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$\delta(0)$-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with nonzero identity, let $\mathcal{I}(R)$ be the set of all ideals of R and $\delta : \mathcal{I}(R) \to \mathcal{I}(R)$ be a function. Then δ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. In this paper, we present the concept of $\delta(0)$-ideals in commutative rings. A proper ideal I of R is called a $\delta(0)$-ideal if whenever $a,\ b \in R$ with $ab \in I$ and $a \notin \delta(0)$, we have $b \in I$. Our purpose is to extend the concept of n-ideals to $\delta(0)$-ideals of commutative rings. Then we investigate the basic properties of $\delta(0)$-ideals and also, we give many examples about $\delta(0)$-ideals.

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1. Introduction

Throughout this study, all rings are assumed to be commutative with nonzero identity. Let R be a ring. If I is an ideal of R with $I \neq R$, then I is called a proper ideal. Let I be an ideal of R. We denote the set of regular elements of R by $\mathrm{Reg}(R)$ and the Jacobson ideal of R by $J(R)$. Also, we denote the radical of I by $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$. In particular, we mean by $\sqrt{0}$ the set of all nilpotents in R; i.e., $\{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$. Let S be a nonempty subset of R. Then the ideal $\{a \in R : aS \subseteq I\}$, which contains I, will be designated by $(I : S)$.

Zhao in [11] introduced the concept of expansions of ideals: a function δ from $\mathcal{I}(R)$ to $\mathcal{I}(R)$ is an ideal expansion if it has the following properties: $I \subseteq \delta(I)$ and if $I \subseteq J$ for some ideals $I,\ J$ of R, then $\delta(I) \subseteq \delta(J)$. For example, δ_0 is the identity function, where $\delta_0(I) = I$ for all ideals I of R, and δ_1 is defined by $\delta_1(I) = \sqrt{I}$. For other examples, consider the functions δ_+ and δ_* of $\mathcal{I}(R)$ defined with $\delta_+(I) = I + J$, where $J \in \mathcal{I}(R)$ and $\delta_*(I) = (I : P)$, where $P \in \mathcal{I}(R)$, for all $I \in \mathcal{I}(R)$, respectively (see [4]).

The notion of the prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied. See, for example, [1,6]. Recall from [3] that a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In [9], Mohamadian defined a proper ideal I of R as an r-ideal if whenever $a, b \in R$ with $ab \in I$ and $Ann(a) = 0$, we have $b \in I$. In a recent study, the authors of [10] introduce the notion of n-ideals. A proper ideal I of R is called an n-ideal of R if whenever a, $b \in R$ such that $ab \in I$ with $a \notin \sqrt{0}$, we have $b \in I$. Motivated by this concept, we give the notion of $\delta(0)$-ideals and we investigate many properties of $\delta(0)$-ideals similar to those of r-ideals and n-ideals. A proper ideal I of R is said to be a $\delta(0)$-ideal if the condition $ab \in I$ with $a \notin \delta(0)$ implies $b \in I$ for every $a, b \in R$. It is clear that if $\delta(I) = \delta_1(I) = \sqrt{I}$, then I is an n-ideal if and only if I is a $\delta(0)$-ideal. Among many results in this paper, it is shown (Corollary 2.5) that if a prime ideal $I \subseteq \delta(0)$, then I is a $\delta(0)$-ideal. In Theorem 2.12 we show that a proper ideal I of R is a $\delta(0)$-ideal of R if and only if $I = (I : a)$ for every $a \notin \delta(0)$. In Proposition 2.16 for a δ-reduced von Neumann ring R, we show that 0 is a $\delta(0)$-ideal if and only if R is a field.

2. $\delta(0)$-Ideals of commutative rings

In this part, we will define $\delta(0)$-ideal of commutative rings and we will give some fundamental theorems and examples about them.

Definition 2.1. A proper ideal I of R is called a $\delta(0)$-*ideal* if whenever a, $b \in R$ with $ab \in I$ and $a \notin \delta(0)$, we have $b \in I$.

Remark 2.2. Let R be a ring, and δ, γ two expansions of ideals.

(1) If I is a $\delta(0)$-ideal of R with $\delta(0) \subseteq \gamma(0)$, then I is a $\gamma(0)$-ideal. In particular if $\sqrt{0} \subseteq \delta(0)$, then every n-ideal is a $\delta(0)$-ideal.
(2) If $\sqrt{0} = \delta(0)$, then an ideal I is a $\delta(0)$-ideal if and only if I is an n-ideal.

