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# New Asymptotic Properties for Solutions of Fractional Delay Neutral Differential Equations

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**Abstract**

In this note, we consider new asymptotic stability properties for solutions of several fractional delay neutral differential equations of a certain type. To obtain the desired properties, we use Lyapunov's direct method, which has a wide range of applications. Finally, we draw the reader's attention to some examples supporting the obtained asymptotic stability properties and their plots under different initial conditions. With this note, we extend and improve some results previously considered in the relevant literature.

***Keywords:*** *Asymptotic stability, Continuous function, Fractional derivative, Lyapunov direct method, Neutral differential equation, Variable delay*

***AMS Subject Classification (2020):*** *34K20; 34A08; 34K40*

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## 1. Introduction

The subject of fractional calculus, which began with an exchange of information between two famous scientists, L' Hospital and Leibnitz, at the end of the 17th century, has spread widely in the scientific world and attracted the attention of many scientists. Control theory, model of neurons in biology, fluid mechanics, viscoelasticity, meteorology, biology, communication etc. fractional calculus modeled with differential equations in fields still maintains its currency today. This subject, which has become an important area of mathematics, physics, medicine, biology and engineering, is highly developed in terms of numerical and analytical solutions for mathematical nonlinear dynamic modeling. For this reason, this subject, which has become the focus of attention of the international academic community, has been addressed by many researchers and many studies published on this subject have taken their place in the relevant literature. We recommend that interested researchers examine the studies referred to in the bibliography and the sources in these studies.

Neutral delay differential equations, which have a wide range of applications in various fields such as applied mathematics, physics, engineering and ecology, are expressed as equations that include delays in both state variables

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and their time-dependent derivatives. Due to these wide areas of application, many studies have been conducted on these equations that have attracted the attention of scientists. We recommend that readers who want to learn this information in a broader context and learn the advantages of considering these equations in more detail examine the references in our study and the references included in them.

In 2014, Aguila-Camacho et al [1] presented a new lemma for the Caputo fractional derivative of a quadratic function, which allows the use of classic quadratic Lyapunov functions in many stability analyses of fractional order systems. Alkhazzan et al [2] investigated a new class of nonlinear fractional stochastic differential equations with fractional integrals and discussed existence, uniqueness, continuity of solutions and Ulam-Hyers stability with the help of Banach contraction theorem. Altun, investigated the asymptotic stability of Riemann-Liouville fractional neutral systems with variable delays by using the Lyapunov-Krasovskii functional in [3]. Altun and Tunç [4] discussed the asymptotic stability of a nonlinear fractional-order system with multiple variable delays. The authors proved a new result on the subject by means of Lyapunov-Krasovskii functional. Diethelm [5] introduced the Caputo derivative, which is close to the Riemann-Liouville derivative with different definitions of fractional derivatives. Graef et al [6] investigated Stability of nonlinear system of fractional-order volterra integro delay differential equations with Caputo fractional derivative. The authors of [6] proved some sufficient conditions for the stability of the zero solution of these equations with the help of Lyapunov and Razumikhin methods and gave explanatory examples of these conditions. Hristova and Tunç, obtained some new conditions for the stability of the solutions of the nonlinear Caputo fractional derivative and limited delay volterra integro-differential equations with the help of Lyapunov method in [7]. Kilbas et al [8] made an important contribution to the literature with a valuable work on the theory and applications of fractional differential equations. Krol, investigated the asymptotic properties of d-dimensional linear fractional differential equations with time delay in [9]. The author presented some necessary and sufficient conditions by using the inverse method. He also supported his work with two examples. Liu et al [10] discussed stability analysis of fractional nonlinear differential systems with Riemann-liouville derivative. The authors presented several sufficient conditions on asymptotic stability of fractional nonlinear systems. They supported their work with some examples. Moulai-Khatir discussed the asymptotic properties of some neutral delay differential equations, including the Riemann-Liouville fractional derivative by means of Lyapunov functions in [11]. He also supported his work with two examples. Podlubny [12] provided a valuable resource to the relevant literature in order to provide an overview of the solution methods of fractional differential equations and their applications. Tunç and Tunç proved some qualitative results of Caputo proportional integro differential equations [13] and volterra integro differential equations [14]. Stability analysis was performed on delayed bidirectional associative memory neural networks by Yang and Zhang [15] and on singular systems by Yiğit et al [16]. Yiğit and Tunç [17] proved the asymptotical stability of zero solution of a nonlinear fractional neutral system with unbounded delay by using Lyapunov-Krasovskii functionals. They also supported theır work with two examples. Some similar results were also obtained on the stability of certain type equations and systems by Yiğit [18], [19] and Zhang et al [20].

In this note, inspired by the above discussions and motivated by the paper of Kilbas et al [8], Moulai-Khatir [11] and Yiğit [18] and the references in these papers, we study the new asymptotic properties for solutions of fractional delay neutral differential equations. We use Lyapunov's direct method, which is widely used in practice, to obtain the properties we seek. By constructing new Lyapunov functions, we obtain three new asymptotic stability properties for three different equations. We draw the readers' attention to three examples that show the practical applicability of these properties we obtained theoretically, with their annotated solutions and graphs.

