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Some Finite Summation Identities Comprising Binomial Coefficients for Integrals of the Bernstein Polynomials and Their Applications

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Abstract

Certain finite sums, including the Catalan numbers, factorial functions, binomial coefficients, and their computational formulas are of indispensable importance both in probability and statistics applications and in other branches of science. The primary aim of this article is to give the integral representation of the finite sum containing the products of the Bernstein polynomials, given in our article, by applying the Beta function and the Euler gamma functions. Other aims of this paper are to bring to light novel finite sum formulae containing binomial coefficients by analyzing and unifying this integral representation. Finally, some relations among these sums, binomial coefficients, and the Catalan numbers are given. We also give the Wolfram language codes. By applying these codes to the finite sums, we give some numerical values.

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1. INTRODUCTION

Finite sums appear in almost all scientific fields. In view of the literature that it has good harmony applications with probability theory, especially in areas such as probability, statistics, engineering and electronic engineering, especially the probability of signal detection. It is also well known that cumulative probability distribution can be represented by finite sum of elementary functions. Finite sums have many applications in the discrete probability distribution, and cumulative distribution function, real world problems involving expected value, moments, and variance (see also Chattamvelli & Shanmugam, 2020; Yalcin & Simsek, 2022; Kelly, 1981).

The motivation of this article is give finite sums involving the Catalan numbers C_a, which are related to the Ballot problem associated with probability theory. This problem shows that for the probability that candidate X is always ahead of candidate Y during a tallying process if they respectively end up with a and b votes where $a > b$. Without restrictions, the number of ways the votes are tallied is the binomial coefficient

$$\binom{a+b}{b} = \frac{(a+b)!}{a!\,b!}.$$

The numerator of the desired probability in which X remains ahead can counted recursively by the Catalan numbers, denoted by C_a. The numbers C_a are defined by

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$$C_a = \frac{1}{a+1}\binom{2a}{a}, \tag{1}$$

where $a \in \{0,1,2,\ldots\}$ and

$$\binom{2a}{a} = \frac{(2a)!}{(a!)^2}$$

(Koshy, 2008; Stanley, 2015; 2021).

All of the results to be obtained in this article are based on integrals of the Bernstein polynomials, which we will only briefly define below. These polynomials take part momentous role in mathematics, approximation theory, theory of Computer Geometric Design, construction of the Bezier curves, spline and the B-spline theory, etc.

Let $v \in \{0,1,2,3\ldots\}$ and $b \in \{0,1,2,\ldots,v\}$. The Bernstein polynomials of degree d with real variable y are defined by

$$B_b^v(y) = \binom{v}{b} y^b (1-y)^{v-b}, \tag{2}$$

where

$$\binom{v}{b} = \frac{v!}{b!\,(v-b)!},$$

(Bernstein, 1912, Gradshteyn & Ryzhik, 2007; Lorentz, 1986).

In this article, the following results will be taken into account when calculating the values of finite sums:

If $v < b$, then assuming that $\binom{v}{b} = 0$.

If $v < 0$, then assuming that $v! = 0$.

and $0^0 = 1$.

The above restrictions are used when writing the codes with the Wolfram language in Section 3.

Here we note that under the following restrictions, the Bernstein polynomials can be reduced to the Binomial distribution with parameters v independent the Bernoulli trials. Assuming that y is a random variable of this distribution with $0 \le y \le 1$. Thus, the probability of getting completely b successes in v independent the Bernoulli trials, given by the equation (2) (Chattamvelli & Shanmugam, 2020; Lorentz, 1986; Simsek, 2019; 2020; Simsek, 2014; Yalcin & Simsek, 2022).

The (Euler) gamma functions are defined by

$$\Gamma(\gamma) = \int_0^\infty w^{\gamma-1} e^{-w} dw,$$

where $\gamma > 0$, and provides

$$\Gamma(\gamma+1) = \gamma\Gamma(\gamma)$$

and, for nonnegative integer a,

$$\Gamma(a+1)=a!$$

(Gradshteyn & Ryzhik, 2007; Moll, 2014; Srivastava & Choi, 2012).

