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RESEARCH ARTICLE

Extending Natural Mates in Euclidean 3-space and Applications to Bertrand Pairs

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In Euclidean 3-space, a family of curves, the co-successor, is motivated and then introduced in relation to the natural mate. A complete characterization of co-successors is proved, followed by an application of the co-successor towards describing Bertrand curves and their mates.

Keywords: *Natural mates, co-successor curves, Bertrand curves, conjugate mates.*
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Introduction

Consider a unit-speed curve $\alpha : I \to \mathbb{E}^3$. α has Frenet-Serret apparatus $\{\kappa, \tau, T, N, B\}$, which satisfies the Frenet-Serret equations, expressed in matrix form as:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $\kappa > 0$.

Definition 0.1. *We say a curve is Frenet if its curvature is nowhere zero ($\tau \neq 0$).*

Definition 0.2. *We define the natural mate of the unit speed curve α, $\overline{\alpha} : I \to \mathbb{E}^3$, to be the unit speed curve whose tangent vector $\overline{T}(s) = N(s)$.[5]*

It has been shown that the Frenet-Serret apparatus $\{\overline{\kappa}, \overline{\tau}, \overline{T}, \overline{N}, \overline{B}\}$ of the natural mate is given by

$$\overline{\kappa} = \omega, \quad \overline{\tau} = \sigma\omega = \frac{\tau'\kappa - \kappa'\tau}{\omega^2}, \quad \overline{T} = N, \quad \overline{N} = \frac{\delta^*}{\omega}, \quad \overline{B} = \frac{\delta}{\omega},$$

where $\omega = \sqrt{\kappa^2 + \tau^2}$, $\sigma = \frac{\tau'\kappa - \kappa'\tau}{\omega^3}$, $\delta = \tau T + \kappa B$, and $\delta^* = -\kappa T + \tau B$. Extensive studies have been done into the relationships of curves with their natural mates. However, a direction that has been relatively unexplored concerns the relationships between curves that have the same natural mate. We motivate and then define the co-successor of a curve, and after giving several immediate relationships between co-successors, we look at a simple application to Bertrand pairs.

Throughout this paper, we will assume that α is a unit speed curve with $\kappa > 0$ (equivalently, $\alpha'' \neq 0$) unless otherwise stated. We take the convention that if a curve is distinguished from a second curve by a tilde, overbar, superscript, or other mark, we distinguish the Frenet-Serret apparatus' of the curves using the same mark. We consider two curves to be the same up to rigid translation and rotation.

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1. Motivating co-successors

The Frenet-Serret apparatus of the natural mate can be expressed completely in terms of the Frenet-Serret apparatus of the curve of which it is the natural mate, thus, every curve has a single unique natural mate. A natural question is then, given the curvature and torsion of the natural mate, what information can be extracted about the curvature and torsion of the curve of which it is the natural mate. This first lemma provides a first attempt at a description of the Frenet-Serret apparatus in terms of that of its natural mate, and are easily verified.

Lemma 1.1. *Suppose α and $\overline{\alpha}$ are unit speed curves. Then $\overline{\alpha}$ is the natural mate of α if and only if their Frenet-Serret apparatus' satisfy:*

$$\begin{aligned} \kappa' &= \frac{\overline{\kappa}'}{\overline{\kappa}}\kappa - \overline{\tau}\tau, \\ \tau' &= \frac{\overline{\kappa}'}{\overline{\kappa}}\tau + \overline{\tau}\kappa, \\ T &= \frac{\tau}{\overline{\kappa}}\overline{B} - \frac{\kappa}{\overline{\kappa}}\overline{N}, \\ B &= \frac{\kappa}{\overline{\kappa}}\overline{B} + \frac{\tau}{\overline{\kappa}}\overline{N}, \\ N &= \overline{T}. \end{aligned} \tag{1.1}$$

Remark 1.1. The first two equations of (1.1) provide a pair of coupled first-order differential equations for the curvature and torsion of α in terms of the curvature and torsion of $\overline{\alpha}$. As such, if we try to solve these differential equations for κ and τ, we expect to obtain more than one solution, dependent on the initial conditions used for κ and τ, under the constraint that $\kappa > 0$. So, we anticipate that there are more than one curve that have the same natural mate.