The next flow of our note is as follows. The second Section contains some definitions and lemmas. In the third Section, asymptotic stability conditions are obtained for some neutral delay differential equations. In the fourth Section, some application examples are given to show the applicability of the obtained conditions. The fifth Section is the conclusion section.

## 2. Preliminaries

We now present some definitions and lemmas to be used in the processes or applications for sufficient criteria to be obtained in the details of the our work.

**Definition 2.1.** [8] The Riemann-Liouville fractional derivative and integral of order $\alpha$ for a function $x(t)$ are defined as

$$\begin{aligned} {}_{t_0}D_t^{\alpha}x(t) =& \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\alpha-1}x(s)ds, \\ & 0 \le n-1 \le \alpha < n, n \in Z^+, \\ {}_{t_0}D_t^{-\alpha}x(t) =& \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds, \alpha > 0, t > t_0, \end{aligned} \tag{2.1}$$

where $\Gamma$ denotes the Gamma function and is defined as

$$\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}e^{-s}ds.$$

**Lemma 2.1.** *[11] If $\beta > \alpha > 0$ and $x(t)$ is integrable, then*

$${}_{t_0}D_t^{\alpha}({}_{t_0}D_t^{-\beta}x(t)) = {}_{t_0}D_t^{\alpha-\beta}x(t) \tag{2.2}$$

*is satisfied.*

**Lemma 2.2.** *Assume that $x(t) \in R$ be a continuous and differentiable function. If the derivative of $x(t)$ is integrable, then the following relationship is satisfied as:*

$$0.5{}_{t_0}D_t^{\alpha}x^2(t) \le x(t){}_{t_0}D_t^{\alpha}x(t), \forall \alpha \in (0,1). \tag{2.3}$$

*Proof.* To claim inequality (2.3) is equivalent to prove only that

$$x(t){}_{t_0}D_t^{\alpha}x(t) - 0.5{}_{t_0}D_t^{\alpha}x^2(t) \ge 0, \forall \alpha \in (0,1). \tag{2.4}$$

According to Newton-Leibnitz formula, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds = x(t_0) + {}_{t_0}D_t^{-1}x(t). \tag{2.5}$$

Substituting (2.5) into (2.1) and applying (2.2), we have

$$\begin{aligned} {}_{t_0}D_t^{\alpha}x(t) &= {}_{t_0}D_t^{\alpha}x(t_0) + {}_{t_0}D_t^{\alpha-1}x(t) \\ &= \frac{1}{\Gamma(1-\alpha)}[\frac{x(t_0)}{(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}x'(s)ds]. \end{aligned}$$

From here, we get

$$x(t){}_{t_0}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{x(t)x(t_0)}{(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}x(t)x'(s)ds].$$

Also, a similar calculation shows that

$$0.5{}_{t_0}D_t^{\alpha}x^2(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{x^2(t_0)}{2(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}x(s)x'(s)ds].$$

Therefore, inequality (2.4) can be rewritten as

$$\frac{1}{\Gamma(1-\alpha)}[\frac{x(t)x(t_0) - \frac{1}{2}x^2(t_0)}{(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}(x(t)-x(s))x'(s)ds] \ge 0. \tag{2.6}$$

Let us integrate by parts the second term of inequality (2.6), then we have

$$\int_{t_0}^{t}(t-s)^{-\alpha}(x(t)-x(s))x'(s)ds = \frac{(x(t)-x(t_0))^2}{2(t-t_0)^{\alpha}} + \frac{\alpha}{2}\int_{t_0}^{t}\frac{(x(t)-x(s))^2}{(t-s)^{\alpha+1}}ds.$$

Therefore, inequality (2.6) is reduced to the following form

$$\frac{1}{\Gamma(1-\alpha)}[\frac{x^2(t)}{2(t-t_0)^{\alpha}} + \frac{\alpha}{2}\int_{t_0}^{t}\frac{(x(t)-x(s))^2}{(t-s)^{\alpha+1}}ds] \ge 0. \tag{2.7}$$

This result shows that inequality (2.7) is clearly true. This completes the proof of Lemma 2.2. □

## 3. Analysis of asymptotic stability conditions for fractional neutral equations

In this section, we will establish asymptotic stability criteria for some neutral equations with mixed delays. For this, we will use the Lyapunov's direct method and some inequalities. We will also give a brief evaluation of the equations we have examined at the end of this section.

