



Research Article

Characterizations of tztzeica curves using FLC frame

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ABSTRACT

In this study, we investigate the condition that a given polynomial curve with respect to the Frenet like curve (Flc) frame is a Tztzeica curve. We also show that any planar polynomial curve cannot be a Tztzeica curve. Finally, the condition to be the Tztzeica curve for each of the spherical indicatrix curves defined according to the Flc frame is expressed, separately.

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INTRODUCTION

The Tztzeica curve is characterized by a unique condition that pertains to the proportion between the curve's torsion. This ratio remains a nonzero constant. The defining characteristic of the Tztzeica curve is that the mentioned ratio, involving torsion and distance, is a constant value. It has been proven that no planar polynomial curve can be a Tztzeica curve. This result provides insights into the limitations and specific conditions for the existence of Tztzeica curves.

New results have been obtained for spherical indicatrix curves defined with respect to the Flc frame, demonstrating their qualification as Tztzeica curves. This extends the understanding of Tztzeica curves to specific instances in spherical geometry. Researchers have explored various aspects of Tztzeica curves, including obtaining parametric, open, and closed equations for Tztzeica hypersurfaces. The study involves mathematical formulations that describe the shape and behavior of these curves. Tztzeica curves

have been studied in different mathematical spaces, such as Minkowski space and Euclidean space. The characterization of Tztzeica curves varies based on the geometric properties of the specific space in which they are examined. The geometric information about Tztzeica curves has been investigated with respect to various frames of reference. Different frames provide different perspectives on the curve's behavior and properties.

In summary, the geometric information about Tztzeica curves encompasses their defining characteristics, the invariance of certain ratios, non-existence for certain types of curves, and exploration of their behavior in different mathematical spaces and frames of reference, [1-10]. This study aims to examine Tztzeica curves in Euclidean space using the Flc frame. The condition for a curve to be a Tztzeica curve has been reinterpreted and defined for given polynomial curves. It has been established that no planar polynomial curve qualifies as a Tztzeica curve. Additionally, new results have been obtained for spherical

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indicatrix curves defined with respect to the Flc frame, meeting the criteria to be Tzitzeica curves.

PRELIMINARIES

Suppose $\alpha = \alpha(u)$ is a regular space curve. Three mutually orthogonal vector fields, referred to as the Frenet frame, are defined as [11]

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad B = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \quad N = B \wedge T. \tag{1}$$

The Frenet apparatus are given by

$$T' = \kappa\nu N, \quad N' = -\kappa\nu T + \tau\nu B, \quad B' = -\tau\nu N, \quad \|\alpha'\| = \nu, \tag{2}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\alpha''}, \quad \tau = \frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\|\alpha' \wedge \alpha''\|^2}. \tag{3}$$

In relation to these, Consider $\alpha = \alpha(u)$ as a polynomial space curve. The definition of the Flc frame for [12],

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad D_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\|\alpha' \wedge \alpha^{(n)}\|}, \quad D_2 = D_1 \wedge T \tag{4}$$

Here $^{(n)}$ indicates the n^{th} order derivative of the curve. D_1 is called as binormal-like vector and D_2 is normal-like vector. The curvatures of the Flc-frame are given by

$$d_1 = \frac{\langle T', D_2 \rangle}{\nu}, \quad d_2 = \frac{\langle T', D_1 \rangle}{\nu}, \quad d_3 = \frac{\langle D_2', D_1 \rangle}{\nu}. \tag{5}$$

The Frenet-like apparatus are given by:

$$\begin{bmatrix} T' \\ D_2' \\ D_1' \end{bmatrix} = \nu \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}, \tag{6}$$

$$d_1 = \kappa \cos \theta, \quad d_2 = -\kappa \sin \theta, \quad d_3 = \frac{d\theta}{\nu} + \tau.$$