2. Co-successor characterization and properties

Definition 2.1. *Let α and $\overline{\alpha}$ be unit speed curves. If $\overline{\alpha}$ is the natural mate of α, we say that α is a successor of $\overline{\alpha}$ [8]. Similarly, if α_1 and α_2 are both successors of the same curve $\overline{\alpha}$, then we say that α_1 and α_2 are co-successors of each other.*

Remark 2.1. It can be seen that the relationship of being co-successors is in fact an equivalence relation. This is because every curve has a unique natural mate, and so if α_1 and α_2 are co-successors, and α_2 and α_3 are co-successors, then all of $\alpha_1, \alpha_2, \alpha_3$ have the same natural mate. As such, this allows us to talk about the family of co-successors without referring to a particular pair, and to talk about two co-successors without necessarily refering to the curve of which they are both successors, since they each only have a single unique natural mate, which is common between them.

Proposition 2.1. *Every co-successor of a generalized helix is a generalized helix.*

Proof. This follows from the fact that a curve is a generalized helix if and only if its natural mate is planar. [5], [6] □

Proposition 2.2. *Every co-successor of a slant helix is a slant helix.*

Proof. This follows from the fact that a curve is a slant helix if and only if its natural mate is a generalized helix. [5], [6] □

While these propositions give trivial relationships between two co-successors, a more general characterization of co-successors would be more useful. We find there is a simple characterization of co-successors in terms of a rotation of their curvatures and torsions.

Theorem 2.1. *Suppose α_1 and α_2 are unit speed curves, then α_1 and α_2 are co-successors if and only if then there exists constant $\nu \in \mathbb{R}$ such that the Frenet-Serret apparatus of α_2 is given in terms of the Frenet-Serret apparatus of α_1 by:*

$$\begin{bmatrix} T_2 \\ N_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ B_1 \end{bmatrix}, \qquad \begin{bmatrix} \kappa_2 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \cos\nu & -\sin\nu \\ \sin\nu & \cos\nu \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \tau_1 \end{bmatrix}. \tag{2.1}$$

We call the number ν the phase separation between α_1 and α_2.

Proof. Suppose α_1 and α_2 are co-successors. Then they both have the same natural mate $\overline{\alpha}$, with Frenet-Serret apparatus $\{\overline{\kappa}, \overline{\tau}, \overline{T}, \overline{N}, \overline{B}\}$. Let $\kappa_1 = r_1 \cos\theta_1$, $\tau_1 = r_1 \sin\theta_1$, $\kappa_2 = r_2 \cos\theta_2$, $\tau_2 = r_2 \sin\theta_2$, where $r_1, r_2, \theta_1, \theta_2$ are functions of the arc length parameter, and $r_1, r_2 > 0$. This is possible since the curvatures are strictly positive. Noting that $\sqrt{\kappa_1^2 + \tau_1^2} = \overline{\kappa} = \sqrt{\kappa_2^2 + \tau_2^2}$, and so $r_1^2 = \kappa_1^2 + \tau_1^2 = \kappa_2^2 + \tau_2^2 = r_2^2$, and so $r_1 = r_2$, which we from here on call r. We additionally have that $\frac{\tau_1'\kappa_1 - \kappa_1'\tau_1}{\overline{\kappa}^2} = \overline{\tau} = \frac{\tau_2'\kappa_2 - \kappa_2'\tau_2}{\overline{\kappa}^2}$, which, after substitution, simplifies to $\theta_1' = \theta_2'$. Thus, there exists some constant ν such that $\theta_2 = \theta_1 + \nu$. After applying sine and cosine identities, we obtain

$$\kappa_2 = \kappa_1 \cos\nu - \tau_1 \sin\nu,$$
$$\tau_2 = \kappa_1 \sin\nu + \tau_1 \cos\nu.$$

Noting that $N_1 = \overline{T} = N_2$, the other two vector relations follow from the curvature and torsion relation above and the expressions for the normal and binormal vectors of the natural mate. From this, we have both matrix relations.

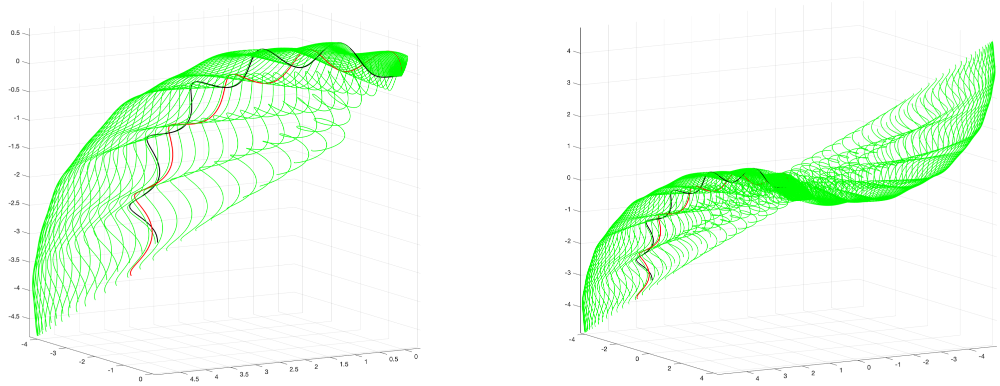
The converse follows from the fact that $N_1 = N_2$, and so their natural mates have equal tangent vectors, and so are congruent. □

