RESTRICTED-FINITE GROUPS WITH SOME APPLICATIONS IN GROUP RINGS

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ABSTRACT. We carry out a study of groups G in which the index of any infinite subgroup is finite. We call them restricted-finite groups and characterize finitely generated not torsion restricted-finite groups. We show that every infinite restricted-finite abelian group is isomorphic to $\mathbb{Z} \times K$ or $\mathbb{Z}_{p^\infty} \times K$, where K is a finite group and p is a prime number. We also prove that a group G is infinitely generated restricted-finite if and only if $G = AT$ where A and T are subgroups of G such that A is normal quasi-cyclic and T is finite. As an application of our results, we show that if G is not torsion with finite G' and the group-ring RG has restricted minimum condition, then R is a semisimple ring and $G \cong T \rtimes \mathbb{Z}$, where T is finite whose order is unit in R. The converse is also true with certain conditions including $G = T \times \mathbb{Z}$.

1. Introduction

Throughout the paper, all rings will have unit elements, groups are not necessarily abelian (unless stated) and all modules will be right unitary. The study of rings with chain conditions (maximal and minimal conditions) on ideals was started by E. Noether and E. Artin in the 1920s and 1930s. Subsequently, the maximal and minimal conditions on classes of subgroups of a group were studied by R. Baer, S. N. Černikov, K. A. Hirsch, O. J. Schmidt, and others. A famous result by Hirsch states that a group is polycyclic if and only if it is solvable and satisfies max [17, 5.4.12]. The structure of solvable groups with min was determined by Černikov. He proved that a solvable group satisfies min if and only if it is an extension of a direct product of finitely many quasi-cyclic groups by a finite group [17, 5.4.23].

With slight modifications in the definition of chain conditions, one can obtain new classes of groups. For example in [16] and its references, soluble groups and locally nilpotent groups satisfying the conditions for subgroups, normal subgroups,

and subnormal subgroups were studied. A group G is said to satisfy max-∞ if and only if each nonempty set of subgroups of infinite order of G has a maximal element or equivalently if and only if there does not exist an infinite properly ascending chain $G_1 < G_2 < \cdots$ of subgroups of infinite order. Similarly, G is said to satisfy min-∞ if and only if each nonempty set of subgroups of the infinite index of G has a minimal element or equivalently if and only if there does not exist an infinite properly descending chain $G_1 > G_2 > \cdots$ of subgroups of the infinite index. The class of groups satisfying max-∞ strictly contains the class of groups satisfying max and the class of groups with min strictly contained in the class of groups with min-∞.

Recall that the Baer radical of a group is the subgroup generated by all the cyclic subnormal subgroups. For an infinite group in which all non-trivial normal subgroups have finite index, named *just-infinite* group, the Baer radical, the Fitting subgroup and the maximal normal nilpotent subgroup coincide. McCarthy [10,11] and Wilson [19] studied just-infinite groups with non-trivial Baer radical and they determined that these groups are extensions of a free abelian group by finite groups. Wilson also studied the groups with trivial Baer radical. In this paper, we study the class of restricted-finite groups.

Definition 1.1. A group G is called restricted-finite if $|G : H|$ is finite for any infinite non-trivial subgroup H of G.

Example 1.2. (i) The infinite cyclic group $\mathbb{Z}$ is an example of restricted-finite group. The infinite dihedral group is non-abelian and restricted-finite. Furthermore, a direct product of a restricted-finite group with a finite group is restricted-finite.
(ii) For any finite group K, the groups $\mathbb{Z} \times K$ and $\mathbb{Z}_{p^\infty} \times K$ are restricted-finite, where $\mathbb{Z}_{p^\infty}$, p a prime, is the *quasi-cyclic* group (of type p^∞) and generated by elements $a_1, a_2, \ldots$ such that $a_1^p = 1$, $a_{i+1}^p = a_i$, $i = 1, 2, \ldots$.

Note that, groups with no infinite proper subgroups are obviously restricted-finite, we call them "*proper-finite*" and give some non-trivial examples of proper-finite groups in the next example.

Example 1.3. A *Tarski group* is an infinite group in which every non-trivial subgroup is a cyclic group of order a prime number p. A Tarski group G is necessarily simple and two-generated. These groups are torsion but they may contain elements of arbitrarily large orders. If there exists a prime number p such that all non-trivial proper subgroups are of order p, then the group is called *Tarski Monster*. In a

series of work published in the 1980s, Ol'šhanskiĭ [12,13,14] proved the existence of Tarski (Monsters) groups and constructed Tarski Monster for all primes $p > 10^{75}$.