Definition 2.1. Let the α polynomial curve $\{T, D_2, D_1\}$ be given with the Flc frame. The path followed by the vector T centered at unit sphere is called T – is defined as

$$\alpha_T = T. \tag{7}$$

Definition 2.2. Let the α polynomial curve $\{T, D_2, D_1\}$ be given with the Flc frame. The path followed by the vector D_2 centered at unit sphere is called D_2 – is defined as

$$\alpha_{D_2} = D_2. \tag{8}$$

Definition 2.3. Let the α polynomial curve $\{T, D_2, D_1\}$ be given with the Flc frame. The path followed by the vector D_1 centered at unit sphere is called D_1 – is defined as

$$\alpha_{D_1} = D_1. \tag{9}$$

TZITZEICA CURVES USING FLC FRAME

In this section of the research, we present the characterization of the Tzitzeica curve employing the Flc frame within 3-dimensional Euclidean space. Essential and comprehensive conditions are articulated for the spherical indicatrix curves to qualify as Tzitzeica curves.

Theorem 3.1. Let $\alpha : I \subset \mathbb{R} \rightarrow E^3$ be a polynomial curve and $\{T, D_2, D_1, d_1, d_2, d_3\}$ be apparatus of the Flc frame, then there exists the following equation:

$$\frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2} = \frac{(d_1^2 + d_2^2)(\nu d_3 - \theta')}{\nu \langle \alpha, d_2 D_2 - d_1 D_1 \rangle^2}.$$

Proof . The differentials of the polynomial curve α can be readily obtained as follows:

$$\begin{aligned} \alpha' &= \nu T, \\ \alpha'' &= \nu' T + \nu^2 d_1 D_2 + \nu^2 d_2 D_1, \\ \alpha''' &= (\nu'' - d_1^2 \nu^3 - d_2^2 \nu^3) T + (3\nu\nu' d_1 + \nu^2 d_1' - \nu^3 d_2 d_3) D_2 \\ &\quad + (3\nu\nu' d_2 + \nu^2 d_2' + \nu^3 d_1 d_3) D_1. \end{aligned}$$

With the help of these equations, we get

$$\begin{aligned} \alpha' \wedge \alpha'' &= \begin{vmatrix} T & D_2 & D_1 \\ \nu & 0 & 0 \\ \nu' & \nu^2 d_1 & \nu^2 d_2 \end{vmatrix} \\ &= -\nu^3 d_2 D_2 + \nu^3 d_1 D_1, \\ \langle \alpha, \alpha' \wedge \alpha'' \rangle^2 &= \langle \alpha, -\nu^3 d_2 D_2 + \nu^3 d_1 D_1 \rangle^2, \\ \langle \alpha' \wedge \alpha'', \alpha''' \rangle &= \left\langle -\nu^3 d_2 D_2 + \nu^3 d_1 D_1, (\nu'' - d_1^2 \nu^3 - d_2^2 \nu^3) T + \right. \\ &\quad \left. (3\nu\nu' d_1 + \nu^2 d_1' - \nu^3 d_2 d_3) D_2 + (3\nu\nu' d_2 + \nu^2 d_2' + \nu^3 d_1 d_3) D_1 \right\rangle, \\ &= \nu^5 (d_1 d_2' - d_1' d_2) + \nu^6 d_3 (d_1^2 + d_2^2), \\ &= \nu^5 (d_1^2 + d_2^2) \left(\arctan \left(\frac{d_2}{d_1} \right) \right)' + \nu^6 d_3 (d_1^2 + d_2^2), \\ &= \nu^5 (d_1^2 + d_2^2) (\nu d_3 - \theta'). \end{aligned}$$

Therefore, it is evident that the following equation exists:

$$\frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2} = \frac{(d_1^2 + d_2^2)(\nu d_3 - \theta')}{\nu \langle \alpha, d_2 D_2 - d_1 D_1 \rangle^2}.$$

The demonstration is concluded.

Corollary 3.1.1. If α is a Tzitzeica curve, then

$$\frac{(d_1^2 + d_2^2)(vd_3 - \theta')}{v\langle \alpha, d_2D_2 - d_1D_1 \rangle^2}$$

is a nonzero constant.