Remark 2.2. The above theorem has the potential to be generalized beyond the context with $\kappa_1 > 0$ and $\kappa_2 > 0$, where we assume for all unit speed curves α that $\alpha'' \neq 0$. In particular, just looking at the forms of (2.1), we first note that if T_1, N_1, and B_1 form a right-handed orthonormal frame, then for arbitrary choice of ν, then T_2, N_2, and B_2 will also be right-handed and orthonormal. Additionally, it can be shown by direct computation that if α_1 satisfies the Frenet-Serret equations, and the Frenet-Serret apparatus of α_2 is related to that of α_1 by equations (2.1), then it follows that the Frenet-Serret apparatus of α_2 will also satisfy the Frenet-Serret equations for any constant ν. However, for many choices of ν, this would lead to κ_2 no longer being a strictly positive function, sometimes being zero or negative, and as such, the frame obtained would no longer be a Frenet frame as it is usually defined.

Definition 2.2. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve. We say that the unit speed curve $\widetilde{\alpha}$ is the conjugate mate of α if the tangent vector of $\widetilde{\alpha}$ is equal to the binormal vector of α.*

Corollary 2.1. *The conjugate mate $\widetilde{\alpha}$ of a Frenet curve α is a co-successor of α with phase separation $\pm\frac{\pi}{2}$.*

Corollary 2.2. *Every co-successor of a curve α can be expressed as a linear combination of α and $\widetilde{\alpha}$, the integral curve of its binormal vector B (If α is Frenet, then $\widetilde{\alpha}$ is its conjugate mate). In particular, the co-successor α_ν of α with phase separation ν can be expressed as $\alpha_\nu = \cos\nu\alpha + \sin\nu\widetilde{\alpha}$.*



(Figure 1) A unit speed Frenet curve (black), its conjugate mate (red) and its family of co-successors (green). The left plot is restricted to positive curvature, while the right plot is not holding this restriction. The curvature and torsion of the black Frenet curve are given by $\kappa = \frac{17s^2}{7s^2+1}$ and $\tau = 2.3 + 0.6\sin(3.3s)$.

3. Equivalent and mirror curves

Before we can apply the notion of co-successors to Bertrand curves, we need to define several other types of associated curves.

Definition 3.1. *Let $\alpha_1 : I \to \mathbb{E}^3$ and $\alpha_2 : J \to \mathbb{E}^3$ be two regular curves. If there is a diffeomorphism $h : I \to J$ such that T_1, N_1, and B_1 are parallel to $T_2 \circ h$, $N_2 \circ h$, and $B_2 \circ h$ respectively, and $T_1 \circ h^{-1}$, $N_1 \circ h^{-1}$, and $B_1 \circ h^{-1}$ are parallel to T_2, N_2, and B_2 respectively, then we say α_1 and α_2 are equivalent.* [3]

This is an equivalence relation, and additionally, if two curves are equivalent, then their natural mates are equivalent.

Corollary 3.1. *Every natural mate is equivalent to a spherical curve.*

Proof. Let α be a unit speed curve. We define $\widetilde{\alpha}(\tilde{s}) = T(s(\tilde{s}))$, which is clearly a spherical curve, where $s(\tilde{s}) = \int \frac{1}{\kappa} d\tilde{s}$. We see that $\widetilde{\alpha}(\tilde{s})$ is unit speed, since

$$\begin{aligned}\frac{d\widetilde{\alpha}(\widetilde{s})}{d\widetilde{s}} &= \frac{d\widetilde{\alpha}(\widetilde{s})}{ds}\frac{ds}{d\widetilde{s}} \\ &= \frac{dT(s(\widetilde{s}))}{ds}\frac{1}{\kappa} \\ &= \kappa(s(\widetilde{s}))N(s(\widetilde{s}))\frac{1}{\kappa(s(\widetilde{s}))} \\ &= N(s(\widetilde{s})).\end{aligned}$$

