International Electronic Journal of Algebra
Volume 36 (2024) 66-72
DOI: 10.24330/ieja.1438742

A CATEGORIES EQUIVALENCE OF ASSOCIATIVE BIMODULES

Victor López Solís and Marlennhi Moreno Villanueva

Received: 24 May 2023; Revised: 6 November 2023; Accepted: 26 December 2023
Communicated by Tuğçe Pekacar Çalcı

Abstract. In this paper we use the classical Wedderburn's Kronecker Factorization Theorem to prove that category of bimodules over B and the category of bimodules over $M_n(B)$ are equivalent, where B is some unital associative algebra. In addition to this, we classify the irreducible bimodules over $M_n(F)$.

Mathematics Subject Classification (2020): 16D20, 16D70, 16D90
Keywords: Kronecker factorization theorem, bimodule, irreducible bimodule

1. Introduction

An *associative algebra* is an F-vector space A with a bilinear binary operation $(x, y) \mapsto xy$ satisfying the following identity:

$$(x, y, z) = 0,$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator of the elements $x, y, z \in A$.

The description of the structure of algebras and superalgebras that contain certain finite-dimensional algebras and superalgebras has a rich history, which has important applications in representation theory and category theory (for example, see [2,3,6,7,8,9,10,11,12,14]). The classical Wedderburn Theorem says that if a unital associative algebra A contains a central simple subalgebra of finite dimension B with the same identity element, then A is isomorphic to a Kronecker product $S \otimes_F B$, where S is the subalgebra of the elements that commute with each $b \in B$. In particular, if A contains $M_n(F)$ as a subalgebra with the same identity element, we have $A \cong M_n(S)$ "coordinated" by S. Kaplansky in Theorem 2 of [5] generalized the Wedderburn result to the alternative algebras A and the split Cayley algebra

The paper is a part of the "Tesis de Licenciatura" at the National University of Santiago Antúnez de Mayolo of the second author. The first author was supported by Resolution of the Organizing Commission N° 629-2022-UNAB of the National University of Barranca. Also the authors gratefully acknowledge financial support by CONCYTEC-PROCIENCIA within the framework of the call "Proyecto Investigación Básica 2019-01" [380-2019-FONDECYT].

B. Jacobson in Theorem 1 of [2] gave a new proof of the result of Kaplansky using his classification of completely reducible alternative bimodules over a field of characteristic different of 2 and finally V. López-Solís in [8] proved that this result is valid for any characteristic. Using this result, Jacobson [2] proved a Kronecker Factorization Theorem for Jordan algebras that contain the Albert algebra with the same identity element. The statements of this type are usually called *Kronecker factorization theorems*.

In [13], K. McCrimmon says that Wedderburn's Kronecker Factorization Theorem (KFT) is the grandfather of all Kronecker factorization theorems. Despite its great importance, we certainly have not found bibliographical references of this result in ring representation theory or algebras. Motivated by this lack, we thought it would be useful to describe some applications of the KFT and thus see its utility.

In this note we use the KFT to prove that the category of bimodules over B and the category of bimodules over $M_n(B)$ are equivalent, where B is some unital associative algebra. In addition to this, we classify the irreducible bimodules over $M_n(F)$.

2. Preliminaries

Let A be an associative algebra over F. A vector space V over F is called an A-bimodule if there are bilinear mappings $A \times V \to V$ and $V \times A \to V$ sending (a, v) to av and (v, a) to va, respectively. We say that V is an *associative bimodule* for A if the algebra $E = A \oplus V$ with the multiplication given by

$$(a + v) \cdot (b + w) = ab + (vb + aw),$$

for all a, b in A and v, w in V, is associative. The algebra $E = A \oplus V$ is called the *split null extension* of A by bimodule V where A is a subalgebra and V is an ideal of E such that $V^2 = 0$. Specifically, we have $E = A \oplus V$ is associative if and only if

$$(a, b, v) = (a, v, b) = (v, a, b) = 0,$$

for all $a, b \in A$ and $v \in V$, where $(x, y, z) := (xy)z - x(yz)$ is the associator of x, y and z. Therefore this definition of associative bimodule coincides with the usual one.