Beta function, which is symmetric and member of special function, is defined by

$$B(\varepsilon,\gamma)=\int_0^1 w^{\varepsilon-1}(1-w)^{\gamma-1}dw,$$

where $\varepsilon>0$ and $\gamma>0$, and provides the following relation between $\Gamma(\gamma)$:

$$B(\varepsilon,\gamma)=\frac{\Gamma(\varepsilon)\Gamma(\gamma)}{\Gamma(\varepsilon+\gamma)}$$

(Gradshteyn & Ryzhik, 2007; Kaur & Shrivastav, 2020, Moll, 2014; Srivastava & Choi, 2012). This function is also so-called the first kind Euler integral.

For instance, when v and b are positive integers, then we have

$$B(v,b)=\frac{(v-1)!\,(b-1)!}{(v+b-1)!}.$$

If we integrate both sides of the equation (2) from $y=0$ to $y=1$, the following well-known integral formulas for the Bernstein polynomials can be obtained:

$$(v+1)\int_0^1 B_b^v(y)dy-1=0$$

and

$$\frac{1}{\binom{v}{b}}\int_0^1 B_b^v(y)dy=\sum_{j=0}^{v-b}(-1)^{v-b-j}\frac{\binom{v-b}{j}}{v+1-j}$$

(Acikgoz & Araci, 2010; Lorentz, 1986; Simsek, 2019; 2020; Simsek, 2014;2015, Yalcin & Simsek, 2022).

Using the above integral formulas for the Bernstein polynomials, we also gave many formulas and finite sums (Simsek, 2019; 2020; Simsek, 2014; Yalcin & Simsek, 2022).

In order to reach the main formulas of this article, the finite sum expressed by the following our theorem involving the multiplication of Bernstein polynomials plays a vital role.

Theorem 1.1. Let $d\in\{0,1,2,3\ldots\}$ and $e\in\{0,1,2,\ldots,d\}$. Then

$$\sum_{j=0}^{d}\binom{d}{j}B_e^j(s)B_e^{d-j}(-s)=\frac{(-1)^e(2e)!\,2^{d-2e}\binom{d}{2e}}{(e!)^2}s^{2e}. \tag{3}$$

Proof of the above theorem was given by Simsek and Yardimci (2016).

By applying integral to (3), we gave the following combinatorial sums, see for detail (Simsek and Yardimci 2016):

$$\sum_{j=e}^{d}\sum_{b=0}^{j-e}\sum_{c=0}^{d-j-e}\binom{d}{j}\binom{j}{e}\binom{d}{j}\binom{j-e}{b}\binom{d-j-e}{c}\binom{d-j}{e}\frac{(-1)^{j-b}}{2j+c-2b+1}=\frac{(-1)^{e}(2e)!\,2^{d-2e}\binom{d}{2e}}{(2e+1)(e!)^{2}}.$$

If $2d$ is substituted for d and d is substituted for e, respectively, the following result is obtained:

$$\sum_{j=d}^{2d}\sum_{b=0}^{j-d}\sum_{c=0}^{d-j}\binom{2d}{j}\binom{j}{d}\binom{2d}{j}\binom{j-d}{b}\binom{d-j}{c}\binom{2d-j}{d}\frac{(-1)^{j-b}}{2j+c-2b+1}=\frac{(-1)^{d}(2d)!}{(2d+1)(d!)^{2}}.$$

Now it is time to briefly summarize the results obtained in this article:

In Section 2, with the help of the equation (1), equation (2) and equation (3), in the section 2, we derive many formulas for finite sums including binomial coefficients, the Catalan numbers, and the Euler gamma functions.

In Section 3, we give the Wolfram language codes. By using these codes, we give numerical values of finite and combinatorial sums.

This article ends with the conclusion section.

2. FINITE SUMS INVOLVING GAMMA FUNCTION AND THE CATALAN NUMBERS

In this section, new combinatorial finite sums and formulas that calculate their sums are given, using the finite sum containing the gamma function, beta function, and Bernstein polynomials. In addition, the relationships between these formulas and the Catalan numbers are also be proven.