Restricted-finite groups also appeared in the study of group-rings with "*restricted minimum condition*". We recall below a history of restricted minimum condition for rings and then we state our results on restricted-finite groups. Some applications of our results are then given for group-rings. A widely used result by Hopkins and Levitzki states that "every right Artinian ring is a right Noetherian ring", see for instance, [3, Theorem 4.15]. Motivated by this, Cohen(1950) studied commutative ring R for which R/I is an Artinian ring for every non-zero ideal I of R [2]. Later on, Ornstein(1968) continued Cohen's works for any ring (not necessarily commutative) [15]. In 1972, Camillo and Krause asked whether a ring R is right Noetherian if for every non-zero right ideal I of R the cyclic right R-module R/I is Artinian [4]. Since in a non-Artinian ring R with latter property, $I \cap J \neq 0$ for all non-zero right ideals I, J of R, hence Camillo-Krause's question leads us to the study of rings R for which R/I is an Artinian R-module where I is an essential right ideal of R. Such property was called in [1,4] the "restricted minimum condition", see [5], [6], [7] and [8] for recent works on the subject. In [6, Theorem 3.5], it is shown that if the group-ring RG has restricted minimum condition, then $|G/N| < \infty$ for any normal infinite subgroup N of G. The latter property on G was called "restricted-finite" in [6], we call that "*normally restricted-finite*".

For any group G, we use $\mathrm{rad}_\infty(G)$ to denote the intersection of all infinite subgroups. It is easy to verify that $\mathrm{rad}_\infty(G)$ is a characteristic subgroup of G and if $\mathrm{rad}_\infty(G)$ is finite, then $\mathrm{rad}_\infty(G/\mathrm{rad}_\infty(G))$ is trivial. Note that $\mathrm{rad}_\infty(G)$ is a proper-finite group. Also $\mathrm{rad}_\infty(G) = G$ if and only if G is proper-finite. Thus $\mathrm{rad}_\infty(\mathrm{rad}_\infty(G)) = \mathrm{rad}_\infty(G)$. Various research questions can be raised about restricted-finite groups; for example: what is the relationship between $\mathrm{rad}_\infty(G)$ and $\mathrm{Frat}(G)$? or determine the Baer radical for a restricted-finite group. However, we are trying to investigate *when a restricted-finite group is a product (semi-product) of an infinite cyclic group with a finite group*? This helps us to express some interesting applications for group-rings with restricted minimum condition.

We shall give a characterization of infinitely generated restricted-finite groups in Theorem 2.19 and finitely generated not torsion restricted-finite groups in Theorems 2.14 and 2.17. Torsion finitely generated restricted-finite groups are investigated in Proposition 2.7. Examples are presented to describe our results and some applications are given for group-rings in Section 3. In most parts, the notations are standard, and we give a partial list for the reader's convenience. $\mathbb{Z}$ is the set of

integers and $\mathbb{Q}$ is the set of rational numbers. $x^g = g^{-1}xg$ and $H^g = g^{-1}Hg$. $\langle X \rangle$ is the subgroup generated by a set X. $C_G(x)$ (or simply $C(x)$) is the centralizer of an element x in a group G. $C_G(H)$ (or simply $C(H)$) is the centralizer of a subgroup H in a group G. $N_G(H)$ is the normalizer of a subgroup H in a group G. $Z(G)$ is the center of a group G. $G = K \rtimes H$ means that G is a split extension (semidirect product) of a normal subgroup K of G by a complement H. $G^{(n)} =$ the nth term of the derived series of G.

Any unexplained terminology and all the basic results on groups, rings, and modules that are used in the sequel can be found in [9] and [17].

2. Main results

We first record some general properties of restricted-finite groups and show that restricted-finite groups satisfy the maximal condition on infinite subgroups; see for instance [16] where such groups were studied.

Proposition 2.1. *Let G be a restricted-finite group.*

  (a) *All subgroups and quotient groups of G are also restricted-finite.*
  (b) *G satisfies the maximal condition on infinite subgroups.*
  (c) *If K and L are infinite subgroups of G, then $K \cap L$ is infinite.*
  (d) *Every infinite subgroup of G contains an infinite normal subgroup of G.*

Proof. (a) Straightforward.

(b) Let $K_1 \leq K_2 \leq \cdots$ be an ascending chain of infinite proper subgroups of G. Then we have $|K_2 : K_1| \leq |K_3 : K_1| \leq \cdots \leq |K_i : K_1| \leq |K_{i+1} : K_1| \leq \cdots \leq |G : K_1| \leq \infty$ for all $i \geq 1$. Therefore there exists $n \geq 1$ such that $|K_i : K_1| = |K_{i+1} : K_1|$ for all $i \geq n$. It follows that $K_n = K_{n+i}$ for all $i \geq 1$.

(c) This is true because $|G : K \cap L| \leq |G : K||G : L|$.