Proposition 2.3. *If I is a $\delta(0)$-ideal of R, then $I \subseteq \delta(0)$.*

Proof. Assume that I is a $\delta(0)$-ideal but $I \nsubseteq \delta(0)$. Then there exists $a \in I$ such that $a \notin \delta(0)$. Since $a.1 = a \in I$ and I is a $\delta(0)$-ideal, we conclude that $1 \in I$, so that $I = R$, a contradiction. Hence $I \subseteq \delta(0)$. □

Proposition 2.4. *Let R be a ring, I be an ideal of R and δ be an expansion function. If I is a primary ideal and $\sqrt{I} \subseteq \delta(0)$, then I is a $\delta(0)$-ideal.*

Proof. Let I be a primary ideal and $ab \in I$ with $a \notin \delta(0)$. Then $a \notin \sqrt{I}$. Since I is primary, we have $b \in I$, as desired. □

Corollary 2.5. *Let I be a prime ideal. If $I \subseteq \delta(0)$, then I is a $\delta(0)$-ideal.*

Next, we give an example of a $\delta(0)$-ideal that is not an n-ideal.

Example 2.6. Assume that $R = \mathbb{Z}_6$ and $\delta_*(J) = (J : 3\mathbb{Z}_6)$. Let $I = 2\mathbb{Z}_6$. Then, I is a $\delta_*(0)$-ideal, since I is prime and $\delta_*(0) = I$. But $I \nsubseteq \sqrt{0}$, which implies it is not an n-ideal.

Proposition 2.7. *Let $\{I_i\}_{i\in\Delta}$ be a nonempty set of $\delta(0)$-ideals of R. Then $\bigcap_{i\in\Delta} I_i$ is a $\delta(0)$-ideal of R.*

Proof. Let $ab \in \bigcap_{i\in\Delta} I_i$ with $a \notin \delta(0)$ for $a, b \in R$. Then $ab \in I_i$ for every $i \in \Delta$. Since I_i is a $\delta(0)$-ideal of R, we get the result that $b \in I_i$ and so $b \in \bigcap_{i\in\Delta} I_i$. □

Recall from [11] that a proper ideal Q of R is a δ-primary ideal if whenever $a, b \in R$ with $ab \in Q$, we have $a \in Q$ or $b \in \delta(Q)$. In the following proposition, we show that every $\delta(0)$-ideal is also a δ-primary ideal.

Proposition 2.8. *Let R be a ring.*

(1) *If I is a $\delta(0)$-ideal of R, then it is a δ-primary ideal.*
(2) *Assume that $\delta^2(0) \subseteq \delta(0)$. Then I is a $\delta(0)$-ideal of R if and only if it is a δ-primary ideal and $I \subseteq \delta(0)$.*

Proof. (1) Suppose that I is a $\delta(0)$-ideal of R and $ab = ba \in I$ with $b \notin \delta(I)$ for $a, b \in R$. Then $b \notin \delta(0)$. Since I is a $\delta(0)$-ideal, we conclude that $a \in I$. Consequently, I is a δ-primary ideal of R.
(2) Suppose that I is a δ-primary ideal and $I \subseteq \delta(0)$. Let $ab = ba \in I$ with $b \notin I$. This implies $a \in \delta(I) \subseteq \delta^2(0)$. Since $\delta^2(0) \subseteq \delta(0)$, we have $a \in \delta(0)$. This implies that I is a $\delta(0)$-ideal of R. Conversely, suppose that I is a $\delta(0)$-ideal. By (1) and Proposition 2.3, we have I is a δ-primary ideal and $I \subseteq \delta(0)$. □

In the next example, we show that the condition $\delta^2(0) \subseteq \delta(0)$ in (2) of Proposition 2.8 is crucial.

Example 2.9. Let $R = \mathbb{Z}_6$ and $\delta_*(J) = (J : 3\mathbb{Z}_6)$. Let $I = 3\mathbb{Z}_6$. It's clear that $\delta_*(I) = R$. Thus, for all $ab \in I$, we have $b \in \delta_*(I)$ which implies that I is a δ_*-primary ideal. Since $I \nsubseteq \delta_*(0)$, we find that I is not a $\delta_*(0)$-ideal by Proposition 2.3.

Proposition 2.10. *Assume that 0 is a $\delta(0)$-ideal such that $\sqrt{\delta(0)} = \delta(0)$. Then $\sqrt{0}$ is a $\delta(0)$-ideal.*

Proof. Suppose $ab \in \sqrt{0}$ and $a \notin \sqrt{0}$. Then there exists $n \in \mathbb{Z}^+$ such that $(ab)^n = 0$. Since $a^n b^n = 0$, $a^n \neq 0$ and 0 is a $\delta(0)$-ideal, we get $b^n \in \delta(0)$. Then $b \in \sqrt{\delta(0)} = \delta(0)$. Hence we conclude that $\sqrt{0}$ is a $\delta(0)$-ideal. □

Recall that a proper ideal I of R is called a J-ideal if whenever $ab \in I$ for $a, b \in R$, we have either $a \in J(R)$ or $b \in I$.