Modification of the equation (3) is given as follows:

$$\sum_{j=e}^{d}\binom{d}{j}B_{e}^{j}(s)B_{e}^{d-j}(-s)=\frac{(-1)^{e}(2e)!\,2^{d-2e}}{(e!)^{2}}\binom{d}{2e}s^{2e}.$$

Combining equation (1) with the previous equation yields:

Corollary 2.1. Let $d\in\{0,1,2,3\ldots\}$ and $e\in\{0,1,2,\ldots,d\}$. Then

$$\sum_{j=e}^{d}\binom{d}{j}B_{e}^{j}(s)B_{e}^{d-j}(-s)=(-1)^{e}(e+1)2^{d-2e}\binom{d}{2e}s^{2e}C_{e}. \tag{4}$$

For $s\neq 0$, combining equation (2) with (4) yields:

Corollary 2.2. Let $d\in\{0,1,2,3\ldots\}$ and $e\in\{0,1,2,\ldots,d\}$. Then

$$\sum_{j=e}^{d}\binom{d}{j}\binom{j}{e}\binom{d-j}{e}(1-s)^{j-e}(1+s)^{d-j-e}=(e+1)2^{d-2e}\binom{d}{2e}C_{e}. \tag{5}$$

Theorem 2.3. Let $d\in\{0,1,2,3\ldots\}$ and $e\in\{0,1,2,\ldots,d\}$. Then

$$\sum_{j=e}^{d}\binom{d}{j}\binom{j}{e}\binom{d-j}{e}\frac{\Gamma(j-e+1)\Gamma(d-j-e+1)}{\Gamma(d-2e+2)}=(e+1)2^{d-2e}\binom{d}{2e}C_e. \quad (6)$$

Proof. If we integrate both sides of equation (5) from $s=-1$ to $s=1$, the following equation can easily reach:

$$\sum_{j=e}^{d}\binom{d}{j}\binom{j}{e}\binom{d-j}{e}\int_{-1}^{1}(1-s)^{j-e}(1+s)^{d-j-e}ds=(2e+2)2^{d-2e}\binom{d}{2e}C_e.$$

Combining the above equation with following formula (Gradshteyn & Ryzhik, 2007; Moll, 2014; Simsek 2015, Srivastava & Choi, 2012):

$$\Gamma(\varepsilon+\delta+2)\int_{-1}^{1}(1-s)^{\varepsilon}(1+s)^{\delta}ds=2^{\varepsilon+\delta+1}\Gamma(\varepsilon+1)\Gamma(\delta+1),$$

where $\varepsilon>0$ and $\delta>0$, gives us

$$\sum_{j=e}^{d}\binom{d}{j}\binom{j}{e}\binom{d-j}{e}\frac{2^{d-2e+1}\Gamma(j-e+1)\Gamma(d-j-e+1)}{\Gamma(d-2e+2)}=2^{d-2e}\binom{d}{2e}(2e+2)C_e.$$

After some calculations proof of the theorem is ended.

Theorem 2.4. Let $d\in\{0,1,2,3\ldots\}$ and $e\in\{0,1,2,\ldots,d\}$. Then

$$\sum_{j=e}^{d}\binom{d}{j}\binom{j}{e}\binom{d-j}{e}(d-e-j)!\,(j-e)!=\binom{d}{2e}(d-2e+1)!\,(e+1)C_e. \quad (7)$$

Proof. Since $\Gamma(j+1)=j!$ for a nonnegative integer j, afterwards (6) reduces to

$$\sum_{j=e}^{d}\binom{d}{j}\binom{j}{e}\binom{d-j}{e}\frac{(d-e-j)!\,(j-e)!}{(d-2e+1)!}=\binom{d}{2e}(e+1)C_e.$$

After some calculations proof of the theorem is ended.

Theorem 2.5. Let $d\in\{0,1,2,3\ldots\}$. Then

$$\sum_{j=d}^{4d}\binom{4d}{j}\binom{j}{d}\binom{4d-j}{d}(3d-j)!\,(j-d)!=(d+1)(2d+1)^2(2d)!\,C_dC_{2d}.$$

Proof. Replacing d by $4d$ and e by d in equation (7), we get

$$\sum_{j=d}^{4d}\binom{4d}{j}\binom{j}{d}\binom{4d-j}{d}(3d-j)!\,(j-d)!=(d+1)\binom{4d}{2d}(2d+1)!\,C_d.$$

Combining the previous equation with (1), then proof of theorem is completed.