(d) Suppose that G contains an infinite subgroup H and let $H_G = \bigcap_{a \in G} H^a$ be the core of H in G, which is the largest normal subgroup of G that contained in H. Clearly, $H_G \subseteq H \subseteq N_G(H)$. If H is not normal, then $|G : N_G(H)| = n$ is finite by the restricted-finite condition on G. It follows that H has exactly n conjugates H^{a_i}, $i = 1, \ldots, n$, in G. Hence by (c), $H_G = \bigcap_{i=1}^n H^{a_i}$ is an infinite subgroup of G. $\square$

Corollary 2.2. *Every restricted-finite group is countable.*

Proof. This follows by Proposition 2.1(b) and Proposition 2.9 of [16]. $\square$

We say that a group G is *infinitely-simple* if G has no proper infinite normal subgroup.

Proposition 2.3. *A group G is infinitely-simple restricted-finite if and only if it is a proper-finite group.*

Proof. Sufficiency is clear. Necessity is obtained by Proposition 2.1(c). $\qquad\square$

Corollary 2.4. *Let G be a simple group. Then G is restricted-finite if and only if G is a proper-finite group.*

Proof. By Proposition 2.3. $\qquad\square$

Corollary 2.5. *If G is restricted-finite, then $\mathrm{rad}_\infty(G)$ is the intersection of all normal infinite subgroups of G.*

Proof. This follows from Proposition 2.1(d). $\qquad\square$

Proposition 2.1 shows that every restricted-finite group has ascending chain condition on infinite subgroups. The *descending* chain condition on infinite subgroups of a group G is equivalent to the descending chain condition on all subgroups of G, in this case, G is said to satisfy "*min*" [17, page 67]. Every group which satisfies min is torsion. Restricted-finite groups that satisfy min are close to the proper-finite groups, as we see Proposition 2.7.

Lemma 2.6. *If one of the following cases occur for a group G, then G is restricted-finite.*

   (i) *G has a subgroup C of finite index such that C is a restricted-finite group.*
  (ii) *G has a normal finite subgroup F such that G/F is a restricted-finite group.*

Proof. (i) Since $|G : C|$ is finite, there exists a normal subgroup contained in C of finite index. Thus we may assume that C is normal. Let H be any infinite subgroup of G and $A = H \cap C$. Since G/C is finite, $HC/C \leq G/C$ is finite. Thus $H/A = H/H \cap C$ is finite and so A is infinite. Now by the restricted-finite condition on C, $|C : A|$ is finite. Since $|G : A| = |G : C||C : A|$, $|G : A|$ is finite. It follows that $|G : H|$ is finite, proving that G is restricted-finite.

(ii) Let K be an infinite subgroup of G. Then KF/F is also an infinite subgroup of G/F. Hence by our assumption, $|G/F : KF/F| = |G : KF|$ is finite. Clearly, $|KF : K|$ is finite. Thus $|G : K| = |G : KF||KF : K|$ is finite, as desired. $\qquad\square$

Proposition 2.7. *The following conditions are equivalent for an infinite group G.*

   (i) *G is restricted-finite which satisfies min.*
  (ii) *G contains an infinite proper-finite subgroup of finite index.*
 (iii) *G contains an infinite proper-finite normal subgroup of finite index.*

(iv) $G/\mathrm{rad}_\infty(G)$ *is finite.*

Proof. (i)$\Rightarrow$(ii) If G is proper-finite, then we have nothing to prove. Otherwise, since G satisfies min, there exists a subgroup K which is minimal among all proper infinite subgroups of G. Thus K is proper-finite and $|G : K|$ is finite by the restricted-finite condition.

(ii)$\Rightarrow$(iii) By Proposition 2.1(d).

(iii)$\Rightarrow$(iv) By Lemma 2.6, G is restricted-finite. Let N be any infinite proper-finite normal subgroup of G. We show that $N = \mathrm{rad}_\infty(G)$. Clearly, $N \supseteq \mathrm{rad}_\infty(G)$. Now if H is an infinite subgroup of G, then by Proposition 2.1(c), $H \cap N$ is infinite. But since every proper subgroup of N is finite, $H \cap N = N$ and so $N \le H$. Therefore, $N \le \mathrm{rad}_\infty(G)$ and so $N = \mathrm{rad}_\infty(G)$.

(vi)$\Rightarrow$(i) This is true by Lemma 2.6(i). $\qquad\square$

By Corollary 2.5, if G is an infinite restricted-finite group, then $\mathrm{rad}_\infty(G) = 1$ if and only if G is *residually finite*. Recall that a group G is called residually finite if for every $1 \ne g \in G$, there exists a normal subgroup N of finite index such that $g \notin N$ (see page 55 of [17]). Among the main consequences of the theory of Zelmanov (see [20], [21]) is that if a finitely generated residually finite group has *finite exponent*, then the group is finite. Here, finite exponent means there exists $n > 0$ such that $g^n = 1$ for every g in the group.