Now, we describe a new fractional neutral differential equation with constant and variable delays as:

$$\begin{aligned} {}_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t))) \\ &\quad -e(t)h(x(t-\tau_2(t)))-u(t)\int_{t-\delta_1}^{t}x(s)ds-s(t)\int_{t-\delta_2}^{t}x(s)ds, \\ {}_{t_0}D_t^{-(1-\alpha)}x(t) &= \vartheta(t), t\in[-\rho,0], \rho>0, \rho\in R, \end{aligned} \tag{3.1}$$

for $\alpha \in (0,1)$ and for all $t \geq t_0+\rho$, where $c(t), d(t), e(t), u(t), s(t), f(x(t)), g(x(t))$ and $h(x(t))$ are continuous functions in their respective arguments, with $a+b<1$ and $f(0)=g(0)=h(0)=0$. The time variable delays $\tau_1(t)$ and $\tau_2(t)$ are continuous and differentiable functions and satisfying

$$0 \leq \tau_1(t) \leq \tau_k \text{ and } \tau_1'(t) \leq \tau_K,$$
$$0 \leq \tau_2(t) \leq \tau_n \text{ and } \tau_2'(t) \leq \tau_N,$$

where $\tau_k, \tau_n, \sigma_1, \sigma_2, \delta_1$ and $\delta_2$ are real positive numbers and $\vartheta \in C([-\rho,0];R)$ with $\rho = max\{\tau_k, \tau_n, \sigma_1, \sigma_2, \delta_1, \delta_2\}$. Moreover, we assume that $f'(x(t)), g'(x(t))$ and $h'(x(t))$ are exist and continuous.

Now, we describe the operator $N$ by:

$$N(t) = x(t)+ax(t-\sigma_1)+bx(t-\sigma_2),$$

then the equation (3.1) can be rewritten as in the form below:

$$\begin{aligned} {}_{t_0}D_t^{\alpha}N(t) &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t))) \\ &\quad -e(t)h(x(t-\tau_2(t)))-u(t)\int_{t-\delta_1}^{t}x(s)ds-s(t)\int_{t-\delta_2}^{t}x(s)ds, \\ {}_{t_0}D_t^{-(1-\alpha)}x(t) &= \upsilon(t), t\in[-\rho,0], \rho>0, \rho\in R, \end{aligned} \tag{3.2}$$

Before going into the details of our study, let us assume that the following sufficient criteria are met.

**A. Assumptions**

(A1) We assume that there exist positive numbers $c_j, d_j, e_j, u_j, s_j, f_j, g_j$ and $h_j, (j=1,2)$ and $\forall x \in R-\{0\}$, such that

i) $c_1 \leq c(t) \leq c_2, d_1 \leq d(t) \leq d_2, e_1 \leq e(t) \leq e_2$, $u_1 \leq u(t) \leq u_2, s_1 \leq s(t) \leq s_2$

ii) $|f'(x)| \leq f_2, \frac{f(x)}{x} \geq f_1$

iii) $|g'(x)| \leq g_2, \frac{g(x)}{x} \geq g_1$

iv) $|h'(x)| \leq h_2, \frac{h(x)}{x} \geq h_1$

v) $2c_1f_1 > \chi$

where

$$\chi = d_2+e_2+u_2+s_2+(c_2f_2^2+c_2+d_2+e_2+u_2+s_2)(a+b)+(\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N}+\delta_1u_2+\delta_2s_2)(1+a+b).$$

**Theorem 3.1.** *We suppose that the assumptions (A1) are met, then the zero solution of fractional neutral differential equation* (3.1) *is asymptotically stable.*

*Proof.* Let us choose a new suitable Lyapunov function that can clearly be seen to be positive definite by

$$
\begin{aligned}
V(t) &= 0.5{}_{t_0}D_t^{\alpha-1}N^2(t)+\mu_1\int_{t-\sigma_1}^{t}x^2(s)ds+\mu_2\int_{t-\sigma_2}^{t}x^2(s)ds\\
&+ \lambda_1\int_{t-\tau_1(t)}^{t}x^2(s)ds+\lambda_2\int_{t-\tau_2(t)}^{t}x^2(s)ds\\
&+ \eta_1\int_{-\delta_1}^{0}\int_{t+s}^{t}x^2(\theta)d\theta ds+\eta_2\int_{-\delta_2}^{0}\int_{t+s}^{t}x^2(\theta)d\theta ds,
\end{aligned}
$$

where $\mu_1,\mu_2,\lambda_1,\lambda_2,\eta_1$ and $\eta_2$ are positive numbers.

In light of the fact that Lemma 2.1 and Lemma 2.2, by the time-derivative of $V(t)$ on the solution of equation (3.2), we can write the inequality given by

