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IDEMPOTENTS AND ZERO DIVISORS IN COMMUTATIVE ALGEBRAS SATISFYING AN IDENTITY OF DEGREE FOUR

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Abstract. We study commutative algebras satisfying the identity $((wx)y)z + ((wy)z)x + ((wz)x)y - ((wy)x)z - ((wx)z)y - ((wz)y)x = 0$. We assume characteristic of the field $\neq 2, 3$. We prove that given any $\lambda \in F$, there exists a commutative algebra with idempotent e, which satisfies the identity, and has λ as an eigen value of the multiplication operator L_e. For algebras with idempotent, the containment relations for the product of the eigen spaces are not as precise as they are for Jordan or power-associative algebras. A great part of this paper is calculating these containment relations.

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1. Introduction

Let A be a commutative algebra over a field F satisfying the identity

$$((wx)y)z + ((wy)z)x + ((wz)x)y - ((wy)x)z - ((wx)z)y - ((wz)y)x = 0 \quad (1)$$

and characteristic of the field $\neq 2, 3$.

In the study of degree four identities not implied by commutativity, Osborn [7] classified those that were compatible with the possession of a unit element. Carini, Hentzel and Piacentini Cattaneo [2] extended this work by dropping this restriction. The identity (1) appeared as one of the additional degree four identities.

These algebras have been studied by Correa, Hentzel [3] and by Rojas-Bruna [8]. In [3] the authors assume the additional identity $((xx)x)x = 0$ and prove the algebra is nilpotent. In [8] the author proved the existence of trace form. Moreover if he assumes the existence of a non degenerate trace form, then the algebra satisfies the identity $((yx)x)x = y((xx)x)$.

The paper is structured as follows. Section 2 will be devoted to give examples and preliminary results. Section 3 deals with idempotent elements. Section 4 is devoted to zero divisors.

2. Examples and preliminaries results

Let A be a commutative algebra satisfying identity (1). Using the associator notation $(a,b,c) = (ab)c - a(bc)$ and commutativity, identity (1) can be written as

$$(x,w,y)z + (y,w,z)x + (z,w,x)y = 0, \tag{2}$$

or

$$(y,wx,z) + (z,wy,x) + (x,wz,y) = 0. \tag{3}$$

For every $w \in A$, define the map $G_w : A \times A \times A \longrightarrow A$, by

$$G_w(x,y,z) = (x,w,y)z + (y,w,z)x + (z,w,x)y, \ \forall\, x,y,z \in A.$$

Remark 2.1. It is clear that identity (2) is equivalent to $G_w \equiv 0$ for every $w \in A$. Also, for each $w \in A, G_w$ is clearly a tri-linear map. If two variables among x,y,z are equal, then $G_w = 0$. In fact, take $x = y$ for example then using commutativity for any $w \in A$, we have $G_w(x,x,z) = (x,w,x)z + (x,w,z)x + (z,w,x)x = 0 + (x,w,z)x - (x,w,z)x = 0$, and $G_w = 0$. This means that G_w is a tri-linear alternating map for every $w \in A$.

This remark implies the following result.

Lemma 2.2. *Every* 2 *dimensional algebra over a field F satisfies identity (2).*

Lemma 2.3. *For every $\lambda \in F$, there exists a two dimensional algebra satisfying identity (2) with idempotent e and where λ is an eigen value of the linear operator L_e.*

Proof. In fact the F-vector space A generated by e,u with multiplication given by

$$e^2 = e, eu = ue = \lambda u, u^2 = \alpha e + \beta u, \ \alpha,\beta \in F$$

is a two dimensional algebra which (by Lemma 2.2) satisfies identity (2). □

Remark 2.4. The above Lemma implies that for commutative algebras satisfying identity (1), the eigen spaces are not restricted to specific values of λ as they are in Jordan algebras (see Schafer [9], page 97) or Power-Associative algebras (see Albert [1], Gerstenhaber [4], Schafer [9] page 131).