Corollary 2.8. *If G is a finitely generated restricted-finite with finite exponent, then G satisfies min.*

Proof. By Proposition 2.7, we shall show that $G/\mathrm{rad}_\infty(G)$ is finite. If $\mathrm{rad}_\infty(G)$ is infinite, then we are nothing to prove. Otherwise, $\mathrm{rad}_\infty(G/\mathrm{rad}_\infty(G))$ is trivial. Thus by Corollary 2.5, the group $G/\mathrm{rad}_\infty(G)$ is residually finite and so by the Zelmanov's Theorem (mentioned above), $G/\mathrm{rad}_\infty(G)$ is finite, as desired. $\qquad\square$

Remarks 2.9. (i) There is a finitely generated (non-trivial) restricted-finite torsion group that satisfies min, but it is not of finite exponent. Let $\mathfrak{M}$ be a Tarski group of infinite exponent, and $a \in \mathfrak{M}$ be an element of order prime p. Let τ_a be the inner automorphism of $\mathfrak{M}$ induced by a. Then $G = \mathfrak{M} \rtimes \langle \tau_a \rangle$ is a finitely generated torsion restricted-finite group of infinite exponent which is not proper-finite and satisfies min.

(ii) Let G be a finitely generated torsion restricted-finite group that does not satisfy min. Then by Proposition 2.7, G has no infinite subgroup K such that K is proper-finite. Thus $\mathrm{rad}_\infty(G)$ is finite. Also, G has an infinite normal proper subgroup by

Proposition 2.1(d). It follows by Zorn's Lemma, G has an infinite maximal normal subgroup G_1. Then G/G_1 is a simple finite group. Again by Lemma 2.10(a) and Proposition 2.1(a), G_1 is a finitely generated torsion restricted-finite group. Proceeding in this way, we obtain an infinite descending series $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ such that $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} is simple finite. If further G is a p-group for some prime number p, then $G_i/G_{i+1} \cong \mathbb{Z}_p$ for all i and G' is an infinite proper subgroup. It follows that for every $n \geq 0$, $G^{(n+1)}$ is an infinite proper subgroup of $G^{(n)}$.

(iii) Find a finitely generated torsion restricted-finite group not satisfying min.

In the following, we investigate not torsion finitely generated restricted-finite groups in Theorems 2.14 and 2.17. Also, we show that infinitely generated restricted-finite groups are locally finite and we shall give a characterization of them in Theorem 2.19. A group G is called *locally finite* if every finitely generated subgroup of G is finite, see 14.3 of [17] for further information.

Lemma 2.10. *Let G be a restricted-finite group.*

    (a) *If G is finitely generated, then every subgroup of G is finitely generated. In particular, G satisfies the maximal condition on subgroups.*

    (b) *If G is infinitely generated, then G is locally finite. In particular, G is torsion.*

Proof. (a) By 1.6.11 of [17], in a finitely generated group, every subgroup of the finite index is finitely generated. Thus by the restricted-finite condition on G, all subgroups of G are finitely generated. The last statement is now clear.

(b) Suppose that G is not finitely generated. If H is a finitely generated subgroup of G that is not finite, by the restricted-finite condition on G, $|G : H|$ is finite. Hence, G must be finitely generated, a contradiction. Therefore, G is locally finite. Locally finite groups are torsion. $\qquad\square$

Proposition 2.11. *A group G is not torsion restricted-finite if and only if G has a normal infinite cyclic subgroup of finite index.*

Proof. ($\Rightarrow$) By Proposition 2.1(d).
($\Leftarrow$) By Lemma 2.6. $\qquad\square$

Example 2.12. Let p be an odd prime number and $A = \langle a_1, a_2, \ldots \rangle \cong \mathbb{Z}_{p^\infty}$, i.e. $a_1^p = 1$, $a_{i+1}^p = a_i$, $i = 1, 2, \ldots$.

    (i) Let G be a group generated by $x, a_1, a_2, \ldots$ such that $x^2 = 1$, $a_i^x = a_i^{-1}$, $i = 1, 2, \ldots$. Then $G = A\langle x \rangle = A \rtimes \langle x \rangle$ is restricted-finite by Lemma 2.6.

Also, $Z(G) = 1$, $A \cap \langle x \rangle = 1$ and G is not the direct product of A and $\langle x \rangle$ for $a_i^x = a_i^{-1} \neq a_i$.

(ii) Let G be a group generated by $x, y, a_1, a_2, \ldots$ such that $x^2 = y^2 = 1$, $xy = yx$, $a_i^x = a_i$, $a_i^y = a_i^{-1}$, $i = 1, 2, \ldots$. Then $G = A\langle x, y \rangle$ is restricted-finite. Note that $Z(G) = \langle x \rangle$.