Corollary 3.1.2. If α is a planar polynomial curve, it is established that α cannot be the Tzitzeica curve. Now, let's determine the essential conditions for spherical indicatrix curves to meet the criteria for being Tzitzeica curves:

Theorem 3.2. Let $\alpha : I \subset IR \rightarrow E^3$ be a polynomial curve with the Flc frame. Then tangent spherical indicatrix curve α_T of the curve α is a Tzitzeica curve if and only if

$$\frac{\langle \alpha_T' \wedge \alpha_T'', \alpha_T''' \rangle}{\langle \alpha_T, \alpha_T' \wedge \alpha_T'' \rangle^2} = \frac{-v^2(v^3d_3(d_1^2 + d_2^2) + d_1(vd_2)' - d_2(vd_1)') (2v'(d_1^2 + d_2^2) + v(d_1^2 + d_2^2)' + d_1(vd_1)' - d_2(vd_2)') + v^3(d_1^2 + d_2^2)(d_1((vd_2)') + v^2d_1d_3)' - d_2((vd_1)' - v^2d_2d_3)'}{v^4d_3(d_1^2 + d_2^2)(d_1((vd_1)' - v^2d_2d_3)' + d_2((vd_2)' + v^2d_1d_3)')} = \lambda_1, \\ \langle \alpha_T, (v^3d_3(d_1^2 + d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')T \\ (v^3d_2(d_1^2 + d_2^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1 \rangle >^2$$

where $\lambda_1 \neq 0$.

Proof. Let α_T be a tangent spherical indicatrix curve of the curve α , then differentiations of α_T are

$$\alpha_T' = v(d_1D_2 + d_2D_1), \\ \alpha_T'' = -v^2(d_1^2 + d_2^2)T + ((vd_1)' - v^2d_2d_3)D_2 + ((vd_2)' + v^2d_1d_3)D_1, \\ \alpha_T''' = -v(2v'(d_1^2 + d_2^2) + v(d_1^2 + d_2^2)' + d_1(vd_1)' - d_2(vd_2)')T \\ + (-v^3d_1(d_1^2 + d_2^2) + ((vd_1)' - v^2d_2d_3)' - vd_3((vd_2)' + v^2d_1d_3))D_2 \\ + (-v^3d_2(d_1^2 + d_2^2) + ((vd_2)' + v^2d_1d_3)' + vd_3((vd_1)' - v^2d_2d_3))D_1.$$

Using these equations, we have

$$\alpha_T' \wedge \alpha_T'' = \begin{vmatrix} T & D_2 & D_1 \\ 0 & vd_1 & vd_2 \\ -v^2(d_1^2 + d_2^2) & (vd_1)' - v^2d_2d_3 & (vd_2)' + v^2d_1d_3 \end{vmatrix} \\ = (v^3d_3(d_1^2 + d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')T \\ - (v^3d_2(d_1^2 + d_2^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1, \\ \langle \alpha_T, \alpha_T' \wedge \alpha_T'' \rangle >^2 = \langle \alpha_T, (v^3d_3(d_1^2 + d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')T \\ - (v^3d_2(d_1^2 + d_2^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1 \rangle >^2,$$

$$\langle \alpha_T' \wedge \alpha_T'', \alpha_T''' \rangle = -v^2(v^3d_3(d_1^2 + d_2^2) + d_1(vd_2)' - d_2(vd_1)') \\ (2v'(d_1^2 + d_2^2) + v(d_1^2 + d_2^2)' + d_1(vd_1)' - d_2(vd_2)') \\ + v^3(d_1^2 + d_2^2)(d_1((vd_2)') + v^2d_1d_3)' - d_2((vd_1)' - v^2d_2d_3)'} \\ + v^4d_3(d_1^2 + d_2^2)(d_1((vd_1)' - v^2d_2d_3)' + d_2((vd_2)' + v^2d_1d_3)').$$