Example 2.5. The following is a three dimensional algebra with an idempotent e, satisfying identity (1). The basis is $\{e,x,y\}$. The element e is an idempotent and

x and y are in the eigen space for $\lambda \in F$. The parameters $b_1, c_1, b_2, c_2, b_3, c_3$ and $\lambda \neq \frac{1}{2}$ are independent. The multiplication table is given by

	e	x	y
e	e	λx	λy
x	λx	$a_1 e + b_1 x + c_1 y$	$a_3 e + b_3 x + c_3 y$
y	λy	$a_3 e + b_3 x + c_3 y$	$a_2 e + b_2 x + c_2 y$

where $a_1 = \frac{c_1(-c_2+b_3)+c_3(-b_1+c_3)}{-1+2\lambda}$, $a_2 = \frac{b_3(-c_2+b_3)+b_2(-b_1+c_3)}{-1+2\lambda}$, $a_3 = \frac{b_2c_1-b_3c_3}{-1+2\lambda}$.

Recall that an algebra A is flexible if and only if for every $x, y \in A$, $(x, y, x) = 0$. It is immediate that every commutative algebra is flexible.

Proposition 2.6. *Let A be a commutative algebra satisfying identity* (1). *Then the nucleus of A, $N(A) = \{x \in A \mid (x, A, A) = (A, x, A) = 0\}$ is an ideal of A.*

Proof. It is clear that $N(A)$ is a subspace of A. Since A is flexible, we need only to prove that for every $x \in N(A)$ and for every $y, z, w \in A$, we have

$$(xy, z, w) = 0, \text{ and } (w, xy, z) = 0.$$

The following identity is known as the Teichmuller's identity and it is satisfied in every non associative algebra.

$$(xy, z, w) + (x, yz, w) + (x, y, zw) = x(y, z, w) + (x, y, z)w. \tag{4}$$

Using that $x \in N(A)$, we get $(xy, z, w) = x(y, z, w)$ and flexibility and identity (2) imply that $x(y, z, w) = 0$, so $(xy, z, d) = 0$. On the other hand, flexibility and identity (3) imply that for every $x \in N(A)$, we get that $(w, xy, z) = 0$. Therefore $N(A)$ is an ideal of A. □

Proposition 2.7. *Let A be a commutative algebra satisfying identity* (1). *Then the set $H = \{x \in A \mid x(A, A, A) = (x, A, A) = 0\}$ is an ideal of A.*

Proof. It is clear that H is a subspace of A. Let w be an element in H and $x \in A$. Then from identity (4) we get

$$(x, y, zw) = 0 \text{ for every } y, z \in A.$$

Let (a, b, c) be an element in (A, A, A) and let $w \in H$. Then identity (2) implies that

$$(wx)(a, b, c) = -(wx, b, a)c - (c, b, wx)a = 0$$

and the result follows. □

Corollary 2.8. *Let A be a commutative algebra satisfying identity* (1) *and let I be the subspace generated by the associators* (a,b,c)*. Then $H(AI)=0$.*

Proof. Let consider the elements $h \in H, x \in A, y \in I$, then $h(xy) = (hx)y - (h,x,y) = 0 - 0 = 0$ and $H(AI) = 0$. □

Remark 2.9. In [8] it is proved that if we assume that the algebra A has a non degenerate trace form, then the subspace generated by the associators (a,b,c) is an ideal.

3. Idempotent elements

Let A be a commutative algebra satisfying identity (1) over a field F of characteristic $\neq 2,3$. Suppose that A has a Peirce decomposition relative to an idempotent element e. This means that the operator L_e given by $L_e(x) = ex$ is diagonalizable.

For each $\lambda \in F$, let $A_\lambda = \{a \in A \mid ea = \lambda a\}$ be the eigen space associated to the linear operator L_e.

Next we introduce the following notation to improve the reading of the paper. Define

$$B = \bigcup_{\lambda \in F} A_\lambda.$$

Our assumption implies that there is a basis contained in B.