(iii) Let $G = A \times S$, where S is a finite simple group. Then G is restricted-finite and $Z(G) = A$.

Proposition 2.13. *Let G be a non-zero torsion-free restricted-finite group. Then $G \cong \mathbb{Z}$.*

Proof. By Lemma 2.10(b), $G = \langle g_1, \ldots, g_n \rangle$ is finitely generated. Since G is torsion-free, $C_G(x)$ is infinite, for all $x \in G$. By Proposition 2.1(c), $Z(G) = \bigcap_{i=1}^n C_G(g_i)$ is infinite. Therefore, $G/Z(G)$ is finite and by a result of Schur (see 10.1.4 of [17]) G' is finite. Thus $G' = 1$ and so G is abelian. Now the structure theorem for finitely generated abelian groups states that $G \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n-\text{times}}$ and by the restricted-finite condition, we must have $G \cong \mathbb{Z}$. $\qquad\square$

We say that a group G is *c-torsion*, if $T(G)$, the set of all finite order elements of G, is a subgroup of G. Clearly, $T(G)$ will be a normal subgroup when it is a subgroup.

Theorem 2.14. *The following statements are equivalent for an infinite group G.*

(i) *G is a finitely generated restricted-finite group with finite G'.*

(ii) *G is a c-torsion restricted-finite group with finite $T(G)$.*

(iii) *G is a restricted-finite group with $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.*

(iv) *$G = T(G) \rtimes C$ where $\mathbb{Z} \cong C \leq G$ and $T(G)$ is a finite normal subgroup.*

Proof. (i)$\Rightarrow$(ii) It is well known that if G' is finite, then G is c-torsion (see 14.5.9 of [17]). We shall show that $T(G)$ is finite. By our assumption, the set $\Lambda = \{H \leq G \mid G' \subseteq H, |H| < \infty\}$ is nonempty. By Lemma 2.10(a), Λ has a maximal member N. Clearly, N is a normal subgroup of G and we see that G/N is a torsion-free group. For $a \in G \setminus N$, $\langle a \rangle N$ is an infinite subgroup, by the maximality of N. Hence, $o(aN)$ is infinite. Since now G/N is a restricted-finite group, $G/N \cong \mathbb{Z}$ by Proposition 2.13. This shows that $N = T(G)$, as desired.

(ii)$\Rightarrow$(iii) This follows by Proposition 2.13 and the fact that $G/T(G)$ is a non-zero torsion-free restricted-finite group.

(iii)$\Rightarrow$(iv) By (iii), there exists a normal subgroup T of G such that $G/T \cong \mathbb{Z}$. Since $\mathbb{Z}$ is a free group, by Exercise 6 on page 50 of [17], there exists a subgroup

$C \cong \mathbb{Z}$ such that $G = TC$, $T \cap C = 1$. Thus $|T| \leq |G : C|$ and so T must be finite by the restricted-finite condition. Now it is easy to verify that $T = T(G)$.

(iv)$\Rightarrow$(i) By Lemma 2.6. $\square$

An abelian group is said to be *reduced* if it has no non-trivial divisible subgroups.

Proposition 2.15. *Let G be an infinite abelian group. Then G is restricted-finite if and only if either $G \cong \mathbb{Z} \times K$ or $G \cong \mathbb{Z}_{p^\infty} \times K$ where K is a finite group and p is a prime number.*

Proof. The sufficiency follows from Lemma 2.6. Conversely, suppose that G is infinite abelian restricted-finite. If G is finitely generated, then we are done by Theorem 2.14. Thus suppose that G is not finitely generated, hence by Lemma 2.10, G is torsion. By 4.1.4 of [17], $G = D \times K$ where D is divisible and K is reduced.

Since D is infinite, by the restricted-finite condition on G, $D \cong \mathbb{Z}_{p^\infty}$, for some prime p and $G/D \cong K$ is finite. Thus $G \cong \mathbb{Z}_{p^\infty} \times K$. $\square$

Corollary 2.16. *Let G be an infinite group with finite G'. Then G is restricted-finite if and only if G/G' is isomorphic to $\mathbb{Z} \times K$ or $\mathbb{Z}_{p^\infty} \times K$, where K is a finite group and p is a prime number.*

Proof. This follows from Proposition 2.15 and Lemma 2.6(ii). $\square$

Theorem 2.17. *The following statements are equivalent for a group G.*

 (i) *G is not torsion, restricted-finite with infinite G'.*
 (ii) *There exists a finite normal subgroup F of G such that $G/F \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ is the infinite dihedral group.*