Thus, we can readily derive the condition for curve α_T to be a Tzitzeica curve

$$\frac{\langle \alpha_T' \wedge \alpha_T'', \alpha_T''' \rangle}{\langle \alpha_T, \alpha_T' \wedge \alpha_T'' \rangle^2} = \frac{-v^2(v^3d_3(d_1^2 + d_2^2) + d_1(vd_2)' - d_2(vd_1)') (2v'(d_1^2 + d_2^2) + v(d_1^2 + d_2^2)' + d_1(vd_1)' - d_2(vd_2)') + v^3(d_1^2 + d_2^2)(d_1((vd_2)') + v^2d_1d_3)' - d_2((vd_1)' - v^2d_2d_3)'}{v^4d_3(d_1^2 + d_2^2)(d_1((vd_1)' - v^2d_2d_3)' + d_2((vd_2)' + v^2d_1d_3)')} \\ = \frac{\langle \alpha_T, (v^3d_3(d_1^2 + d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')T \\ (v^3d_2(d_1^2 + d_2^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1 \rangle >^2}{\langle \alpha_T, (v^3d_3(d_1^2 + d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')T \\ (v^3d_2(d_1^2 + d_2^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1 \rangle >^2}.$$

Theorem 3.3. Let $\alpha : I \subset IR \rightarrow E^3$ be a polynomial curve with the Flc frame. Then the normal like spherical indicatrix curve α_{D_2} of the curve α is a Tzitzeica curve if and only if

$$\frac{\langle \alpha_{D_2}' \wedge \alpha_{D_2}'', \alpha_{D_2}''' \rangle}{\langle \alpha_{D_2}, \alpha_{D_2}' \wedge \alpha_{D_2}'' \rangle^2} = \frac{(v^3d_1(d_1^2 + d_2^2) - ((vd_1)' + d_2d_3v^2) - vd_1((vd_2)' - d_1d_2v^2))(v^3d_3(d_1^2 + d_2^2)) \\ + ((vd_1)((vd_2)' + d_2d_3v^2) + (vd_3)((vd_2)' - d_1d_2v^2) \\ + (v(d_1^2 + d_2^2))) (vd_3((vd_1)' + d_2d_3v^2) + vd_2((vd_2)' - d_1d_2v^2) \\ - (v^3d_3(d_1^2 + d_2^2) - ((vd_2)' - d_1d_2v^2)' + vd_2((vd_1)' + d_2d_3v^2))(v^3d_1(d_1^2 + d_2^2)) \\ - (v^3d_2(d_1^2 + d_2^2) - ((vd_3)' - d_1d_2v^2)' + vd_1((vd_3)' - d_1d_2v^2))D_2 \\ + (v^3d_1(d_1^2 + d_2^2))D_1 >^2}{\langle \alpha_{D_2}, (v^3d_3(d_1^2 + d_2^2))T + (-vd_3((vd_1)' + d_2d_3v^2) + vd_1((vd_2)' - d_1d_2v^2))D_2 \\ + (v^3d_1(d_1^2 + d_2^2))D_1 \rangle >^2} = \lambda_2,$$

where $\lambda_2 \neq 0$.

Proof. Let α_{D_2} be a normal spherical indicatrix curve of the curve α , then differentiations of the curve α_{D_2} are found as

$$\alpha_{D_2}' = v(-d_1T + d_3D_1), \\ \alpha_{D_2}'' = -((vd_1)' + d_2d_3v^2)T - v^2(d_1^2 + d_2^2)D_2 + ((vd_3)' - d_1d_2v^2)D_1, \\ \alpha_{D_2}''' = (v^3d_1(d_1^2 + d_2^2) - ((vd_1)' + d_2d_3v^2)' - vd_2((vd_3)' - d_1d_2v^2))T \\ - ((vd_1)((vd_1)' + d_2d_3v^2) + (vd_3)((vd_3)' - d_1d_2v^2) + (v(d_1^2 + d_2^2)))D_2 \\ - (v^3d_3(d_1^2 + d_2^2) - ((vd_3)' - d_1d_2v^2)' + vd_2((vd_1)' + d_2d_3v^2))D_1.$$