Let $f\colon B \setminus \{0\} \longrightarrow F$ be the map defined by

$$f(x) = \lambda \Longleftrightarrow x \in A_\lambda.$$

Theorem 3.1. *If e is an idempotent element in A, then for all $a,b \in B \setminus \{0\}$,*

$$e(ab) = (f(a) + f(b) - 1)ab \text{ when } f(a) \neq f(b).$$

Proof. From identity (1) we get:

$$\begin{aligned}
0 &= ((ea)b)e + ((eb)e)a + ((ee)a)b - ((eb)a)e - ((ea)e)b - ((ee)b)a \\
0 &= f(a)(ab)e + f(b)f(b)ba + f(a)ab - f(b)(ba)e - f(a)f(a)ab - f(b)ba \\
0 &= f(a)(ab)e + f(b)f(b)ab + f(a)ab - f(b)(ab)e - f(a)f(a)ab - f(b)ab.
\end{aligned}$$

Then

$$\begin{aligned}
0 &= (f(a) - f(b))(ab)e + (f(a) - f(b) - f(a)f(a) + f(b)f(b))ab \\
0 &= (f(a) - f(b))(ab)e + (f(a) - f(b) - (f(a) - f(b))(f(a) + f(b)))ab \\
0 &= (f(a) - f(b))(e(ab) + (1 - f(a) - f(b))ab).
\end{aligned}$$

If $f(a) \neq f(b)$, then $e(ab) = (f(a) + f(b) - 1)ab$. □

Corollary 3.2. *If $\lambda \neq \mu \in F$, then $A_\lambda A_\mu \subseteq A_{\lambda+\mu-1}$, and for every $\lambda \neq 1$, we get $A_\lambda A_1 \subseteq A_\lambda$.*

Theorem 3.3. *Let $a, b, c \in B \setminus \{0\}$. Then*

(i) $0 = (f(a) - f(b))(ab)c + (f(b) - f(c))(bc)a + (f(c) - f(a))(ca)b.$

(ii) *If $f(a) = f(c) \neq f(b)$, then $(a, b, c) = 0$.*

(iii) *If $f(a), f(b), f(c)$ are distinct, then*

$$\frac{(a,b,c)}{f(a)-f(c)} = \frac{(a,c,b)}{f(a)-f(b)} = \frac{(b,c,a)}{f(b)-f(a)} \tag{5}$$
$$= \frac{(b,a,c)}{f(b)-f(c)} = \frac{(c,a,b)}{f(c)-f(b)} = \frac{(c,b,a)}{f(c)-f(a)}.$$

Proof. Item (ii) is immediate from item (i), so let us prove the first one. From identity (1) we obtain: $0 = ((ea)b)c + ((eb)c)a + ((ec)a)b - ((eb)a)c - ((ea)c)b - ((ec)b)a$, then

$$0 = f(a)(ab)c + f(b)(bc)a + f(c)(ca)b - f(b)(ba)c - f(a)(ac)b - f(c)(cb)a. \tag{6}$$

Thus, $0 = (f(a) - f(b))(ab)c + (f(b) - f(c))(bc)a + (f(c) - f(a))(ca)b.$

This proves item (i).

In order to prove item (iii) we deduce from (6) that

$$0 = f(a)(b, a, c) + f(b)(c, b, a) + f(c)(a, c, b).$$

Since $(b, a, c) = -(a, c, b) - (c, b, a)$ and $(c, b, a) = -(a, b, c)$ replacing in the above identity, we obtain

$$0 = -f(a)(a, c, b) - f(a)(c, b, a) + f(c)(a, c, b)$$

and

$$0 = (-f(a) + f(c))(a, c, b) + (f(a) - f(b))(a, b, c),$$

that is,

$$(f(a) - f(b))(a, b, c) = (f(a) - f(c))(a, c, b).$$

If $f(a), f(b), f(c)$ are distinct, we obtain identities appearing in (5). □

Theorem 3.4. *Let $a, b, c \in B \setminus \{0\}$. If $f(a) \neq 1, f(b) \neq 1, f(a) \neq f(b)$, then $(ab)c \in B$ and $f((ab)c) = f(a) + f(b) + f(c) - 2$ whenever $(ab)c \neq 0$.*

Proof. We have that $ab \neq 0$ (otherwise $(ab)c = 0$). We separate the proof in two cases.