$$
\begin{aligned}
V'(t) &\le N(t){}_{t_0}D_t^{\alpha}N(t)+\mu_1x^2(t)-\mu_1x^2(t-\sigma_1)+\mu_2x^2(t)-\mu_2x^2(t-\sigma_2)+\lambda_1x^2(t)-\lambda_1(1-\tau_1'(t))x^2(t-\tau_1(t))\\
&+ \lambda_2x^2(t)-\lambda_2(1-\tau_2'(t))x^2(t-\tau_2(t))\\
&+ \delta_1\eta_1x^2(t)-\eta_1\int_{t-\delta_1}^{t}x^2(s)ds+\delta_2\eta_2x^2(t)-\eta_2\int_{t-\delta_2}^{t}x^2(s)ds\\
&\le (\mu_1+\mu_2+\lambda_1+\lambda_2+\delta_1\eta_1+\delta_2\eta_2)x^2(t)-\mu_1x^2(t-\sigma_1)\\
&- \mu_2x^2(t-\sigma_2)-\lambda_1(1-\tau_K)x^2(t-\tau_1(t))-\lambda_2(1-\tau_N)x^2(t-\tau_2(t))\\
&- \eta_1\int_{t-\delta_1}^{t}x^2(s)ds-\eta_2\int_{t-\delta_2}^{t}x^2(s)ds-c(t)f(x(t))x(t)\\
&- d(t)g(x(t-\tau_1(t)))x(t)-e(t)h(x(t-\tau_2(t)))x(t)\\
&- u(t)\int_{t-\delta_1}^{t}x(s)dsx(t)-s(t)\int_{t-\delta_2}^{t}x(s)dsx(t)-ac(t)f(x(t))x(t-\sigma_1)\\
&- ad(t)g(x(t-\tau_1(t)))x(t-\sigma_1)-ae(t)h(x(t-\tau_2(t)))x(t-\sigma_1)\\
&- au(t)\int_{t-\delta_1}^{t}x(s)dsx(t-\sigma_1)-as(t)\int_{t-\delta_2}^{t}x(s)dsx(t-\sigma_1)\\
&- bc(t)f(x(t))x(t-\sigma_2)-bd(t)g(x(t-\tau_1(t)))x(t-\sigma_2)\\
&- be(t)h(x(t-\tau_2(t)))x(t-\sigma_2)-bu(t)\int_{t-\delta_1}^{t}x(s)dsx(t-\sigma_2)\\
&- bs(t)\int_{t-\delta_2}^{t}x(s)dsx(t-\sigma_2).
\end{aligned}
$$

With the help of the inequality $2|\varpi\nu|\le\varpi^2+\nu^2$ and the assumptions given in (A1), the following result is reached:

$$
\begin{aligned}
V'(t) &\le \frac{1}{2}(-2c_1f_1+d_2+e_2+c_2f_2^2(a+b)+2\mu_1+2\mu_2+2\lambda_1+2\lambda_2+2\delta_1\eta_1+2\delta_2\eta_2+u_2+s_2)x^2(t)\\
&+ \frac{1}{2}(-2\mu_1+a(c_2+d_2+e_2+u_2+s_2))x^2(t-\sigma_1)\\
&+ \frac{1}{2}(-2\mu_2+b(c_2+d_2+e_2+u_2+s_2))x^2(t-\sigma_2)\\
&+ \frac{1}{2}(-2\lambda_1(1-\tau_K)+d_2g_2^2(1+a+b))x^2(t-\tau_1(t))\\
&+ \frac{1}{2}(-2\lambda_2(1-\tau_N)+e_2h_2^2(1+a+b))x^2(t-\tau_2(t))\\
&+ \frac{1}{2}(-2\eta_1+u_2(1+a+b))\int_{t-\delta_1}^{t}x^2(s)ds\\
&+ \frac{1}{2}(-2\eta_2+s_2(1+a+b))\int_{t-\delta_2}^{t}x^2(s)ds.
\end{aligned}
$$

Let

$$\begin{aligned} 2\mu_1 &= a(c_2+d_2+e_2+u_2+s_2), 2\mu_2 = b(c_2+d_2+e_2+u_2+s_2), \\ 2\lambda_1 &= \frac{d_2 g_2^2(1+a+b)}{(1-\tau_K)}, 2\lambda_2 = \frac{e_2 h_2^2(1+a+b)}{(1-\tau_N)}, \\ 2\eta_1 &= u_2(1+a+b), 2\eta_2 = s_2(1+a+b). \end{aligned}$$

From here, we can deduce

$$\begin{aligned} V'(t) &\leq \frac{1}{2}[(-2c_1 f_1 + d_2 + e_2 + u_2 + s_2 \\ &+ (a+b)(c_2 f_2^2 + c_2 + d_2 + e_2 + u_2 + s_2) \\ &+ (1+a+b)(\frac{d_2 g_2^2}{1-\tau_K} + \frac{e_2 h_2^2}{1-\tau_N} + \delta_1 u_2 + \delta_2 s_2]x^2(t). \end{aligned}$$

Therefore, we have

$$V'(t) \leq -m_0 x^2(t),$$

where

$$m_0 = 2c_1 f_1 - \chi > 0.$$

with

$$\begin{aligned} \chi &= d_2 + e_2 + u_2 + s_2 + (c_2 f_2^2 + c_2 + d_2 + e_2 + u_2 + s_2)(a+b) \\ &+ (\frac{d_2 g_2^2}{1-\tau_K} + \frac{e_2 h_2^2}{1-\tau_N} + \delta_1 u_2 + \delta_2 s_2)(1+a+b). \end{aligned}$$

From here, we can deduce that the zero solution of fractional neutral differential equation (3.1) is asymptotically stable. This completes the proof. □