Thus $\widetilde{T}(\widetilde{s}) = N(s(\widetilde{s})) = (\overline{T} \circ s)(\widetilde{s})$. Differentiating this expression with respect to $\widetilde{s}$, we get that $\widetilde{\kappa}\widetilde{N} = \frac{d\widetilde{T}}{d\widetilde{s}} = \frac{ds}{d\widetilde{s}}\frac{dN}{ds} = \frac{ds}{d\widetilde{s}} - \kappa T + \tau B = \omega\frac{1}{\kappa}\frac{\delta^*}{\omega}$, and so $\widetilde{N} = (\overline{N} \circ s)(\widetilde{s})$. This gives by taking the cross product $\widetilde{B} = (\overline{B} \circ s)(\widetilde{s})$. Thus $\overline{\alpha}$ is equivalent to $\widetilde{\alpha}$, which is spherical. □

Definition 3.2. *Given a unit speed curve α, the mirror of α across a unit vector M is the curve given by $\alpha_M = \alpha - 2M\langle M, \alpha\rangle$.*

Geometrically, this represents reflecting α across the perpendicular plane to M to obtain the new curve α_M. Since every mirror of α can be obtained from any other mirror by a rotation and translation, we consider them congruent, and will simply refer to α_M as the mirror of α, without referencing any particular unit vector M.

Lemma 3.1. *A curve α_M is the mirror of a unit speed curve α if and only if its curvature and torsion are given by $\kappa_M = \kappa$, $\tau_M = -\tau$.*

Additionally, the Frenet frame of α_M is given by

$$\begin{aligned} T_M &= T - 2M\langle M, T\rangle, \\ N_M &= N - 2M\langle M, N\rangle, \\ B_M &= -B + 2M\langle M, B\rangle. \end{aligned} \tag{3.1}$$

Proof. Suppose α_M is the mirror of α, then $\alpha_M = \alpha - 2M\langle M, \alpha\rangle$. Differentiating with respect to the arclength parameter of α yields $\alpha_M{}' = T - 2M\langle M, T\rangle$, which is unit, and so α^M has the same arclength parameter as α (This is geometrically obvious). Thus, $\alpha_M{}' = T_M = T - 2M\langle M, T\rangle$. Differentiating again gives $\kappa_M N_M = \kappa N - 2\kappa M\langle M, N\rangle = \kappa(N - 2M\langle M, N\rangle)$. One can easily show that $N - 2M\langle M, N\rangle$ is unit, so we conclude that $N_M = N - 2M\langle M, N\rangle$ and $\kappa_M = \kappa$. We then have

$$\begin{aligned} B_M &= T_M \times N_M \\ &= (T - 2M\langle M, T\rangle) \times (N - 2M\langle M, N\rangle) \\ &= B - 2M \times (\langle M, T\rangle N - \langle M, N\rangle T) \\ &= B - 2M \times (M \times (N \times T)) \\ &= B + 2M \times (M \times B) = B + 2M\langle M, B\rangle - 2B\langle M, M\rangle) \\ &= -B + 2M\langle M, B\rangle. \end{aligned}$$

Differentiating, we obtain

$$-\tau_M N_M = B_M{}' = \tau N - 2\tau M\langle M, N\rangle = \tau N_M,$$

and so conclude that $\tau_M = -\tau$.

Suppose the converse holds, then since the mirror of α has the same curvature and torsion of α_M, then α_M must be the mirror of α. □

Lemma 3.2. *The natural mate of the mirror of a curve is congruent to the mirror of the natural mate.*

Proof. Let α be a unit speed curve with natural mate $\overline{\alpha}$, mirror α_M, let the mirror of the natural mate be $\overline{\alpha_M}$, and the natural mate of the mirror be $\overline{\alpha}_M$.

The result follows by direct calculation

$$\begin{aligned} \overline{\kappa_M} &= \sqrt{\kappa_M{}^2 + \tau_M{}^2} = \sqrt{\kappa^2 + \tau^2} = \overline{\kappa} = \overline{\kappa}_M, \\ \overline{\tau_M} &= \frac{\tau_M{}'\kappa_M - \kappa_M{}'\tau_M}{\overline{\kappa_M}^2} = -\frac{\tau'\kappa - \kappa'\tau}{\overline{\kappa}^2} = -\overline{\tau} = \overline{\tau}_M. \end{aligned}$$

□

The following two lemmas similarly follow quickly from the definitions.