Case 1 $f(ab) \neq f(c)$**.** In this case we have by Theorem 3.1

$$f((ab)c) = f(ab) + f(c) - 1 = f(a) + f(b) + f(c) - 2.$$

Case 2 $f(ab) = f(c)$**.** This condition implies

$$f(c) = f(a) + f(b) - 1. \tag{7}$$

Since $f(b) \neq 1$, identity (7) implies that $f(c) \neq f(a)$. The same way, since $f(a) \neq 1$, identity (7) also implies that $f(b) \neq f(c)$. If we suppose $bc \neq 0$ and $f(bc) = f(a)$, we get

$$f(a) = f(b) + f(c) - 1.$$

Adding to (7), we get

$$f(c) + f(a) = f(a) + f(c) + 2f(b) - 2.$$

Dividing by 2, we get $f(b) - 1 = 0$, which contradicts the hypothesis of the theorem. Finally, we conclude that $bc = 0$ or $f(bc) \neq f(a)$. In the same way if we suppose $ac \neq 0$ and $f(ac) = f(b)$, we get $f(a) - 1 = 0$, which also contradicts the hypothesis of the theorem. We conclude that $\{ac = 0$ or $f(ac) \neq f(b)\}$ and $\{bc = 0$ or $f(bc) \neq f(a)\}$. Using the same argument as in Case 1, we conclude that $\{(ac)b = 0$ or $(ac)b$ belongs to $A_\mu\}$ and $\{(bc)a = 0$ or $(bc)a$ belongs to $A_\mu\}$ where μ is given by:

$$\mu = f(a) + f(b) + f(c) - 2.$$

Finally, Theorem 3.4 item (i) implies

$$(ab)c = \frac{(f(c) - f(b))}{f(a) - f(b)}(bc)a + \frac{(f(a) - f(c))}{f(a) - f(b)}(ac)b,$$

then we get that $(ab)c \in A_\mu$. □

Corollary 3.5. *If* $\lambda, \mu, \beta \in F, \lambda \neq \mu, \lambda \neq 1, \mu \neq 1, a \in A_\lambda, b \in A_\mu, c \in A_\beta$, *then* $(ab)c \in A_{\lambda+\mu+\beta-2}$.

We do not know if A_1 is closed under multiplication, but we can prove the following result.

Theorem 3.6. *The subspace* S *spanned by the set* $\{ab \mid a \in A_x, b \in A_{2-x},\ x \in F \setminus \{1\}\}$ *is a sub-ring of* A *contained in* $A_1 \cup \{0\}$ *which absorbs multiplication from* A_1. *Furthermore* $(A_1, S, A_1) = 0$.

Proof. Suppose that $f(a) \neq f(b)$ and $f(a) + f(b) - 1 = 1$. Note that this is the case when $f(a) = x \neq 1$ and $f(b) = 2 - x$. Note also that $f(b) \neq 1$.

Since $f(a) \neq f(b)$ from Theorem 3.3 item (i), we have that for every $c \in B \setminus \{0\}$, we get:

$$(ab)c = \frac{-1}{f(a) - f(b)}\Big[(f(b) - f(c))(bc)a + (f(c) - f(a))(ca)b\Big].$$

If $f(c) = 1$, then

$$(ab)c = \frac{-1}{f(a) - f(b)}\Big[(f(b) - 1)(bc)a + (1 - f(a))(ca)b\Big].$$

Let $x = f(a), y = f(b)$. Then $(A_x A_y)A_1 \subseteq (A_y A_1)A_x + (A_1 A_x)A_y \subseteq A_x A_y \subseteq A_1$ where last two contentions are implied by Corollary 3.2.