Moreover, if the integral terms given in system (3.1) are taken to be zero then the following neutral mixed delay equation is obtained. We define the neutral mixed delay equation as:

$$\begin{aligned} {}_{t_0}D_t^{\alpha}[x(t) + ax(t-\sigma_1) + bx(t-\sigma_2)] &= -c(t)f(x(t)) - d(t)g(x(t-\tau_1(t))) - e(t)h(x(t-\tau_2(t))), \quad (3.3) \\ {}_{t_0}D_t^{-(1-\alpha)}x(t) &= \vartheta(t), t \in [-\rho, 0], \rho > 0, \rho \in R, \end{aligned}$$

for $\alpha \in (0,1)$ and for all $t \geq t_0 + \rho$, where $c(t), d(t), e(t), f(x(t)), g(x(t))$ and $h(x(t))$ are continuous functions in their respective arguments, with $a + b < 1$ and $f(0) = g(0) = h(0) = 0$. The time variable delays $\tau_1(t)$ and $\tau_2(t)$ are continuous and differentiable functions and satisfying

$$0 \leq \tau_1(t) \leq \tau_k \text{ and } \tau_1'(t) \leq \tau_K,$$
$$0 \leq \tau_2(t) \leq \tau_n \text{ and } \tau_2'(t) \leq \tau_N,$$

where $\tau_k, \tau_n, \sigma_1$ and $\sigma_2$ are real positive numbers and $\vartheta \in C([-\rho, 0]; R)$ with $\rho = max\{\tau_k, \tau_n, \sigma_1, \sigma_2\}$. Moreover, we assume that $f'(x(t)), g'(x(t))$ and $h'(x(t))$ are exist and continuous.

For simplicity, we describe the operator $N$ by:

$$N(t) = x(t) + ax(t-\sigma_1) + bx(t-\sigma_2),$$

then the equation (3.3) can be rewritten as in the form below:

$${}_{t_0}D_t^{\alpha}N(t) = -c(t)f(x(t)) - d(t)g(x(t-\tau_1(t))) - e(t)h(x(t-\tau_2(t))). \quad (3.4)$$

Before going into the details of proof of Theorem 3.2, let us assume that the following sufficient criteria are met.

**B. Assumptions**

(B1) We assume that there exist positive numbers $c_j, d_j, e_j, f_j, g_j$ and $h_j, (j = 1, 2)$ and $\forall x \in R - \{0\}$, such that

i) $c_1 \le c(t) \le c_2, d_1 \le d(t) \le d_2, e_1 \le e(t) \le e_2$

ii) $|f'(x)| \le f_2, \frac{f(x)}{x} \ge f_1$

iii) $|g'(x)| \le g_2, \frac{g(x)}{x} \ge g_1$

iv) $|h'(x)| \le h_2, \frac{h(x)}{x} \ge h_1$

v) $2c_1 f_1 > \chi$

where

$$\chi = d_2 + e_2 + (c_2 f_2^2 + c_2 + d_2 + e_2)(a+b) + (\frac{d_2 g_2^2}{1-\tau_K} + \frac{e_2 h_2^2}{1-\tau_N})(1+a+b).$$

**Theorem 3.2.** *We suppose that the assumptions (B1) are met, then the zero solution of fractional neutral differential equation* (3.3) *is asymptotically stable.*

*Proof.* Let us choose a new suitable Lyapunov function that can clearly be seen to be positive definite by

$$\begin{aligned} V(t) &= 0.5{}_{t_0}D_t^{\alpha-1}N^2(t) + \mu_1 \int_{t-\sigma_1}^t x^2(s)ds + \mu_2 \int_{t-\sigma_2}^t x^2(s)ds \\ &+ \lambda_1 \int_{t-\tau_1(t)}^t x^2(s)ds + \lambda_2 \int_{t-\tau_2(t)}^t x^2(s)ds, \end{aligned}$$

where $\mu_1, \mu_2, \lambda_1$ and $\lambda_2$ are positive numbers.

In light of the fact that Lemma 2.1 and Lemma 2.2, by the time-derivative of $V(t)$ on the solution of equation (3.4), we can write the inequality given by

$$\begin{aligned} V'(t) &\le N(t){}_{t_0}D_t^{\alpha}N(t) + \mu_1 x^2(t) - \mu_1 x^2(t-\sigma_1) + \mu_2 x^2(t) \\ &- \mu_2 x^2(t-\sigma_2) + \lambda_1 x^2(t) - \lambda_1(1-\tau_1'(t))x^2(t-\tau_1(t)) \\ &+ \lambda_2 x^2(t) - \lambda_2(1-\tau_2'(t))x^2(t-\tau_2(t)) \\ &\le (\mu_1+\mu_2+\lambda_1+\lambda_2)x^2(t) - \mu_1 x^2(t-\sigma_1) \\ &- \mu_2 x^2(t-\sigma_2) - \lambda_1(1-\tau_K)x^2(t-\tau_1(t)) - \lambda_2(1-\tau_N)x^2(t-\tau_2(t)) \\ &- c(t)f(x(t))x(t) - d(t)g(x(t-\tau_1(t)))x(t) - e(t)h(x(t-\tau_2(t)))x(t) \\ &- ac(t)f(x(t))x(t-\sigma_1) - ad(t)g(x(t-\tau_1(t)))x(t-\sigma_1) \\ &- ae(t)h(x(t-\tau_2(t)))x(t-\sigma_1) - bc(t)f(x(t))x(t-\sigma_2) \\ &- bd(t)g(x(t-\tau_1(t)))x(t-\sigma_2) - be(t)h(x(t-\tau_2(t)))x(t-\sigma_2). \end{aligned}$$