Suppose A has an identity element 1, then the associative bimodule V is called a *unital associative bimodule* for A if $1v = v1 = v$ for all $v \in V$. For the definition of *unital right modules*, see [4].

Let us recall some elementary facts about matrix units in $M_n(R)$, where R is a ring with identity. For $i, j = 1, \dots, n$, we define e_{ij} as a matrix whose entry (i, j)

is 1 and the other entries are 0. Also, we have the multiplication table

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \text{and} \quad \sum e_{ii} = 1.$$

Hereinafter right modules will mean unital right modules. Similarly, bimodules will mean unital associative bimodules.

Denote by $\mathbf{mod}-R$ the category of right modules for a fixed ring R. Ob($\mathbf{mod}-R$) is the class of right modules for R and the morphisms are R-module homomorphisms. Products are composites of maps. Similarly, $\mathbf{mod} - M_n(R)$ denote the category of right modules over $M_n(R)$.

Proposition 2.1. *Let R be a ring and $M_n(R)$ be the ring of matrices of order $n \times n$ with entries in R. Then the categories $\mathbf{mod} - R$ and $\mathbf{mod} - M_n(R)$ of modules to the right over R and $M_n(R)$ respectively, are equivalent.*

Proposition 2.1 is proved in the Jacobson's book (see Proposition 1.4 in [4]). Our aim is to prove this result for bimodules using the famous KFT.

Above we have defined the e_{ij} as the matrices with 1 in the input (i, j) and 0 in the others. We call the set of these elements a **system of unitary matrices** that satisfy

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad \sum e_{ii} = 1$$

where δ is the Kronecker delta.

As mentioned in the Introduction, the KFT points out a very interesting feature of central simple algebras, namely, whenever they sit in a larger algebra they do so in a very particular way. In fact the property of the theorem characterizes finite dimensional central simple algebras. See the Herstein's book (see Theorem 4.4.2 in [1]).

Theorem 2.2. *Let A be a unital associative algebra that contains a central simple subalgebra of finite dimension S with the same identity element, then A is isomorphic to a Kronecker product $S \otimes_F B$, where B is the subalgebra of the elements that commute with each $b \in S$.*

Define $[a, b] := ab - ba$ the *commutator* of the elements $a, b \in A$. In Theorem 2.2 we have

$$B = \{a \in A : [a, S] = 0\},$$

that is, $B = C_A(S)$ is the *centralizer* of S in A. In particular:

Corollary 2.3. *Let A be an algebra with identity element 1 such that A contains a system of n^2 unitary matrix elements. Then $A \cong M_n(B)$, where B is the subalgebra of the elements that commute with each e_{ij} of the system.*

Denote by F a field of arbitrary characteristic. Corollary 2.3 says that if A contains $M_n(F)$ as a subalgebra with the same identity element, then $A \cong M_n(B)$, that is, A is "coordinated" by B and acquires the matrix structure of $M_n(F)$.

3. A categories equivalence

Next, we state and prove the most important results of the article. The first result is an equivalence of categories. Indeed, it is an analogue of Proposition 2.1 for bimodules and it is given using the KFT. The second application of KFT is related to the classification of irreducible bimodules.

Let B be an arbitrary unital associative algebra over the base field F. Denote by

- $\mathbf{Bimod}-B$ the category of bimodules for B, where $\mathrm{Ob}(\mathbf{Bimod}-B)$ is the class of bimodules over B.
- $\mathbf{Bimod}-M_n(B)$ the category of bimodules for $M_n(B)$, where $\mathrm{Ob}(\mathbf{Bimod}-M_n(B))$ is the class of bimodules over $M_n(B)$.