If $s \in S, a, b \in A_1$, then identity (1) implies

$$0 = ((sa)b)e + ((sb)e)a + ((se)a)b - (sb)a)e - ((sa)e)b - ((se)b)a.$$

Thus, $0 = (sa)b + (sb)a + (sa)b - (sb)a - (sa)b - (sb)a = (a, s, b)$. □

4. Zero divisors

Zero divisors are important in order to find relations among the different eigen spaces. The following theorem is important because allows us to found zero divisors, as is showed in Corollary 4.2 and Corollary 4.3.

Theorem 4.1. *For every triplet a, b, c of nonzero elements in B, we get:*

(i) *If $f(a), f(b), f(c)$ are distinct, then $(ab)c, (bc)a, (ca)b$ are all in the eigen space with $\lambda = f(a) + f(b) + f(c) - 2$.*

(ii) *If $f(a), f(b), f(c)$ are distinct, then*

$$f(bc)(ab)c = f(ab)(bc)a \ \wedge \ f(ca)(ab)c = f(ab)(ca)b, \tag{8}$$

which implies

$$\frac{(ab)c}{f(ab)} = \frac{(bc)a}{f(bc)} = \frac{(ca)b}{f(ca)} \tag{9}$$

provided that all denominators are not zero.

Proof. Item (i) Without loss of generality it is enough to prove that $(ab)c$ is in the eigen space with value $\lambda = f(a) + f(b) + f(c) - 2$. Since $f(a), f(b), f(c)$ are all different, at most only one of them is 1. If $1 \notin \{f(a), f(b)\}$, the result follows from Theorem 3.4. Let us suppose that $f(a) = 1$ (the case $f(b) = 1$ is analogous). Theorem 3.1 says $ab = 0$ or $ab \in B$ with $f(ab) = f(a) + f(b) - 1 = f(b)$, then $(ab)c = 0$ or $(ab)c \in B$ with $f(ab) = f(b) \neq f(c)$, then $f((ab)c) = f(ab) + f(c) - 1 = f(b) + f(c) - 1 = 1 + f(b) + f(c) - 2 = f(a) + f(b) + f(c) - 2$.

Item (ii) From item (i) and Theorem 3.1, we obtain the following identities:

$$
\begin{aligned}
((ba)c)e &= (f(a)+f(b)+f(c)-2)((ab)c) \\
((bc)e)a &= (f(b)+f(c)-1)((bc)a) \\
((be)a)c &= f(b)((ab)c) \\
-((bc)a)e &= (-f(a)-f(b)-f(c)+2)((bc)a) \\
-((ba)e)c &= (-f(a)-f(b)+1)((ab)c) \\
-((be)c)a &= -f(b)((bc)a)
\end{aligned}
$$

Adding the identities and using identity (1), we get $0 = (f(b)+f(c)-1)((ab)c) + (-f(a)-f(b)+1)((bc)a)$ so that $0 = f(bc)(ab)c - f(ab)(bc)a$, from where the first identity in (8) is immediate. Second identity of (8) is obtained by permuting (a,b,c) in the first one and of course identities in (9) are obvious from (8). □

The previous theorem has two corollaries:

Corollary 4.2. *Let $f(a), f(b), f(c)$ be all distinct. If $f(ab) = 0$ and ($f(bc) \neq 0$ or $f(ca) \neq 0$), then $(ab)c = 0$.*

Proof. It is immediate from identity (8). □

Corollary 4.3. *If $f(a), f(b), f(c)$ are distinct and $f(a)+f(b) = 1$, then $(ab)c = 0$.*

Proof. Indeed the assumption $f(a), f(b), f(c)$ being distinct implies $f(b)+f(c) \neq 1$ and $f(a)+f(c) \neq 1$, because $f(a)+f(b) = 1$, and $f(b)+f(c) = 1$ imply $f(a) = f(c)$. Same argument holds for $f(a)+f(c)$. □

Next lemma gives some particular zero divisors.

