_t\circ B_s+B_s\circ A_t\right)\,d\mu(t)\,d\mu(s).\end{aligned}$$

Using Lemma 3.4, we have

$$\begin{aligned}&\int_\Omega\int_\Omega\Big((A_t\sharp_\alpha B_t)\circ(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\circ(A_s\sharp_\alpha B_s)\Big)\,d\mu(t)\,d\mu(s)\\ =&\int_\Omega(A_t\sharp_\alpha B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_{1-\alpha}B_s)\,d\mu(s)+\int_\Omega(A_t\sharp_{1-\alpha}B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_\alpha B_s)\,d\mu(s)\\ =&2\int_\Omega(A_t\sharp_\alpha B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_{1-\alpha}B_s)\,d\mu(s)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\int_\Omega\left(A_t\circ B_s+B_s\circ A_t\right)d\mu(t)\,d\mu(s)\\ =&\left(\int_\Omega A_t\mu(t)\circ\int_\Omega B_s\,d\mu(s)\right)+\left(\int_\Omega B_s\mu(s)\circ\int_\Omega A_t\,d\mu(t)\right)\\ =&2\int_\Omega A_t\mu(t)\circ\int_\Omega B_s\,d\mu(s).\end{aligned}$$

Hence, we get

$$\int_\Omega(A_t\sharp_\alpha B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_{1-\alpha}B_s)\,d\mu(s)\leq\int_\Omega A_t\mu(t)\circ\int_\Omega B_s\,d\mu(s),$$

as required. □

Remark 3.7. *In the discrete case* $\Omega=\{1,\cdots,n\}$, *for positive invertible operators* $A_1,\cdots,A_n$ *and* $B_1,\cdots,B_n$, *Theorem 3.8 enforces the inequality (1.4) for the Hadamard product.*

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