Utilizing these equations, we can readily compute the subsequent expressions

$$\alpha_{D_2}' \wedge \alpha_{D_2}'' = \begin{vmatrix} T & D_2 & D_1 \\ -vd_1 & 0 & vd_3 \\ -(vd_1)' - v^2d_2d_3 & -v^2(d_1^2 + d_2^2) & (vd_3)' - d_1d_2v^2 \end{vmatrix}, \\ \alpha_{D_2}' \wedge \alpha_{D_2}'' = (v^3d_1(d_1^2 + d_2^2) - ((vd_1)' + d_2d_3v^2)' - vd_2((vd_3)' - d_1d_2v^2))T + (-vd_3((vd_1)' + d_2d_3v^2) + vd_1((vd_3)' - d_1d_2v^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1, \\ \langle \alpha_{D_2}, \alpha_{D_2}' \wedge \alpha_{D_2}'' \rangle >^2 = \langle \alpha_{D_2}, (v^3d_1(d_1^2 + d_2^2) - ((vd_1)' + d_2d_3v^2)' - vd_2((vd_3)' - d_1d_2v^2))T \\ + (-vd_3((vd_1)' + d_2d_3v^2) + vd_1((vd_3)' - d_1d_2v^2))D_2 + (v^3d_1(d_1^2 + d_2^2))D_1 \rangle >^2, \\ \langle \alpha_{D_2}' \wedge \alpha_{D_2}'', \alpha_{D_2}''' \rangle = (v^3d_1(d_1^2 + d_2^2) - ((vd_1)' + d_2d_3v^2)' - vd_2((vd_3)' - d_1d_2v^2))(v^3d_3(d_1^2 + d_2^2)) \\ + ((vd_1)((vd_1)' + d_2d_3v^2) + (vd_3)((vd_3)' - d_1d_2v^2) \\ + (v(d_1^2 + d_2^2))) (vd_3((vd_1)' + d_2d_3v^2) + vd_2((vd_2)' - d_1d_2v^2) \\ - (v^3d_3(d_1^2 + d_2^2) - ((vd_3)' - d_1d_2v^2)' + vd_2((vd_1)' + d_2d_3v^2))(v^3d_1(d_1^2 + d_2^2)) \\ - (v^3d_2(d_1^2 + d_2^2) - ((vd_3)' - d_1d_2v^2)' + vd_1((vd_3)' - d_1d_2v^2))D_2 \\ + (v^3d_1(d_1^2 + d_2^2))D_1 >^2.$$

Therefore, the condition for curve α_{D_2} to be a Tzitzeica curve is articulated as:

$$\begin{aligned} & (v^3 d_1 (d_1^2 + d_3^2) - ((v d_1)' + d_2 d_3 v^2)' - v d_2 ((v d_3)' - d_1 d_2 v^2)) (v^3 d_3 (d_1^2 + d_3^2)) \\ & + ((v d_1)' (v d_3)' + d_2 d_3 v^2) + (v d_3)' (v d_3)' - d_1 d_2 v^2 \\ & + (v (d_1^2 + d_3^2)) (v d_3 ((v d_1)' + d_2 d_3 v^2) + v d_1 ((v d_3)' - d_1 d_2 v^2)) \\ & - (v^3 d_3 (d_1^2 + d_3^2) - ((v d_3)' - d_1 d_2 v^2)' + v d_2 ((v d_1)' + d_2 d_3 v^2)) (v^3 d_1 (d_1^2 + d_3^2)) \\ \frac{\langle \alpha'_{D_2} \wedge \alpha''_{D_2}, \alpha'''_{D_2} \rangle}{\langle \alpha_{D_2}, \alpha'_{D_2} \wedge \alpha''_{D_2} \rangle^2} = & \frac{\langle \alpha_{D_2}, (v^3 d_3 (d_1^2 + d_3^2))' T + (-v d_2 ((v d_1)' + d_2 d_3 v^2) + v d_1 ((v d_3)' - d_1 d_2 v^2)) D_2 \\ & + (v^3 d_1 (d_1^2 + d_3^2))' D_1 \rangle}{\lambda_3^2} \end{aligned}$$