Theorem 3.1. *The categories* $\mathbf{Bimod}-B$ *and* $\mathbf{Bimod}-M_n(B)$ *are equivalent.*

Proof. We want to prove that

$$\mathbf{Bimod}-B \cong \mathbf{Bimod}-M_n(B).$$

Let $N \in \mathrm{Ob}(\mathbf{Bimod}-B)$ and consider the split null extension $E = B \oplus N$ of the associative algebra B by the bimodule N. Since E is an associative algebra we can form the matrix algebra $K = M_n(E)$ containing $M_n(B)$ as a subalgebra. Thus, K contains the ideal $M = M_n(N) \cap K = M_n(N)$ which is the set of matrices of K whose entries are in the ideal N of E. Consequently, M is a bimodule for $M_n(B)$ relative to the multiplication defined in $M_n(E)$. Therefore, $M \in \mathrm{Ob}(\mathbf{Bimod}-M_n(B))$ and will be the $M_n(B)$-bimodule **associated** with the given bimodule N of B. Thus, we have a map $N \mapsto M = M_n(N)$ of $\mathrm{Ob}(\mathbf{Bimod}-B)$ to $\mathrm{Ob}(\mathbf{Bimod}-M_n(B))$. If $f : N \longrightarrow N'$ is a B-bimodule homomorphism, then the map

$$\widetilde{f} : \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \vdots & \ddots & \vdots \\ f(a_{n1}) & \dots & f(a_{nn}) \end{bmatrix}$$

is a homomorphism of $M_n(B)$-bimodules of M to M'. The maps $N \mapsto M = M_n(N)$ and $f \mapsto \widetilde{f}$ constitute a functor

$$\mathbf{T} : \mathbf{Bimod}{-}B \longrightarrow \mathbf{Bimod}{-}M_n(B).$$

If $N \in \mathrm{Ob}(\mathbf{Bimod}{-}B)$, denote

$$M = \mathbf{T}(N).$$

Since $E = B \oplus N$, we have that $K = M_n(B) \oplus M$. The fact that $N^2 = 0$ in E implies that $M^2 = 0$ in K, so K is the split null extension of $M_n(B)$ by its bimodule M.

It can be easily verified that $\mathbf{T}$ is actually a functor of the category $\mathbf{Bimod}{-}B$ to the category $\mathbf{Bimod}{-}M_n(B)$. Furthermore, for every pair of objects N and N' of $\mathbf{Bimod}{-}B$, the following equality holds

$$\mathbf{T}(hom(N, N')) = hom(\mathbf{T}(N), \mathbf{T}(N')).$$

Thus, N and N' are isomorphic if and only if $\mathbf{T}(N)$ and $\mathbf{T}(N')$ are isomorphic.

Similarly, the functor $\mathbf{T}$ offers a lattice isomorphism of the lattice of the submodules of N relative to B on the lattice of the submodules of M over $M_n(B)$.

To complete the reduction of the theory of bimodules for $M_n(B)$ to that of bimodules for B, we will show that each $M_n(B)$-bimodule is isomorphic to some bimodule associated with a bimodule for B. Consider a bimodule V for $M_n(B)$ and let

$$A = M_n(B) \oplus V$$

be the split null extension of $M_n(B)$ by V. Thus A is an associative algebra (with identity element 1, the identity of $M_n(B)$) containing the matrix algebra $M_n(F) \subseteq M_n(B)$ as a unital subalgebra, then by Corollary 2.3 of KFT, there exists a unital associative algebra D such that $A = M_n(D)$, then

$$M_n(D) = M_n(B) \oplus V.$$

Let W be the set of elements of D that appear in the entries of the matrices of V. Then

$$V := M_n(W),$$

where $W \lhd D$ and $W^2 = 0$ in D, since $V \lhd A$ and $V^2 = 0$ in A; so $D = B \oplus W$ is the split null extension of B by its bimodule W, then W is a bimodule over B. Thus $\mathbf{T}(W) = V$. □

Corollary 3.2. *Every bimodule V over B is completely reducible if and only if $T(V)$ is completely reducible over $M_n(B)$.*

We finalize the paper by classifying the irreducible bimodules which was proved in [15], but in the proof there is a mistake, because the authors cite the Corollary 2.3 of KFT as if it were KFT itself. Therefore, here we offer the complete proof of that results and the correction of the mentioned mistake.