3. NUMERICAL APPLICATIONS OF THEOREM 2.5 WITH WOLFRAM LANGUAGE CODES

By using the same method in (Kucukoglu, 2023) and coding the RHS (right-hand side) and the LHS (left-hand side) of the combinatorial sum, given in Theorem 2.5, in the Wolfram language, we get the following code snippets, which gives us the numerical values of the combinatorial sum mentioned above.

```
Unprotect[Power];
Power[0, 0] = 1;
Protect[Power];
Unprotect[Binomial];
Binomial[n_, k_] :=If[n<k, 0 ,n!/(k!  * (n-k)!)];
Protect[Binomial];
Unprotect[Factorial];
Factorial[n_] :=If[n<0, 0 ,Product[j,{j,1,n}]];
Protect[Factorial];
CombinatoralSumLHS[d_]:=Sum[Binomial[4*d,j]*Binomial[j,d]*Binomial[4*d-j,d]*((j-d)!)*((3*d-
j)!),{j,d,4*d}]
```

```
TableForm[Table[CombinatoralSumLHS[d],{d,0,10}],TableHeadings -
>{Table["d="<>ToString[k],{k,0,10}]}]
```

By using the Wolfram language codes, which are given the above, Table of LHS of Theorem 2.5 is given as follows:

d=0	1
d=1	72
d=2	50400
d=3	93139200
d=4	326918592000
d=5	1858466811801600
d=6	15559084148402995200
d=7	180040830860091801600000
d=8	2751564018034783003852800000
d=9	53673842111798500461821952000000
d=10	1301182750011063967595672489164800000

```
CombinatoralSumRHS1[d_]:=(d+1)*Power[2*d+1,2]*Factorial[2*d]*CatalanNumber[d]*CatalanNumber[2
*d]
```

```
TableForm[Table[CombinatoralSumRHS1[d],{d,0,10}],TableHeadings -
>{Table[“d=”<>ToString[k],{k,0,10}]}]
```

By using the Wolfram language codes, which are given the above, Table of RHS of Theorem 2.5 is given as follows:

d=0	1
d=1	72
d=2	50400
d=3	93139200
d=4	326918592000

d=5 1858466811801600
d=6 155590841484402995200
d=7 18004083086009180160000
d=8 275156401803478300385280000
d=9 5367384211179850046182195200000
d=10 130118275001106396759567248916480000

```
CombinatoralSumRHS2[d_]:=(d+1)*Binomial[4*d,2*d]*Factorial[2*d+1]*CatalanNumber[d]
```

```
TableForm[Table[CombinatoralSumRHS2[d],{d,0,10}],TableHeadings -
>{Table["d="<>ToString[k],{k,0,10}]}]
```

By using the above Wolfram language codes, we also give numerical values of the combinatorial sum by the following table:

d=0 1
d=1 72
d=2 50400
d=3 93139200
d=4 326918592000
d=5 1858466811801600
d=6 155590841484402995200
d=7 18004083086009180160000
d=8 275156401803478300385280000
d=9 5367384211179850046182195200000
d=10 130118275001106396759567248916480000

By using computational algorithms with the Wolfram language codes, Kilar (Kilar 2023) gave numerical values for finite sums. With the aid of the Wolfram language codes, Kucukoglu (Kucukoglu, 2023) gave not only numerical values for finite sums, but also gave graphics of the special polynomials.

4. CONCLUSION

In this article, we investigated some properties of the certain family of the finite sums. By applying the Beta function and the Euler gamma functions to Theorem 2.1 in our article Simsek and Yardimci (2016), which involving products of the Bernstein polynomials with the finite sums, we derived some explicit formulas for these sums. These formulas were also covered the Catalan numbers, factorial functions and binomial coefficients. We think that our computational formulas are of indispensable importance both in probability and statistics applications and in other branches of science.

We gave the Wolfram language codes. By using these codes to the finite sums, we also gave some numerical values for the finite sums.

The formulas of this article may also potentially applied in the discrete probability distribution, and cumulative distribution function, involving moments, expected value, and variance.

CONFLICT OF INTEREST

The author declares no conflict of interest.

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