Theorem 3.4. Let $\alpha : I \subset \mathbb{R} \rightarrow E^3$ be a polynomial curve with the Flc frame. Then the binormal like spherical indicatrix curve α_{D_1} of the curve α is a Tzitzeica curve if and only if

$$\begin{aligned} & (v^3 d_2 (d_2^2 + d_3^2) + v d_1 ((v d_3)' + v d_1 d_2) - ((v d_3)' - v d_1 d_2)) (v^3 d_3 (d_2^2 + d_3^2)) \\ & - (v^3 d_3 (d_2^2 + d_3^2) - v d_1 ((v d_2)' - v d_1 d_3) - ((v d_3)' + v d_1 d_2)) (v^3 d_1 (d_2^2 + d_3^2)) \\ & - (v d_2 ((v d_3)' + v d_1 d_2) - v d_3 ((v d_2)' - v d_1 d_3)) ((v d_2 ((v d_3)' - v d_1 d_3) + (v d_3)' (v d_3)' \\ & + v d_1 d_2) + (v^2 (d_2^2 + d_3^2))') \\ \frac{\langle \alpha'_{D_1} \wedge \alpha''_{D_1}, \alpha'''_{D_1} \rangle}{\langle \alpha_{D_1}, \alpha'_{D_1} \wedge \alpha''_{D_1} \rangle^2} = & \frac{\langle \alpha_{D_1}, (v^3 d_3 (d_2^2 + d_3^2))' T - (v^3 d_2 (d_2^2 + d_3^2))' D_2 \\ & + (v d_2 ((v d_3)' - v d_1 d_3) + v^2 d_1 (d_2^2 + d_3^2))' D_1 \rangle}{\lambda_3} \end{aligned}$$

where $\lambda_3 \neq 0$.

Proof. Let α_{D_1} be a binormal spherical indicatrix curve of the curve α , then differentiations of the curve α_{D_1} are found as

$$\begin{aligned} \alpha'_{D_1} &= -v(d_2 T + d_3 D_2), \\ \alpha''_{D_1} &= -((v d_2)' - v d_1 d_3) T - ((v d_3)' + v d_1 d_2) D_2 - v^2 (d_2^2 + d_3^2) D_1, \\ \alpha'''_{D_1} &= (v^3 d_2 (d_2^2 + d_3^2) - ((v d_2)' - v d_1 d_3)' + v d_1 ((v d_3)' + v d_1 d_2)) T \\ &+ (v^3 d_3 (d_2^2 + d_3^2) - (v d_1)' ((v d_2)' - v d_1 d_3) - (v d_3)' ((v d_3)' + v d_1 d_2))' D_2 \\ &- ((v d_2)' ((v d_2)' - v d_1 d_3) + v d_3 ((v d_3)' + v d_1 d_2) + (v^2 (d_2^2 + d_3^2))') D_1. \end{aligned}$$

So, we obtain the following equations

$$\alpha'_{D_1} \wedge \alpha''_{D_1} = \begin{vmatrix} T & D_2 & D_1 \\ -v d_2 & -v d_3 & 0 \\ -(v d_2)' + v d_1 d_3 & -(v d_3)' - v d_1 d_2 & -v^2 (d_2^2 + d_3^2) \end{vmatrix}$$

$$\begin{aligned} \alpha'_{D_1} \wedge \alpha''_{D_1} &= (v^3 d_3 (d_2^2 + d_3^2)) T - (v^3 d_2 (d_2^2 + d_3^2)) D_2 \\ &+ (v d_2 ((v d_3)' - v d_3 (v d_2)' + v^2 d_1 (d_2^2 + d_3^2))) D_1, \\ \langle \alpha_{D_1}, \alpha'_{D_1} \wedge \alpha''_{D_1} \rangle &= \langle \alpha_{D_1}, (v^3 d_3 (d_2^2 + d_3^2)) T - (v^3 d_2 (d_2^2 + d_3^2)) D_2 \\ &+ (v d_2 ((v d_3)' - v d_3 (v d_2)' + v^2 d_1 (d_2^2 + d_3^2))) D_1 \rangle \end{aligned}$$