With the help of the inequality $2|\varpi\nu| \le \varpi^2 + \nu^2$ and the assumptions given in (B1), the following result is reached:

$$\begin{aligned} V'(t) &\le \frac{1}{2}(-2c_1 f_1 + d_2 + e_2 + c_2 f_2^2(a+b) + 2\mu_1 + 2\mu_2 + 2\lambda_1 + 2\lambda_2)x^2(t) \\ &+ \frac{1}{2}(-2\mu_1 + a(c_2+d_2+e_2))x^2(t-\sigma_1) \\ &+ \frac{1}{2}(-2\mu_2 + b(c_2+d_2+e_2))x^2(t-\sigma_2) \\ &+ \frac{1}{2}(-2\lambda_1(1-\tau_K) + d_2 g_2^2(1+a+b))x^2(t-\tau_1(t)) \\ &+ \frac{1}{2}(-2\lambda_2(1-\tau_N) + e_2 h_2^2(1+a+b))x^2(t-\tau_2(t)). \end{aligned}$$

Let

$$\begin{aligned} 2\mu_1 &= a(c_2+d_2+e_2), 2\mu_2 = b(c_2+d_2+e_2), \\ 2\lambda_1 &= \frac{d_2 g_2^2(1+a+b)}{(1-\tau_K)}, 2\lambda_2 = \frac{e_2 h_2^2(1+a+b)}{(1-\tau_N)}. \end{aligned}$$

From here, we can deduce

$$V'(t) \leq \frac{1}{2}[(-2c_1f_1 + d_2 + e_2 + (a+b)(c_2f_2^2 + c_2 + d_2 + e_2) + (1+a+b)(\frac{d_2g_2^2}{1-\tau_K} + \frac{e_2h_2^2}{1-\tau_N}]x^2(t).$$

Therefore, we have

$$V'(t) \leq -m_1x^2(t),$$

where

$$m_1 = 2c_1f_1 - \chi > 0.$$

with

$$\chi = d_2 + e_2 + (c_2f_2^2 + c_2 + d_2 + e_2)(a+b) + (\frac{d_2g_2^2}{1-\tau_K} + \frac{e_2h_2^2}{1-\tau_N})(1+a+b).$$

From here, we can deduce that the zero solution of fractional neutral differential equation (3.3) is asymptotically stable. This completes the proof. □

Further, we define the following fractional neutral equation (3.3) with

$$e(t)h(x(t-\tau_2(t))) = 0, \tau_1(t) = \tau(t)$$

and

$$ax(t-\sigma_1) + bx(t-\sigma_2) = ax(t-\sigma),$$

$$\begin{aligned} {}_{t_0}D_t^{\alpha}[x(t) + ax(t-\sigma)] &= -c(t)f(x(t)) - d(t)g(x(t-\tau(t))), \\ {}_{t_0}D_t^{-(1-\alpha)}x(t) &= \vartheta(t), t \in [-\rho, 0], \rho > 0, \rho \in R, \end{aligned} \tag{3.5}$$

for $\alpha \in (0,1)$ and for all $t \geq t_0 + \rho$, where $c(t), d(t), f(x(t))$ and $g(x(t))$ are continuous functions in their respective arguments, with $a < 1$ and $f(0) = g(0) = 0$. The time variable delay $\tau(t)$ is continuous and differentiable function and satisfying

$$0 \leq \tau(t) \leq \tau_k \text{ and } \tau'(t) \leq \tau_K,$$

where $\tau_k$ and $\sigma$ are real positive numbers and $\vartheta \in C([-\rho, 0]; R)$ with $\rho = max\{\tau_k, \sigma\}$. Moreover, we assume that $f'(x(t))$ and $g'(x(t))$ are exist and continuous.

For simplicity, we describe the operator $M$ by:

$$M(t) = x(t) + ax(t-\sigma),$$

then the equation (3.5) can be rewritten as in the form below:

$${}_{t_0}D_t^{\alpha}M(t) = -c(t)f(x(t)) - d(t)g(x(t-\tau(t))). \tag{3.6}$$

Before going into the details of proof of Theorem 3.3, let us assume that the following sufficient criteria are met.