Theorem 2.11. *Let δ be an expansion function of ideals of R and $\delta(0)$ be a maximal ideal. Then every J-ideal is a $\delta(0)$-ideal.*

Proof. Let I be a J-ideal. Suppose $ab \in I$ and $a \notin I$. Then $b \in J(R) \subseteq \delta(0)$. Thus I is a $\delta(0)$-ideal, as desired. □

Theorem 2.12. *Let R be a ring and I a proper ideal of R. Then the following are equivalent:*

(1) *I is a $\delta(0)$-ideal of R;*
(2) *$I = (I : a)$ for every $a \notin \delta(0)$;*
(3) *For ideals J and K of R, $JK \subseteq I$ with $J \cap (R - \delta(0)) \neq \emptyset$ implies $K \subseteq I$.*

Proof. (1) ⇒ (2) Assume that I is a $\delta(0)$-ideal of R. For every $a \in R$, the inclusion $I \subseteq (I : a)$ always holds. Let $a \notin \delta(0)$ and $b \in (I : a)$. Then we have $ab \in I$. Since I is a $\delta(0)$-ideal, we conclude that $b \in I$ and thus $I = (I : a)$.
(2) ⇒ (3) Suppose that $JK \subseteq I$ with $J \cap (R - \delta(0)) \neq \emptyset$, for ideals J and K of R. Since $J \cap (R - \delta(0)) \neq \emptyset$, there exists an $a \in J$ such that $a \notin \delta(0)$. Then we have $aK \subseteq I$, and so $K \subseteq (I : a) = I$ by (2).
(3) ⇒ (1) Let $ab \in I$ with $a \notin \delta(0)$ for $a, b \in R$. It is sufficient to take $J = aR$ and $K = bR$ to prove the result. □

Proposition 2.13. *Let I be a prime ideal of R such that $\delta(I) = I$. Then I is a $\delta(0)$-ideal of R if and only if $I = \delta(0)$.*

Proof. Since $0 \in I$, it is clear that $\delta(0) \subseteq \delta(I) = I$. If I is a $\delta(0)$-ideal of R, then by Proposition 2.3, we have $I \subseteq \delta(0)$ and so $I = \delta(0)$. For the converse, assume that $I = \delta(0)$. Now we show that I is a $\delta(0)$-ideal. Let $ab \in I$ and $a \notin \delta(0)$ for $a, b \in R$. Since I is a prime ideal and $a \notin \delta(0)$, we get $b \in I$ and so I is a $\delta(0)$-ideal of R. □

Recall from [4] that a ring R is said to be δ-reduced if $\delta(0) = 0$.

Proposition 2.14. (1) *$\delta(0)$ is a $\delta(0)$-ideal of R if and only if it is a prime ideal of R.*
(2) *Any δ-reduced ring which is not an integral domain, has no $\delta(0)$-ideals.*

Proof. (1) Suppose that $\delta(0)$ is a prime ideal of R and $ab \in \delta(0)$ with $a \notin \delta(0)$ for $a, b \in R$. Then $b \in \delta(0)$. Hence $\delta(0)$ is a $\delta(0)$-ideal of R. For the converse, assume that $\delta(0)$ is a $\delta(0)$-ideal of R. Let $ab \in \delta(0)$ and $a \notin \delta(0)$. Since $\delta(0)$ is a $\delta(0)$-ideal of R, we conclude that $b \in \delta(0)$. Hence $\delta(0)$ is a prime ideal of R.
(2) Let R be a δ-reduced ring which is not an integral domain. Then $\delta(0) = 0$ is not a prime ideal of R and so by (1), it is not a $\delta(0)$-ideal. On the other hand, if I is a nonzero $\delta(0)$-ideal of R, then by Proposition 2.3, $I \subseteq \delta(0) = 0$ and so $I = 0$, which is a contradiction. □

Proposition 2.15. *Let R be a ring. If R is an integral domain, then 0 is a $\delta(0)$-ideal. Moreover, if R is a δ-reduced ring, then R is an integral domain if and only if 0 is a $\delta(0)$-ideal of R.*

Proof. Let $ab = 0$ and $a \notin \delta(0)$. Then $a \neq 0$. Since R is an integral domain, we have $b = 0$. It implies that 0 is a $\delta(0)$-ideal. Moreover, suppose that R is a δ-reduced ring. If 0 is a $\delta(0)$-ideal of R, then by Proposition 2.14 (2), R is an integral domain. □

Proposition 2.16. *Let R be a δ-reduced von Neumann ring. Then 0 is a $\delta(0)$-ideal if and only if R is a field.*

Proof. Take $0 \neq a \in R$. Then $a = a^2x$ for some $x \in R$. Since $a(1 - ax) \in 0$ and 0 is a $\delta(0)$-ideal, we get $1 - ax \in \delta(0) = 0$. Thus $ax = 1$, as required.

The converse follows from Proposition 2.15. □

Proposition 2.17. *Let R be a ring and S a nonempty subset of R. If I is a $\delta(0)$-ideal of R with $S \nsubseteq I$, then $(I : S)$ is a $\delta(0)$-ideal of R.*

Proof. It is easy to see that $(I : S) \neq R$. Let $ab \in (I : S)$ and $a \notin \delta(0)$. Then we have $abs \in I$ for every $s \in S$. Since I is a $\delta(0)$-ideal of R, we conclude that $bs \in I$ and thus $b \in (I : S)$. □

Let R be a ring. We call a $\delta(0)$-ideal M of R a maximal $\delta(0)$-ideal if there is no $\delta(0)$-ideal containing M properly.