Lemma 4.4. *If e is an idempotent element in A, then for all $a, b, c, d \in A$, the elements*

$$x_1 = ((((ee)e)b)c) - ((((ee)e)c)b) + ((((be)e)c)e) - ((((ce)e)b)e) - ((((be)e)e)c) + ((((ce)e)e)b), \quad (10)$$

$$x_2 = ((ee)(ad)) - ((de)(ae)) + (((ee)a)d) - (((de)a)e) - (((ae)e)d) + (((ad)e)e) \quad (11)$$

satisfy $x_1x_2 = 0$.

Proof. We want to prove that $x_1x_2 = 0$.

We produced with Albert all possible multiplications of $\{a, b, c, d, e\}$, e appearing 5 times and we assume the relation $xy - yx$ to produce a basis of the resulting algebra. We define the polynomial map $H(x, y, z, w) = (x, y, z)w + (z, y, w)x + (w, y, x)z$ so that identity (2) can be written as $H(x, y, z, w) = 0$, and we create all substitutions into $H(x, y, z, w)$ with up to 5 taps as is $((((H(x, y, z, w)r)s)t)u)v$.

We create a matrix A whose rows (except for the last one) has the coefficients of all substitutions of this 5 taps evaluation in the basis given by Albert of this algebra. The last row of A have the coefficients in the given basis of the identity we wanted to prove to be zero, x_1x_2.

The size of the basis is 55653 that corresponds to the columns of the matrix. All identities have degree nine so the number of rows of A is 29924. The columns corresponding to the elements of the basis of degree 9 are from 25729 to 55653 so the number of them is $55653 - 25729 = 29924$, which coincides with number of rows of A. To prove $x_1x_2 = 0$ it is enough to find a row vector V such that $VA = 0$ whose the last entrance V_{29924} is different from zero in the field F. If this is the case and we denote by $R_1, \ldots, R_{29924}$ the rows of A, we have that

$$\sum_{n=1}^{29924} V_n R_n = 0$$

from where we get

$$R_{29924} = -\left(\frac{1}{V_{29924}}\right)\left(\sum_{n=1}^{29923} V_n R_n)\right).$$

If we assume $H(x, y, z, w) = 0$, we get that $R_n = 0$ for every $1 \leq n \leq 29923$, then $R_{29924} = 0$ and we conclude that $x_1x_2 = 0$. Using Mathematica we found a V such that $VA = 0$ and $V_{29924} = 512$. Therefore assuming $char(F) \neq 2$, we obtain the desired V. So $x_1x_2 = 0$. $\square$

Remark 4.5. In fact we obtained a smaller row vector W satisfies $WS = 0$ where S is a smaller matrix obtain by removing rows of A, but it is easy to construct the desired V from W by adding zeroes in the columns corresponding to the deleted rows of A.

Remark 4.6. In order to have a proof with a smaller matrix A, we can use the fact that the identities included such as $H(ee, a, e, b)$ which expands to zero, can be simplified using $ee = e$. Therefore one should be able to shorten the proof when $ee = e$ is used.

We will use the fact that $x_1x_2 = 0$ to reduce $((ab)e)e$ even when a and b are in the same eigen space. We have the following result:

Theorem 4.7. *Let $a, b, c, d \in B$. If $f(b) \neq 1, f(c) \neq 1, f(b) \neq f(c), f(a) = f(d) = p$, then $(bc)\Big(p(1-2p)ad + (1-p)(ad)e + ((ad)e)e\Big) = 0$. That is, if we put $q(t) = t^2 + (1-p)t + p(1-2p)$, then $(bc)(q(R_e)(ad)) = 0$.*

Proof. We expand the individual factors in the expression of x_1, x_2.