Corollary 3.3. *All irreducible bimodule for $M_n(F)$ is isomorphic to the regular bimodule* $\mathrm{Reg}(M_n(F))$.

Proof. Consider an irreducible bimodule V for $M_n(F)$. Then $E = M_n(F) \oplus V$ is a unital associative algebra containing $M_n(F)$, with the same identity element. Thus by Corollary 2.3 of KFT, there exists a subalgebra B of E such that $E = M_n(B)$. From $E = M_n(B)$, consider the set D of the elements of B that appear in the entries of the matrices of V. Consequently $V = M_n(D)$, where $D \lhd B$ and $D^2 = 0$, since $V \lhd E$ and $V^2 = 0$ in E. Thus $B = F1 \oplus D$ is the split null extension of $F1$ by D, that is, D is an irreducible F-bimodule, because V it is. Therefore $D = F \cdot 1$, which implies that $V = \mathrm{Reg}(M_n(F))$. $\square$

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.
Declarations conflict of interest. There are no competing interests.

References

[1] I. N. Herstein, Noncommutative Rings, Carus Math. Monogr., 15, Mathematical Association of America, Washington, DC, 1994.

[2] N. Jacobson, *A Kronecker factorization theorem for Cayley algebras and the exceptional simple Jordan algebra,* Amer. J. Math., 76 (1954), 447-452.

[3] N. Jacobson, Structure and Representations of Jordan Algebras, Amer. Math. Soc. Colloq. Publ., American Mathematical Society, Providence, RI, 1968.

[4] N. Jacobson, Basic Algebra II, W. H. Freeman and Co., San Francisco, CA, 1980.

[5] I. Kaplansky, *Semi-simple alternative rings,* Portugal. Math., 10 (1951), 37-50.

[6] M. C. López-Díaz and I. P. Shestakov, *Representations of exceptional simple alternative superalgebras of characteristic 3*, Trans. Amer. Math. Soc., 354(7) (2002), 2745-2758.

[7] M. C. López-Díaz and I. P. Shestakov, *Representations of exceptional simple Jordan superalgebras of characteristic 3*, Comm. Algebra, 33(1) (2005), 331-337.

[8] V. H. López Solís, *Kronecker factorization theorems for alternative superalgebras,* J. Algebra, 528 (2019), 311-338.

[9] V. H. López Solís, *On a problem by Nathan Jacobson for Malcev algebras,* arXiv:2106.01155.

[10] V. H. López Solís, *Kronecker factorization theorems for the exceptional Malcev algebra,* Preprint.

[11] V. H. López Solís and I. P. Shestakov, *On a problem by Nathan Jacobson*, Rev. Mat. Iberoam., 38(4) (2022), 1219-1238.

[12] C. Martínez and E. Zelmanov, *A Kronecker factorization theorem for the exceptional Jordan superalgebra,* J. Pure Appl. Algebra, 177(1) (2003), 71-78.

[13] K. McCrimmon, A Taste of Jordan Algebras, Universitext, Springer-Verlag, New York, 2004.

[14] S. V. Pchelintsev, O. V. Shashkov and I. P. Shestakov, *Right alternative bimodules over Cayley algebra and coordinatization theorem,* J. Algebra, 572 (2021), 111-128.

[15] M. A. Yglesias Jauregui, V. H. López Solís, B. M. Cerna Maguiña and V. A. Pocoy Yauri, *Bimódulos asociativos unitarios irreducibles sobre la álgebra de las matrices $n \times n$,* https://repositorio.unasam.edu.pe/handle/UNASAM/4736, (2018).

Victor López Solís (Corresponding Author)
Dep. Académico de Ciencias Básicas y Afines
Universidad Nacional de Barranca
Barranca, Peru
e-mail: vlopez@unab.edu.pe

Marlennhi Moreno Villanueva
Huaraz, Peru
e-mail: marlennhi95@hotmail.com