Theorem 2.18. *If I is a maximal $\delta(0)$-ideal of R with $\delta(I) = I$, then $I = \delta(0)$.*

Proof. We show that I is a prime ideal of R, and so by Proposition 2.13, we have $I = \delta(0)$. Let $ab \in I$ and $a \notin I$ for $a, b \in R$. Since I is a $\delta(0)$-ideal and $a \notin I$, $(I : a)$ is a $\delta(0)$-ideal by Proposition 2.17. Thus $b \in (I : a) = I$ by the maximality of I. Hence I is a prime ideal of R. □

By Proposition 2.14, if $\delta(0)$ is a prime ideal of R, then R admits a $\delta(0)$-ideal $\delta(0)$. For the converse, we have:

Theorem 2.19. *Let R be a ring. If there exists a $\delta(0)$-ideal of R, then there exists a maximal $\delta(0)$-ideal of R. Moreover, if $\delta(I) = I$ for every maximal $\delta(0)$-ideal I of R, then $\delta(0)$ is a prime ideal of R.*

Proof. Suppose that I is a $\delta(0)$-ideal of R and $\Omega = \{J : J \text{ is a } \delta(0)\text{-ideal of } R\}$. Since $I \in \Omega$, $\Omega \neq \emptyset$. It is clear that Ω is a partially ordered set by the set inclusion. Suppose $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is a chain of Ω. Now, we show that $\bigcup_{n=1}^{\infty} I_n$ is a $\delta(0)$-ideal of R. Let $ab \in \bigcup_{n=1}^{\infty} I_n$ with $a \notin \delta(0)$ for $a, b \in R$. Then we have $ab \in I_k$ for some $k \in \mathbb{N}$. Since I_k is a $\delta(0)$-ideal, we conclude $b \in I_k \subseteq \bigcup_{n=1}^{\infty} I_n$. So $\bigcup_{n=1}^{\infty} I_n$ is an upper bound of the chain $\{I_i : i \in \mathbb{N}\}$. By Zorn's Lemma, Ω has a maximal element K. Then by Theorem 2.18, we get the result that $K = \delta(0)$ is a prime ideal of R. $\square$

Proposition 2.20. *Let R be a ring and K an ideal of R with $K \cap (R - \delta(0)) \neq \emptyset$. Then the following hold:*

(1) *If I_1, I_2 are $\delta(0)$-ideals of R with $I_1K = I_2K$, then $I_1 = I_2$.*
(2) *If IK is a $\delta(0)$-ideal of R, then $IK = I$.*

Proof. (1) Since I_1 is a $\delta(0)$-ideal and $I_2K \subseteq I_1$, it follows from Theorem 2.12 (3) that we get the result that $I_2 \subseteq I_1$. Likewise, we get $I_1 \subseteq I_2$.
(2) Since IK is a $\delta(0)$-ideal and $IK \subseteq IK$, we conclude that $I \subseteq IK$, so this completes the proof. $\square$

Let R and S be commutative rings with $1 \neq 0$ and let δ, γ be two expansion functions of $\mathcal{I}(R)$ and $\mathcal{I}(S)$, respectively. Then a ring homomorphism $f : R \to S$ is called a $\delta\gamma$-homomorphism if $\delta\left(f^{-1}(I)\right) = f^{-1}(\gamma(I))$ for all ideals I of S. Let γ_1 be a radical operation on ideals of S and δ_1 a radical operation on ideals of R. A homomorphism from R to S is an example of $\delta_1\gamma_1$-homomorphism. Additionally, if f is a $\delta\gamma$-epimorphism and I is an ideal of R containing $\text{Ker}(f)$, then $\gamma(f(I)) = f(\delta(I))$, see [4].

Theorem 2.21. *Let $f : R \to S$ be a ring $\delta\gamma$-homomorphism. Then the following hold:*

(1) *If f is an epimorphism and I is a $\delta(0_R)$-ideal of R containing* $\text{Ker}(f)$, *then $f(I)$ is a $\gamma(0_S)$-ideal of S.*

(2) *If f is a monomorphism and J is a $\gamma(0_S)$-ideal of S, then $f^{-1}(J)$ is a $\delta(0_R)$-ideal of R.*

Proof. (1) Let $a'b' \in f(I)$ with $a' \notin \gamma(0_S)$ for $a', b' \in S$. Since f is an epimorphism, there exist $a, b \in R$ such that $a' = f(a)$ and $b' = f(b)$. Then $a'b' = f(ab) \in f(I)$. As $\mathrm{Ker}(f) \subseteq I$, we conclude that $ab \in I$. Also, note that $a \notin \delta(0_R)$. Since I is a $\delta(0_R)$-ideal of R, we get the result that $b \in I$ and so $f(b) = b' \in f(I)$ as needed.
(2) Let $ab \in f^{-1}(J)$ and $a \notin \delta(0_R)$. Then $f(ab) = f(a)f(b) \in J$. Since $a \notin \delta(0_R)$ and f is a monomorphism, we get $f(a) \notin \gamma(0_S)$. Since J is a $\gamma(0_S)$-ideal of $S, f(b) \in J$ and so $b \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a $\delta(0_R)$-ideal of R.