For x_1,

$$((((ee)e)b)c) = +f(b)(bc),\ -((((ee)e)c)b) = -f(c)(cb) = -f(c)(bc),$$

$$((((be)e)c)e) = +f(b)f(b)(bc)e,\ -((((ce)e)b)e) = -f(c)f(c)(cb)e,$$

$$-((((be)e)e)c) = -f(b)f(b)f(b)(bc),\ +((((ce)e)e)b) = +f(c)f(c)f(c)(cb).$$

Therefore, $x_1 = [f(b) - f(c) - f(b)^3 + f(c)^3](bc) + [f(b)^2 - f(c)^2](bc)e$ or

$$x_1 = (f(b) - f(c))\left[\left(1 - f(b)^2 - f(b)f(c) - f(c^2)\right)(bc) + (f(b) + f(c))(bc)e\right].$$

For x_2,

$$((ee)(ad)) = +e(ad),\ -((de)(ae)) = -f(a)f(d)(ad),$$

$$(((ee)a)d) = +f(a)(ad),\ -(((de)a)e) = -f(d)(ad)e,$$

$$-(((ae)e)d) = -f(a)f(a)(ad)\ + (((ad)e)e) = +((ad)e)e.$$

We have $x_2 = +[-f(a)f(d) + f(a) - f(a)f(a)](ad) + [1 - f(d)](ad)e + ((ad)e)e$ or

$$x_2 = f(a)[-f(d) + 1 - f(a)](ad) + [1 - f(d)](ad)e + ((ad)e)e.$$

If $f(b) = f(c)$, then x_1 is zero. If $f(b) \neq f(c)$, then by Theorem 3.1 $(bc)e = (f(b) + f(c) - 1)bc$ and

$$x_1 = (f(b)-f(c))\left[\left(1-f(b)^2-f(b)f(c)-f(c^2)\right)(bc)+(f(b)+f(c))(f(b)+f(c)-1)(bc)\right].$$

In general we have that $(b-c)(1-b^2-bc-c^2+(b+c)(b+c-1)) = (b-1)(b-c)(c-1)$, then

$$x_1 = (f(b) - 1)(f(b) - f(c))(f(c) - 1)(bc). \tag{12}$$

Thus if $f(b) = 1$ or $f(c) = 1$ or $f(b) = f(c)$, we have $x_1 = 0$.

We have that

$$x_2 = f(a)[-f(d) + 1 - f(a)](ad) + [1 - f(d)](ad)e + ((ad)e)e.$$

If $f(a) \neq f(d)$, then by Theorem 3.1 $(ad)e = (f(a)+f(d)-1)ad$, and $((ad)e)e) = (f(a) + f(d) - 1)^2(ad)$. Then

$$x_2 = \left[-f(a)[f(d) - 1 + f(a)] + (1 - f(a))(f(a) + f(d) - 1) + (f(a) + f(d) - 1)^2\right]ad$$

$$= (f(a) + f(d) - 1)\left[-f(a) + 1 - f(d) + f(a) + f(d) - 1\right]ad = 0.$$

If $f(a) = f(d) = p$, then

$$x_2 = p(1 - 2p)(ad) + (1 - p)(ad)e + ((ad)e)e. \tag{13}$$

Finally if $f(b) \neq 1, f(c) \neq 1, f(b) \neq f(c), f(a) = f(d) = p$, since $x_1x_2 = 0$, identities (12) and (13) imply that

$$(bc)\Big(p(1-2p)ad + (1-p)(ad)e + ((ad)e)e\Big) = 0.$$

Now if we put $q(t) = t^2 + (1-p)t + p(1-2p)$, then we have $(bc)(q(R_e)(ad)) = 0$. $\square$

Discussion: We have a formula for the eigen space of the product when the elements come from different eigen spaces.

Theorem 4.7 says that when the elements come from the same eigen space, the eigen space of the product satisfies

$$x_2 = p(1-2p)(ad) + (1-p)(ad)e + ((ad)e)e$$

because elements like this which are not zero annihilate a good deal of the rest of the space. Namely, the product of distinct eigen spaces except $f(a) = 1$ or $f(b) = 1$.

Remark 4.8. We use Mathematica to make the numerical calculations in this paper. We use Professor Dave Jacob's program "Albert" for intuition and for an independent check of our results (see [5], [6]).

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Declarations

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