Proof. (i)$\Rightarrow$(ii) Since G is not torsion and restricted-finite, by Lemma 2.10, every subgroup of G is finitely generated. By Proposition 2.11, there exists a normal infinite cyclic subgroup N of G of finite index. Since $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and $|\mathrm{Aut}(N)| = 2$, we have $|G/C_G(N)| \leq 2$. If $G = C_G(N)$, then $N \leq Z(G)$ and so $G/Z(G)$ is finite and by 10.1.4 of [17], G' is finite, a contradiction. Hence $|G/C_G(N)| = 2$. Since N is abelian, $N \leq C_G(N)$, in fact $N \leq Z(C_G(N))$, for if $x \in N$ and $g \in C_G(N)$, then $xg = gx$. Now $C_G(N)$ is an infinite restricted-finite group with an infinite center and as before $C_G(N)'$ is finite. Hence by Proposition 2.14, $C_G(N) = F \rtimes C$ where $\mathbb{Z} \cong C \leq C_G(N)$ and F is the torsion subgroup of $C_G(N)$. Since F has characteristic in $C_G(N)$, $F \trianglelefteq G$. If $T(G) = F$, then G/F is torsion-free, and by Proposition 2.13, $G/F \cong \mathbb{Z}$. Hence by Proposition 2.14, G' is

finite, a contradiction. Therefore $F \subsetneq T(G)$ and there exists $g \in G \backslash C_G(N)$ of finite order and so gF is a non-identity element of G/F. Now $G = C_G(N)\langle g \rangle = CF\langle g \rangle$ and $g^2 \in C_G(N)$. Since $g^2 \in C_G(N) = CF$, $g^2 F \in CF/F \cong K \cong \mathbb{Z}$, and so $g^2 \in F$. Hence $G/F = (CF/F) \rtimes (\langle g \rangle F/F) \cong \mathbb{Z} \rtimes \mathbb{Z}_2$. Because G' is infinite, $(G/F)'$ is also infinite. It follows that G/F is the infinite dihedral group.

(ii)$\Rightarrow$(i) Clearly, G is not torsion. Also G is restricted-finite by Lemma 2.6. Furthermore, since G/F is infinite dihedral, $G'F/F = (G/F)'$ is infinite and therefore G' is infinite. $\qquad \square$

Example 2.18. Let $G = \langle a, x \mid x^2 = 1, \ a^x = a^{-1} \rangle = \langle a \rangle \rtimes \langle x \rangle$ be the infinite dihedral group. Then $G/\langle a \rangle$ is finite of order 2 and by Lemma 2.6, G is restricted-finite. Note that $G' = \langle a^2 \rangle$ is infinite.

Theorem 2.19. *The following statements are equivalent for a group G.*

(i) *G is an infinitely generated restricted-finite group.*

(ii) *G has a normal subgroup A of finite index such that $A \cong \mathbb{Z}_{p^\infty}$, for some prime number p.*

(iii) *$G = AT$, where A and T are subgroups of G such that A is normal quasi-cyclic and T is finite. In this case, $A = \mathrm{rad}_\infty(G)$.*

(iv) *$G = AT$, where A and T are subgroups of G such that A is infinitely generated proper-finite and T is finite.*

Proof. (i)$\Rightarrow$(ii) Suppose that G is infinitely generated restricted-finite. By Lemma 2.10(b), G is locally finite. Hence by 14.3.7 of [17], G contains an infinite abelian subgroup of B_1. Then by Proposition 2.15, there exists a prime p such that $B_1 \cong \mathbb{Z}_{p^\infty} \times K$ where K is finite. Let A be a subgroup of G such that $A \cong \mathbb{Z}_{p^\infty}$. By Proposition 2.1(d), A contains an infinite normal subgroup of G. It follows that A is normal in G and has finite index in G.

(ii)$\Rightarrow$(iii) It is easy to verify that $A = \mathrm{rad}_\infty(G)$. In fact, $\mathrm{rad}_\infty(G) \subseteq A$ and for every infinite subgroup B, we have $A \cap B$ is an infinite subgroup of A by Proposition 2.1(c). Hence $A \cap B = A$. Thus $A \subseteq B$ for every infinite subgroup B of G. Therefore, $A \subseteq \mathrm{rad}_\infty(G)$. Now by our assumption there are $g_1, \ldots, g_n$ in G, such that $G = AT$ with $T = \langle g_1, \ldots, g_n \rangle$. If T is infinite, then $A \subseteq T$ and so $G = T$, a contradiction. Thus T is finite.

(iii)$\Rightarrow$(iv) This is clear.