Let δ be an expansion function of $\mathcal{I}(R)$ and I a proper ideal of R. Then the function $\delta_q : R/I \to R/I$, defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of R/I, see [4]. Consider the natural homomorphism $\pi : R \to R/J$. Then for ideals I of R with $\mathrm{Ker}\,(\pi) \subseteq I$, we have $\delta_q(\pi(I)) = \pi(\delta(I))$. □

Corollary 2.22. *Let R be a ring and $J \subseteq I$ be two ideals of R. Then the following hold:*

(1) *If I is a $\delta(0)$-ideal of R, then I/J is a $\delta_q(0)$-ideal of R/J.*
(2) *If I/J is a $\delta_q(0)$-ideal of R/J and $J \subseteq \delta(0)$, then I is a $\delta(0)$-ideal of R.*
(3) *If I/J is a $\delta_q(0)$-ideal of R/J and J is a $\delta(0)$-ideal of R, then I is a $\delta(0)$-ideal of R.*

Proof. (1) Assume that I is a $\delta(0)$-ideal of R with $J \subseteq I$. Let $\pi : R \to R/J$ be the natural homomorphism. Note that $\mathrm{Ker}(\pi) = J \subseteq I$, and so by Theorem 2.21 (1), it follows that I/J is a $\delta_q(0)$-ideal of R/J.
(2) Let $ab \in I$ with $a \notin \delta(0)$ for $a, b \in R$. Then we have $(a+J)(b+J) = ab+J \in I/J$ and $a+J \notin \delta_q(0_{R/J})$. Since I/J is a $\delta(0)$-ideal of R/J, we conclude that $b+J \in I/J$ and so $b \in I$. Consequently, I is a $\delta(0)$-ideal of R.
(3) It follows from (2) and Proposition 2.3. □

The following example shows that the converse of Corollary 2.22 (1) is not always true.

Example 2.23. Let $R = \mathbb{Z}$, $I = 6\mathbb{Z}$ and $J = 2\mathbb{Z}$. Set δ_+ be an ideal expansion such that $\delta_+(K) = K + 3\mathbb{Z}$ for each ideal K of $\mathbb{Z}$. Clearly, $\delta_+(0) = 3\mathbb{Z}$. Therefore $\delta_{q+}(0_{R/J}) = \mathbb{Z}_2$. Since $I/J = \{0\}$ is a prime ideal of $R/J = \mathbb{Z}_2$ and $I/J \subseteq \delta_{q+}(0_{R/J})$, by Corollary 2.5, I/J is a $\delta_{q+}(0_{R/J})$-ideal of R/J. But $2 \times 3 \in I$ and $2 \notin \delta_+(0)$ and $3 \notin I$. Then I is not a $\delta_+(0)$-ideal of R.

Let S be a multiplicatively closed subset of a ring R and δ an expansion function of $\mathcal{I}(R)$. Note that δ_S is an expansion function of $\mathcal{I}(S^{-1}R)$ such that $\delta_S(S^{-1}I) = S^{-1}\delta(I)$ for each ideal I of R.

Proposition 2.24. *Let R be a ring and S a multiplicatively closed subset of R. Then the following hold:*

(1) *If I is a $\delta(0)$-ideal of R, then $S^{-1}I$ is a $\delta_s(0)$-ideal of $S^{-1}R$.*
(2) *If $S = \mathrm{Reg}(R)$ and J is a $\delta_s(0)$-ideal of $S^{-1}R$, then J^c is a $\delta(0)$-ideal of R.*

Proof. (1) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ with $\frac{a}{s} \notin \delta_s(0_{S^{-1}R})$, where $a, b \in R$ and $s, t \in S$. Then we have $uab \in I$ for some $u \in S$. It is clear that $a \notin \delta(0)$. Since I is a $\delta(0)$-ideal of R, we conclude that $ub \in I$ and so $\frac{b}{t} = \frac{ub}{ut} \in S^{-1}I$. Consequently, $S^{-1}I$ is a $\delta_s(0)$-ideal of $S^{-1}R$.
(2) Let $ab \in J^c$ and $a \notin \delta(0_R)$. Then we have $\frac{a}{1}\frac{b}{1} \in J$. Now we show that $\frac{a}{1} \notin \delta_s(0_{S^{-1}R})$. Suppose $\frac{a}{1} \in \delta_s(0_{S^{-1}R})$. Then $ua = 0$ for some $u \in S$. Since $u \in \mathrm{Reg}(R)$, we conclude that $a = 0 \in \delta(0_R)$, a contradiction. Thus we have $\frac{a}{1} \notin \delta_s(0_{S^{-1}R})$. Since J is a $\delta(0)$-ideal of $S^{-1}R$, we get $\frac{b}{1} \in J$ and so $b \in J^c$. □

Definition 2.25. Let S be a nonempty subset of R with $R - \delta(0) \subseteq S$. Then S is called a $\delta(0)$-*multiplicatively closed subset* of R if $xy \in S$ for all $x \in R - \delta(0)$ and $y \in S$.