**C. Assumptions**

(C1) We assume that there exist positive numbers $c_j, d_j, f_j$ and $g_j, (j = 1,2)$ and $\forall x \in R - \{0\}$, such that

i) $c_1 \leq c(t) \leq c_2, d_1 \leq d(t) \leq d_2$

ii) $|f'(x)| \leq f_2, \frac{f(x)}{x} \geq f_1$

iii) $|g'(x)| \leq g_2, \frac{g(x)}{x} \geq g_1$

iv) $2c_1f_1 > d_2 + a(c_2f_2^2 + c_2 + d_2) + \frac{d_2g_2^2(1+a)}{1-\tau_K}$

**Theorem 3.3.** *We suppose that the assumptions (C1) are met, then the zero solution of fractional neutral differential equation* (3.5) *is asymptotically stable.*

*Proof.* Let us choose a new suitable Lyapunov function that can clearly be seen to be positive definite by

$$V(t) = 0.5_{t_0} D_t^{\alpha-1} M^2(t) + \mu \int_{t-\sigma}^{t} x^2(s)ds + \lambda \int_{t-\tau(t)}^{t} x^2(s)ds,$$

where $\mu$ and $\lambda$ are positive numbers.

In light of the fact that Lemma 2.1 and Lemma 2.2, by the time-derivative of $V(t)$ on the solution of equation (3.6), we can write the inequality given by

$$\begin{aligned} V'(t) &\leq M(t)_{t_0} D_t^{\alpha} M(t) + \mu x^2(t) - \mu x^2(t-\sigma) \\ &+ \lambda x^2(t) - \lambda(1-\tau_1'(t))x^2(t-\tau(t)) \\ &\leq (\mu+\lambda)x^2(t) - \mu x^2(t-\sigma) - \lambda(1-\tau_K)x^2(t-\tau(t)) \\ &- c(t)f(x(t))x(t) - d(t)g(x(t-\tau(t)))x(t) \\ &- ac(t)f(x(t))x(t-\sigma) - ad(t)g(x(t-\tau(t)))x(t-\sigma). \end{aligned}$$

With the help of the inequality $2|\varpi\nu| \leq \varpi^2 + \nu^2$ and the assumptions given in (C1), the following result is reached:

$$\begin{aligned} V'(t) &\leq \frac{1}{2}(-2c_1 f_1 + d_2 + ac_2 f_2^2 + 2\mu + 2\lambda)x^2(t) \\ &+ \frac{1}{2}(-2\mu + a(c_2+d_2))x^2(t-\sigma) \\ &+ \frac{1}{2}(-2\lambda(1-\tau_K) + d_2 g_2^2(1+a))x^2(t-\tau(t)). \end{aligned}$$

Let

$$\begin{aligned} 2\mu &= a(c_2+d_2), \\ 2\lambda &= \frac{d_2 g_2^2(1+a)}{(1-\tau_K)}. \end{aligned}$$

From here, we can deduce

$$V'(t) \leq \frac{1}{2}[(-2c_1 f_1 + d_2 + a(c_2 f_2^2 + c_2 + d_2) + \frac{d_2 g_2^2(1+a)}{1-\tau_K}]x^2(t).$$

Therefore, we have

$$V'(t) \leq -m_2 x^2(t),$$

where

$$m_2 = 2c_1 f_1 - d_2 - a(c_2 f_2^2 + c_2 + d_2) - \frac{d_2 g_2^2(1+a)}{1-\tau_K} > 0.$$

From here, we can deduce that the zero solution of fractional neutral differential equation (3.5) is asymptotically stable. This completes the proof. □

*Remark* 3.1. If $\tau(t) = r$ is taken, then the equation (3.5) we discussed turns into equation (1) discussed in article [11]. Similarly, if $bx(t-\sigma_2) = 0$,

$$d(t)g(x(t-\tau_1(t))) + e(t)h(x(t-\tau_2(t))) = b(t)f(x(t-r))$$

and

$$u(t)\int_{t-\delta_1}^{t} x(s)ds + s(t)\int_{t-\delta_2}^{t} x(s)ds = e(t)\int_{t-\delta}^{t} x(s)ds,$$

then the equation (3.1) we discussed turns into equation (2) discussed in article [11]. It is clear from here that the sufficient conditions we obtained include the conditions obtained in the article [11]. In addition, it should be noted that some delay terms in our study are variable dependent. This shows that our article is more general. Furthermore, in the Numerical applications section, i.e. in the next section examples that embody the sufficient conditions we have obtained theoretically and images of different initial conditions will be included.

## 4. Numerical applications

In this section, we will give examples and explanatory solutions showing that the sufficient conditions we have obtained for asymptotic stability are applicable in practice. We will also include graphs showing that asymptotic stability is achieved at different initial conditions with the help of MATLAB-Simulink.