(iv)$\Rightarrow$(i) By Lemmas 2.6(i) and 2.10(a). $\qquad \square$

Corollary 2.20. *Every infinitely generated restricted-finite group satisfies min.*

Proof. By Theorem 2.19 and Proposition 2.7. $\qquad \square$

3. Some applications

Let R be a ring and M be an R-module. An R-submodule N of M is called *essential* if $N \cap K \neq 0$ for any non-zero R-submodule K of M. The R-module M is said to have the "restricted minimum condition" (RMC for short) if the R-module M/N is Artinian for every essential R-submodule N. The ring R is said to have the right RMC (r.RMC) if R has RMC as a right R-module. In Theorem 3.9 of [5], for a not torsion abelian group G, it is determined when the ring RG has r.RMC. As an application of our results, we show that the same result holds for every not torsion group G with finite G'. We first investigate when a not torsion restricted-finite group is abelian. If $G = HK$ and $H \cap K = 1$, then we say that H is a complement to K in G.

Proposition 3.1. *Let G be a not torsion restricted-finite with $T = T(G)$. Then G is abelian if and only if every infinite cyclic complement to T in G lies in a normal torsion-free subgroup.*

Proof. The necessity is clear. For the sufficiency, note that the hypotheses on G pass to G/F for every finite normal subgroup F. Thus by Theorem 2.17, G' can not be infinite. Hence, by Theorem 2.14, $G = T \rtimes C$, where $C \cong \mathbb{Z}$ and T is finite. By hypothesis, there exists a normal torsion-free subgroup K containing C. Since K is restricted-finite, by Proposition 2.13, K is infinite cyclic. Therefore we must have $G = KT$, where $K = \langle x \rangle \cong \mathbb{Z}$. Thus $K \leq Z(G)$ and $T \leq C_G(x)$. If $a \in T$, then $o(xa)$ is infinite and so there exists an infinite normal cyclic subgroup N such that $xa \in N$. Then $N \cap T = 1$ and so $T \leq C_G(xa)$. Therefore $T \leq C_G(a)$. It follows $T \leq Z(G)$. Hence G is abelian. $\qquad\square$

Proposition 3.2. *Let G be a group. Then $G \cong \mathbb{Z}_{p^\infty}$ for some prime number p if and only if G is infinitely generated restricted-finite which has no maximal subgroup.*

Proof. One direction is well known and the other direction follows from Theorem 2.19. $\qquad\square$

If G_1 and G_2 are two groups and R is a ring, then it is well known that $R(G_1 \times G_2) \cong (RG_1)(G_2)$ as two rings; see for example [18, Theorem 1.4]. In the sequel, we use some results of [6] that we state below for the reader's convenience.

Theorem 3.3. [6, Theorem 3.5] *Let G be a group and M be a non-zero R-module. Suppose that MG has RMC as an RG-module, then we have:*

    (1) *M_R has RMC.*

(2) *Either G is finite or M_R is semisimple.*

(3) *Every normal subgroup H of G is a normally restricted-finite group.*

It is known that finitely generated nilpotent torsion groups are finite, but there are torsion-free cases (such as the unitriangular group $U(n, \mathbb{Z})$); see 5.2.18 and the following notes, of [17]. The next result shows that if G is a finitely generated nilpotent torsion-free group, then the group-ring RG has no r.RMC unless $G \cong \mathbb{Z}$.

Corollary 3.4. *Let R be a ring and G be a finitely generated torsion-free nilpotent group. Then RG has r.RMC if and only if R is semisimple and $G \cong \mathbb{Z}$.*

Proof. The sufficiency follows from Theorem 3.8 of [6]. Suppose that RG has r.RMC. Since G is torsion-free nilpotent, it contains a normal infinite cyclic central subgroup C. Now by Theorem 3.3, G/C is finite. Thus, by Lemma 2.6, G is restricted-finite. Since, by 12.1.5 of [17], maximal subgroups of G are normal, G satisfies the hypotheses in Proposition 3.1. This implies that G is abelian. Therefore, by Proposition 2.13, $G \cong \mathbb{Z}$.   □

Lemma 3.5. *Let G be a torsion-free abelian group such that $H \cap N$ is non-trivial for all non-trivial subgroups H and N of G. Then G is isomorphic to a subgroup of $\mathbb{Q}$.*

Proof. This is well known. Because by hypothesis, the $\mathbb{Z}$-module G is an essential extension of $\mathbb{Z}$ and so it must be embedded in $\mathbb{Q}$, the injective hull of $\mathbb{Z}$.   □

Lemma 3.6. *Let G be a group such that G' is finite and every torsion-free subgroup of G is cyclic. Then $G/T(G)$ is isomorphic to a subgroup of $\mathbb{Q}$.*