Proposition 2.26. *For a proper ideal I of R, I is a $\delta(0)$-ideal of R if and only if $R - I$ is a $\delta(0)$-multiplicatively closed subset of R.*

Proof. Suppose that I is a $\delta(0)$-ideal of R. Then by Proposition 2.3, we have $I \subseteq \delta(0)$ and so $R - \delta(0) \subseteq R - I$. Let $x \in R - \delta(0)$ and $y \in R - I$. Assume that $xy \in I$. Since $x \notin \delta(0)$ and I is a $\delta(0)$-ideal, we conclude that $y \in I$, a contradiction. Thus we get $xy \in R - I$, and so $R - I$ is a $\delta(0)$-multiplicatively closed subset of R. For the converse, suppose that I is an ideal and $R - I$ is a $\delta(0)$-multiplicatively closed subset of R. Now we show that I is a $\delta(0)$-ideal. Let $ab \in I$ with $a \notin \delta(0)$ for $a, b \in R$. Then we have $b \in I$, or else we would have $ab \in R - I$ since $R - I$ is a $\delta(0)$-multiplicatively closed subset of R. So it follows that I is a $\delta(0)$-ideal of R. □

By the above observations, we have the following result analogous to the relations between prime ideals and multiplicatively closed subsets. We remind the reader that if I is an ideal that disjoint from a multiplicatively closed subset S of R, then there exists a prime ideal P of R containing I such that $P \cap S = \emptyset$. The following theorem states that a similar result is true for $\delta(0)$-ideals.

Theorem 2.27. *Let I be an ideal of R such that $I \cap S = \emptyset$ where S is a $\delta(0)$-multiplicatively closed subset of R. Then there exists a $\delta(0)$-ideal J containing I such that $J \cap S = \emptyset$.*

Proof. Consider the set $\Omega = \{I' : I' \text{ is an ideal of } R \text{ with } I' \cap S = \emptyset\}$. Since $I \in \Omega$, we have $\Omega \neq \emptyset$. By using Zorn's Lemma, we get a maximal element J of Ω. Now we show that J is a $\delta(0)$-ideal of R. Suppose not. Then we have $ab \in J$ for some $a \notin \delta(0)$ and $b \notin J$. Thus we get $b \in (J : a)$ and $J \subsetneq (J : a)$. By the maximality of J, we have $(J : a) \cap S \neq \emptyset$ and thus there exists an $s \in S$ such that $s \in (J : a)$. So we have $as \in J$. Also $sa \in S$, because $a \in R - \delta(0)$, $s \in S$ and S is a $\delta(0)$-multiplicatively closed subset of R. Thus we get $S \cap J \neq \emptyset$, and this contradicts by $J \in \Omega$. Hence J is a $\delta(0)$-ideal of R. □

An element a of a ring R is called δ-nilpotent if $a \in \delta(0)$. So the δ_0-nilpotent element is the zero element. Also, δ_1-nilpotent elements are exactly the ordinary nilpotent elements.

Proposition 2.28. *Let R be a ring with $\delta(I) = I$ for every maximal $\delta(0)$-ideal I. Suppose that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$, where $I, I_1, I_2, \cdots, I_n$ are ideals of R. If I_i is a $\delta(0)$-ideal and the others have non δ-nilpotent elements with $I \nsubseteq \bigcup_{j \neq i} I_j$, then $I \subseteq I_i$.*

Proof. We may assume that $i = 1$. Since $I \nsubseteq I_2 \cup \cdots \cup I_n$, there exits $x \in I - \bigcup_{j=2}^{n} I_j$. Thus we have $x \in I_1$. Let $y \in I \cap (I_2 \cap I_3 \cap \cdots \cap I_n)$. Since $x \notin I_k$ and $y \in I_k$ for every $2 \leq k \leq n$, we have $x+y \notin I_k$. Thus we have $x+y \in I - \bigcup_{j=2}^{n} I_j$ and so $x+y \in I_1$. As $x+y \in I_1$ and $x \in I_1$, it follows that $y \in I_1$ and so $I \cap \bigcap_{k=2}^{n} I_k \subseteq I_1$. Since R has a $\delta(0)$-ideal by Theorem 2.19, $\delta(0)$ is a prime ideal. So the product of non δ-nilpotent elements is also a non-nilpotent element. Thus we have $(\prod_{k=2}^{n} I_k) \cap (R - \delta(0)) \neq \emptyset$. Since $I \cdot (\prod_{k=2}^{n} I_k) \subseteq I_1$ and I_1 is a $\delta(0)$-ideal of R, we have $I \subseteq I_1$ by Theorem 2.12. □

Proposition 2.29. *For any ring R, the following are equivalent:*

(1) *Every element of R is either δ-nilpotent or unit.*
(2) *Every proper principal ideal is a $\delta(0)$-ideal.*
(3) *Every proper ideal is a $\delta(0)$-ideal.*
(4) *R is a local ring with maximal ideal $\delta(0)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $< x > \neq R$, where $x \in R$. Let $ab \in < x >$ and $a \notin \delta(0)$. Since a is not δ-nilpotent, by (1), a is a unit in R. Then we have

$b = a^{-1}(ab) \in <x>$ and so $<x>$ is a $\delta(0)$-ideal of R.