$$\begin{aligned} \langle \alpha'_{D_1} \wedge \alpha''_{D_1}, \alpha'''_{D_1} \rangle &= (v^3 d_2 (d_2^2 + d_3^2) + v d_1 ((v d_3)' + v d_1 d_2) - ((v d_2)' - v d_1 d_3)) (v^3 d_3 (d_2^2 + d_3^2)) \\ &- (v^3 d_3 (d_2^2 + d_3^2) - v d_1 ((v d_2)' - v d_1 d_3) - ((v d_3)' + v d_1 d_2)) (v^3 d_1 (d_2^2 + d_3^2)) \\ &- (v d_2 ((v d_3)' + v d_1 d_2) - v d_3 ((v d_2)' - v d_1 d_3)) ((v d_2 ((v d_3)' - v d_1 d_3) + (v d_3)' (v d_3)' \\ &+ v d_1 d_2) + (v^2 (d_2^2 + d_3^2))'). \end{aligned}$$

From here, the condition to be the Tzitzeica curve of the curve α_{D_1} is given by

$$\begin{aligned} & (v^3 d_2 (d_2^2 + d_3^2) + v d_1 ((v d_3)' + v d_1 d_2) - ((v d_2)' - v d_1 d_3)) (v^3 d_3 (d_2^2 + d_3^2)) \\ & - (v^3 d_3 (d_2^2 + d_3^2) - v d_1 ((v d_2)' - v d_1 d_3) - ((v d_3)' + v d_1 d_2)) (v^3 d_1 (d_2^2 + d_3^2)) \\ & - (v d_2 ((v d_3)' + v d_1 d_2) - v d_3 ((v d_2)' - v d_1 d_3)) ((v d_2 ((v d_3)' - v d_1 d_3) + (v d_3)' (v d_3)' \\ & + v d_1 d_2) + (v^2 (d_2^2 + d_3^2))') \\ \frac{\langle \alpha'_{D_1} \wedge \alpha''_{D_1}, \alpha'''_{D_1} \rangle}{\langle \alpha_{D_1}, \alpha'_{D_1} \wedge \alpha''_{D_1} \rangle^2} = & \frac{\langle \alpha_{D_1}, (v^3 d_3 (d_2^2 + d_3^2))' T - (v^3 d_2 (d_2^2 + d_3^2))' D_2 \\ & + (v d_2 ((v d_3)' - v d_3 (v d_2)' + v^2 d_1 (d_2^2 + d_3^2))) D_1 \rangle}{\lambda_3^2} \end{aligned}$$

CONCLUSION

In this study, the condition for a polynomial curve given according to the Flc framework to be a Tzitzeica curve expressed. It was then shown that no planar polynomial curve can be a Tzitzeica curve. As a result, each of the spherical indicatrix curves defined according to the Flc framework is the Tzitzeica curve condition was investigated.

AUTHORSHIP CONTRIBUTIONS

Concept: Ş.S., E.K., A.K.H; Design: Ş.S., E.K., A.K.H; Supervision: Ş.S., E.K., A.K.H; Materials: Ş.S., E.K., A.K.H; Analysis: Ş.S., E.K., A.K.H; Literature search: Ş.S., E.K., A.K.H; Writing: Ş.S., E.K., A.K.H; Critical revision: Ş.S., E.K., A.K.H.

DATA AVAILABILITY STATEMENT

The published document incorporates all visuals and data gathered or generated throughout the course of the study.

CONFLICT OF INTEREST

The author has disclosed no potential conflicts of interest concerning the research, authorship, and/or publication of this article.

ETHICS

No ethical issues are associated with the publication of this manuscript.

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