Lemma 3.3. *If two curves are equivalent, then their mirrors are equivalent.*

Lemma 3.4. *Suppose α_1 and β_1 are equivalent, both having a co-successor α_2 and β_2, respectively, with phase separation ν. Then α_2 and β_2 are equivalent.*

4. Bertrand curves

Definition 4.1. *A curve α is Bertrand if there exists another curve $\hat{\alpha}$ such that $\hat{\alpha}(s) = \alpha(s) + \lambda(s)N(s)$, where the normal vectors of the two curves are parallel ($N = \pm\hat{N}$). We call these two curves together a Bertrand pair and say that $\hat{\alpha}$ is the Bertrand mate of α.* [2], [3]

It is easy to show that λ is a constant. It is also a well-known result that a curve α is Bertrand if and only if there exists constants a, b such that $a\kappa + b\tau = 1$.

When discussing a Bertrand pair α and $\hat{\alpha}$, we will choose our parametrization such that α is unit speed, while $\hat{\alpha}$ is not necessarily unit speed.

Corollary 4.1. *Let α_1 be a Bertrand curve with co-successor α_2, then α_2 is also Bertrand.*

Proof. This follows from

$$\begin{aligned}1 &= a\kappa_1 + b\tau_1\\ &= a(\kappa_2\cos\nu - \tau_2\sin\nu) + b(\kappa_2\sin\nu + \tau_2\cos\nu)\\ &= (a\cos\nu + b\sin\nu)\kappa_2 + (b\cos\nu - a\sin\nu)\tau_2.\end{aligned}$$

□

Because of the two distinct possibilities in the definition of a Bertrand pair allowing the normal vectors of the curves to either be parallel or anti-parallel, it is useful to distinguish between these two cases.

Definition 4.2. *Let α and $\hat{\alpha}$ be a Bertrand pair, then we say that α and $\hat{\alpha}$ are positively (negatively) Bertrand if $N = \hat{N}$ ($N = -\hat{N}$).*

We now apply the notion of a co-successor to obtain an alternative description of the Bertrand mate of a curve, and obtain a simple relationship between the natural mates of a Bertrand pair.

Theorem 4.1. *Suppose α and $\hat{\alpha}$ are a positively Bertrand pair. Then $\hat{\alpha}$ is a co-successor of α up to equivalence.*

Proof. Suppose α and $\hat{\alpha}$ are a positively Bertrand pair. Then $N = \hat{N}$, and in particular, we can write $\hat{T} = \cos\theta T + \sin\theta B$. Differentiating, where $\hat{s}$ is the arc-length parameter of $\hat{\alpha}$, we get that $\hat{s}'\hat{\kappa}\hat{N} = \hat{s}'\hat{\kappa}N = -\theta'\sin\theta T + \kappa\cos\theta N + \theta'\cos\theta B - \tau\sin\theta N$, and so $-\theta'\sin\theta = \theta'\cos\theta = 0$, implying that θ is a constant. Since $\hat{N} = N$ and we have that $\hat{T} = \cos\theta T + \sin\theta B$ and $\hat{B} = -\sin\theta T + \cos\theta B$, which we note are parallel to the Frenet frame of the co-successor of α with phase separation θ after a change in parametrization. Thus, $\hat{\alpha}$ is equivalent to the selected co-successor of α. □

Corollary 4.2. *Let α and $\hat{\alpha}$ be a positively Bertrand pair. Then the natural mates of α and $\hat{\alpha}$ are equivalent.*

Proof. This follows since the natural mate of α and its co-successor are equal, and the fact that natural mates of equivalent curves are themselves equivalent. □

Theorem 4.2. *Suppose α and $\hat{\alpha}$ are a negatively Bertrand pair. Then the mirror of $\hat{\alpha}$ is a co-successor of α up to equivalence.*

Proof. This is proved similarly to Theorem 4.4, except we take $\hat{T} = -\cos\theta T - \sin\theta B$, and then consider the mirror of $\hat{\alpha}$'s relationship with α. □

Corollary 4.3. *Let α and $\hat{\alpha}$ be a negatively Bertrand pair. Then the natural mate of α and the natural mate of the mirror of $\hat{\alpha}$ are equivalent.*

Proof. This follows since α and its co-successors have the same natural mate, the mirror of $\hat{\alpha}$ is equivalent to a co-successor of α, and the fact that the natural mates of equivalent curves are equivalent. □

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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