(2) $\Rightarrow$ (3) Let I be a proper ideal of R and $ab \in I$, where $a \notin \delta(0)$. Since $ab \in <ab>$ and $<ab>$ is a $\delta(0)$-ideal of R, we conclude that $b \in <ab> \subseteq I$. Hence I is a $\delta(0)$-ideal of R.

(3) $\Rightarrow$ (4) Let I be a proper ideal. Then (3) implies that I is a $\delta(0)$-ideal. Since by Proposition 2.3, we have $I \subseteq \delta(0)$. Then R is a local ring with a maximal ideal $\delta(0)$.

(4) $\Rightarrow$ (1) It is straightforward. □

Definition 2.30. A proper ideal I of R is called a *weakly $\delta(0)$-ideal* if whenever $a, b \in R$ with $0 \neq ab \in I$, we have either $a \in I$ or $b \in \delta(0)$.

Definition 2.31. Let I be a weakly $\delta(0)$-ideal. Then (a, b) is called *$\delta(0)$-twin-zero* of I if $ab = 0$, $a \notin I$ and $b \notin \delta(0)$.

Theorem 2.32. *Let I be a weakly $\delta(0)$-ideal of R and suppose (a, b) is a $\delta(0)$-twin-zero of I for some $a, b \in R$. Then $a\delta(0) = b\delta(0) = 0$.*

Proof. Assume that $a\delta(0) \neq 0$. Then there exists $i \in \delta(0)$ such that $ai \neq 0$. Hence $a(b + i) \neq 0$. Since $a \notin I$ and I is a weakly $\delta(0)$-ideal, we have $b + i \in \delta(0)$. This implies $b \in \delta(0)$, a contradiction. So $a\delta(0) = 0$. A similar argument shows that $b\delta(0) = 0$. □

Lemma 2.33. *Let δ be an expansion function of ideals and I be a weakly $\delta(0)$-ideal. Suppose $aJ \subseteq I$ for some element $a \in R$ such that (a, b) is not $\delta(0)$-twin-zero for any $b \in J$. If $a \notin I$, then $J \subseteq \delta(0)$.*

Proof. Suppose that $J \nsubseteq \delta(0)$. Then there exists $j \in J$ such that $j \notin \delta(0)$. Since (a, j) is not $\delta(0)$-twin-zero, $aj \in I$ and $a \notin I$, we get $j \in \delta(0)$. This gives a contradiction. □

Theorem 2.34. *Let δ, γ be expansion functions such that I is a $\gamma(0)$-ideal of R. Then I is a weakly $\gamma o\delta(0)$-ideal if and only if I is a $\gamma o\delta(0)$-ideal.*

Proof. Assume that I is a weakly $\gamma o\delta(0)$-ideal. Take $ab \in I$. If $0 \neq ab$, then either $a \in I$ or $b \in \gamma o\delta(0)$ and so we are done. Now suppose that $ab = 0$ and $a \notin I$. Since $ab = 0 \in I$ and I is a $\gamma(0)$-ideal, we get $b \in \gamma(0) \subseteq \gamma(\delta(0))$. Hence I is a $\gamma o\delta(0)$-ideal. The other direction is clear. □

Theorem 2.35. *Let R_1 and R_2 be commutative rings, δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively. Let I be a proper ideal of R_1 and $R = R_1 \times R_2$. If $I \times R_2$ is a $\delta(0)$-ideal where $\delta(0) = \delta_1(0) \times \delta_2(0)$, then I is a $\delta_1(0)$-ideal.*

Proof. Let $ab \in I$ for some $a, b, \in R$. Since $(a,1)(b,1) \in I \times R_2$ and $I \times R_2$ is a $\delta(0)$-ideal, we have $(a,1) \in \delta(0 \times 0)$ or $(b,1) \in I \times R_2$. This gives $a \in \delta_1(0)$ or $b \in I$, as needed. □

Let A be a ring and E an A-module. Then $A \propto E$, the trivial (ring) extension of A by E, is the ring whose additive structure is coordinate-wise addition and whose multiplication is defined by $(a,e)(b,f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the idealization $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [7], [8]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [2,5]). In addition, for an ideal I of A and a submodule F of E, $I \propto F$ is an ideal of $A \propto E$ if and only if $IE \subseteq F$. Moreover, for an expansion function δ of A, it is clear that $\delta_\propto$ defined as $\delta_\propto(I \propto F) = \delta(I) \propto E$ is an expansion function of $A \propto E$.

Definition 2.36. Let M be an R-module. Then a proper submodule N of M is called a *$\delta(0)$-submodule* if whenever $am \in N$ for $a \in R$, $m \in M$, we have either $a \in \delta(0)$ or $m \in N$.