Proof. If G is torsion, there is nothing to prove. Suppose that G is not torsion and let $T := T(G)$. Since G' is finite, $G/Z(G)$ is torsion by 14.5.6 of [17]. Thus $Z(G)$ contains an infinite cyclic subgroup $\langle a \rangle$. We apply Lemma 3.5 for the torsion-free abelian group G/T. Consider the infinite cyclic subgroup $\langle a \rangle T/T$ in G/T. Let $\bar{g} = gT$, where $g \in G$. We show that for every $b \in G \setminus T$, $\langle \bar{a} \rangle \cap \langle \bar{b} \rangle$ is non-trivial. Indeed, if $\langle \bar{a} \rangle \cap \langle \bar{b} \rangle = \bar{1}$, then $\langle a \rangle \cap \langle b \rangle \subseteq T \cap \langle a \rangle = 1$. Now by our assumption, the torsion-free subgroup $\langle a \rangle \langle b \rangle$ must be cyclic, a contradiction. Therefore the result follows from Lemma 3.5.   □

Lemma 3.7. *Let G be a group and H be a subgroup of the finite index of G. If RH has r.RMC, then $(RG)_{(RH)}$ has RMC.*

Proof. Let $|G : H| = n$ and $G = \bigcup_{i=1}^{n} a_i H$. It follows that $RG = \sum_{i=1}^{n} a_i(RH)$ is a finitely generated RH-module. This follows that RG is a homomorphic image

of a finite direct product of RH, as a right RH-module. Thus RG has RMC as an RH-module, by Proposition 2.1 of [5]. □

Theorem 3.8. *Let R be a ring, G be a not torsion group with finite G' and $T = T(G)$. Consider the following statements.*

(i) *The ring RG has r.RMC.*

(ii) *R is a semisimple ring and $G \cong T \rtimes \mathbb{Z}$, where T is finite whose order is unit in R.*

(iii) *G is finitely generated and for every torsion-free subgroup N of G, the ring $(RT)N$ has r.RMC.*

Then (i)⇒(ii), (ii)⇔(iii). In case (iii), (RG) has RMC as a right S-module where $S = (RT)N$ and N is a torsion-free subgroup of G.

Proof. (i)⇒(ii) Let $\Gamma = G/G'$. First note that the ring $R\Gamma$ has r.RMC by Proposition 3.2 of [6] and so Γ is a restricted-finite group by Theorem 3.3. Thus G is a restricted-finite group by Lemma 2.6(ii). Since G' is finite, $G' \subseteq T$ and G is c-torsion. Also, T must be finite because G is restricted-finite and G/T is a non-zero torsion-free group. Thus $G \cong T \rtimes \mathbb{Z}$ by Theorem 2.14. Now suppose that N is an infinite cyclic in $Z(G)$ (that exists because G' is finite and hence $G/Z(G)$ is finite). Let $H = TN$. Then the ring RH has r.RMC by Proposition 3.2 of [6]. Since $RH \cong (RT)N$, the ring RT must be semisimple by Theorem 3.3. This follows that R is semisimple and $|T|$ is a unit in R by the famous result of Maschke.

(ii)⇒(iii) Clearly, G is finitely generated and restricted-finite by Theorem 2.14. Suppose that N is a torsion-free subgroup of G and let $S = (RT)N$. Since R is assumed to be semisimple and $|T|$ is unit in R, the ring RT is semisimple. Also, since G is restricted-finite, N is a restricted-finite group by Proposition 2.1. Thus $N \cong \mathbb{Z}$ by Proposition 2.13. These follow S has r.RMC by Theorem 3.8 of [6].

(iii)⇒(ii) We use Lemma 3.6. Suppose that N is any torsion-free subgroup of G. Let $S = (RT)N$, by hypothesis S has r.RMS. As we see in the proof of (i)⇒(ii), RT is a semisimple ring and $N \cong \mathbb{Z}$. Thus R is a semisimple ring and $|T|$ is a unit in R. Also, by Lemma 3.6, G/T is isomorphic to a subgroup of $\mathbb{Q}$. Now, since G is assumed to be finitely generated, we must have $G/T \cong \mathbb{Z}$, proving that $G \cong T \rtimes \mathbb{Z}$. For the last statement, suppose that (iii) holds, N is a torsion-free subgroup of G and $(RT)N = S$. We already know that $N \cong \mathbb{Z}$. Let N_1 be an infinite cyclic subgroup in $Z(G)$ (such a subgroup exists because G' is finite), and let $H = TN_1$. Because N_1 is a normal subgroup of G, we have $H \cong T \times N_1$. Hence, $RH \cong (RT)N_1 \cong (RT)N$. Thus the ring RH has r.RMC by (iii). On the other

hand, $|G : H|$ is finite. Hence, (RG) has RMC as a right S-module by Lemma 3.7. The proof is now complete. $\square$

Corollary 3.9. [6, Theorem 3.9] Let R be a ring and G be a not torsion abelian group. Then the following statements are equivalent.

    (a) RG has r.RMC.

    (b) R is a semisimple ring and $G \cong H \times \mathbb{Z}$, where H is a finite group whose order is invertible in R.

Proof. By Theorem 3.8. $\square$

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Declarations

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