Theorem 2.37. *Let A be a ring, E an A-module and δ be an expansion function of $\mathcal{I}(A)$. Let I be an ideal of A and F a submodule of E such that $IE \subseteq F$. Then the following statements hold:*

1. *If $I \propto F$ is a $\delta(0)$-ideal of $A \propto E$, then I is a $\delta(0)$-ideal of A and F is a $\delta(0)$-submodule of E.*
2. *Assume that $(F : c) = F$ for every $c \in A \setminus I$. Then $I \propto F$ is a $\delta(0)$-ideal of $A \propto E$ if and only if I is a $\delta(0)$-ideal of A.*

Proof. (1) Assume that $I \propto F$ is a $\delta(0)$-ideal of $A \propto E$. Let $ab \in I$ with $a \notin \delta(0)$ for a, b two elements of A. Thus $(a,0)(b,0) = (ab,0) \in I \propto F$ and $(a,0) \notin \delta_\propto(0_{A\propto E})$. This implies $(b,0) \in I \propto F$. Therefore $b \in I$. Now suppose that $am \in F$ with $a \notin \delta(0)$. Then $(a,0)(0,m) = (0,am) \in I \propto F$ with $(a,0) \notin \delta_\propto(0)$. This implies that $(0,m) \in I \propto F$ and so $m \in F$, as desired.
(2) By (1), it suffices to prove the "if" assertion. Assume that I is a $\delta(0)$-ideal of A and $(a,e)(b,f) = (ab, af + be) \in I \propto F$, with $(a,e) \notin \delta_\propto(0_{A\propto E})$. Then we have $ab \in I$ with $a \notin \delta(0)$. Then $b \in I$ and by Proposition 2.3, we have $a \notin I$. Since $IE \subseteq F$ and $af + be \in F$, we have $af \in F$. This implies $f \in (F : a) = F$. Therefore $(b,f) \in I \propto F$, as desired. □

Let δ_1 and δ_2 be two ideal expansions, and let $\delta(I) = \delta_1(I) \cap \delta_2(I)$. We can easily check that δ is also an ideal expansion. Generally, the intersection of any collection of ideal expansions is an ideal expansion. In the next example, we show that the converse of Theorem 2.37 (1) is not true in general.

Example 2.38. Consider the $\mathbb{Z}$-module $\mathbb{Z}_9$ and δ an ideal expansion defined by $\delta(I) = \delta_1(I) \cap \delta_2(I)$, with $\delta_1(I) = \sqrt{I}$ and $\delta_2(I) = I$. Then $\delta(0) = \{0\}$. Thus by Proposition 2.15, we have that 0 is a $\delta(0)$-ideal of $\mathbb{Z}$. But $I = (0, \overline{0})$ is not a $\delta_\propto(0_{\mathbb{Z}\propto\mathbb{Z}_9})$-ideal of $\mathbb{Z} \propto \mathbb{Z}_9$. Because $(3; \overline{0})(0; \overline{3}) = (0; \overline{0}) \in I$. $(3; \overline{0}) \notin \delta_\propto(0_{\mathbb{Z}\propto\mathbb{Z}_9})$ and $(0; \overline{3}) \notin I$.

Corollary 2.39. *Let A be a ring, E an A-module and δ be an expansion function of $\mathcal{I}(A)$. Let I be a proper ideal of A. Then I is a $\delta(0)$-ideal of A if and only if $I \propto E$ is a $\delta_\propto(0)$-ideal of $A \propto E$.*

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Declarations

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References

[1] D. D. Anderson and M. Bataineh, *Generalizations of prime ideals*, Comm. Algebra, 36(2) (2008), 686-696.

[2] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra, 1(1) (2009), 3-56.

[3] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, 1969.

[4] A. Badawi and B. Fahid, *On weakly 2-absorbing δ-primary ideals of commutative rings*, Georgian Math. J., 27(4) (2020), 503-516.

[5] D. E. Dobbs, A. El Khalfi and N. Mahdou *Trivial extensions satisfying certain valuation-like properties*, Comm. Algebra, 47(5) (2019), 2060-2077.

[6] M. Ebrahimpour and R. Nekooei, *On generalizations of prime ideals*, Comm. Algebra, 40(4) (2012), 1268-1279.

[7] S. Glaz, Commutative Coherent Rings, Lecture Notes in Math., 1371, Springer-Verlag, Berlin, 1989.

[8] J. A. Huckaba, Commutative Rings with Zero Divisors, Monogr. Textbooks Pure Appl. Math., 117, Marcel Dekker, Inc., New York, 1988.

[9] R. Mohamadian, *r-ideals in commutative rings*, Turkish J. Math., 39(5) (2015), 733-749.

[10] U. Tekir, S. Koc and K. H. Oral, *n-ideals of commutative rings*, Filomat, 31(10) (2017), 2933-2941.

[11] D. Zhao, *δ-primary ideals of commutative rings*, Kyungpook Math. J., 41(1) (2001), 17-22.

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