**Example 4.1.** Let us define the following fractional delay differential equation, which is the special case of fractional neutral differential equation (3.1).

$$\begin{aligned} {}_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t))) \\ &\quad -e(t)h(x(t-\tau_2(t)))-u(t)\int_{t-\delta_1}^{t}x(s)ds-s(t)\int_{t-\delta_2}^{t}x(s)ds. \end{aligned} \tag{4.1}$$

The values in this equation are as follows,

$$\begin{aligned} c_1 &= 8\leq c(t)=8+\frac{1}{5+t^2}\leq 8.2=c_2, \\ d_1 &= 0.2\leq d(t)=0.2+\frac{2}{5+t^2}\leq 0.6=d_2, \\ e_1 &= 0.3\leq e(t)=0.3+\frac{1}{2+t^2}\leq 0.8=e_2, \\ u_1 &= 0.4\leq u(t)=0.4+\frac{1}{10+t^2}\leq 0.5=u_2, \\ s_1 &= 0.6\leq s(t)=0.6+\frac{1}{5+t^2}\leq 0.8=s_2, \\ a &= \frac{1}{100}<1, b=\frac{3}{100}<1, a+b=\frac{1}{25}<1, \alpha\in(0,1), \\ 0 &\leq \tau_1(t)=0.15sin^2t\leq 0.15=\tau_k, \tau_1'(t)=0.15sin2t\leq 0.15=\tau_K, \\ 0 &\leq \tau_2(t)=0.2sin^2t\leq 0.2=\tau_n, \tau_2'(t)=0.2sin2t\leq 0.2=\tau_N, \\ f(x) &= 0.4x+\frac{x}{10+|x|}, g(x)=x+\frac{2x}{10+|x|}, h(x)=0.7x+\frac{2x}{10+|x|}. \end{aligned}$$

It is clear that $f(0)=g(0)=h(0)=0$. Additionally, $\forall x\in R, 0\leq \frac{2}{10+|x|}\leq 1$, we can deduce

$$\forall x\in R-\{0\}, \frac{f(x)}{x}\geq 0.4=f_1, \frac{g(x)}{x}\geq 1=g_1, \frac{h(x)}{x}\geq 0.7=h_1.$$

Furthermore, we can get

$$\begin{aligned} |f'(x)| &= |0.4+\frac{10}{(10+|x|)^2}|\leq 0.5=f_2, \\ |g'(x)| &= |1+\frac{20}{(10+|x|)^2}|\leq 1.2=g_2, \\ |h'(x)| &= |0.7+\frac{20}{(10+|x|)^2}|\leq 0.9=h_2, \end{aligned}$$

With the help of a simple mathematical calculation, the following conclusion is reached.

$$\begin{aligned} &-2c_1f_1+d_2+e_2+u_2+s_2+(c_2f_2^2+c_2+d_2+e_2+u_2+s_2)(a+b) \\ &+(\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N}+\delta_1u_2+\delta_2s_2)(1+a+b)=-0.91. \end{aligned}$$

From the solutions explained above, it can be seen that all criteria of Theorem 3.1. are met. Thus, the zero solution of fractional neutral differential equation (4.1) is asymptotically stable. Also, the graph showing the orbital behavior of the fractional neutral differential equation (4.1) is as follows.

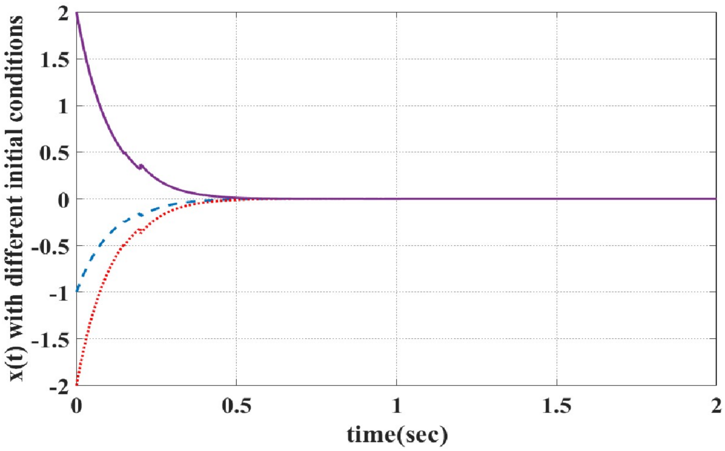


**Figure 1.** Orbital behavior of the fractional neutral differential equation (4.1).

**Example 4.2.** Let us define the following fractional delay differential equation, which is the special case of fractional neutral differential equation (3.3).

$$_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] = -c(t)f(x(t)) - d(t)g(x(t-\tau_1(t))) - e(t)h(x(t-\tau_2(t))). \tag{4.2}$$

The values in this equation are as follows,

$$\begin{aligned}
c_1 &= 6 \le c(t) = 6+\frac{1}{5+t^2} \le 6.2 = c_2,\\
d_1 &= 0.3 \le d(t) = 0.3+\frac{3}{10+t^2} \le 0.6 = d_2,\\
e_1 &= 0.5 \le e(t) = 0.5+\frac{3}{10+t^2} \le 0.8 = e_2,\\
a &= \frac{1}{100} < 1, b = \frac{3}{100} < 1, a+b = \frac{1}{25} < 1, \alpha \in (0,1),\\
0 &\le \tau_1(t) = 0.15sin^2t \le 0.15 = \tau_k, \tau_1'(t) = 0.15sin2t \le 0.15 = \tau_K,\\
0 &\le \tau_2(t) = 0.2sin^2t \le 0.2 = \tau_n, \tau_2'(t) = 0.2sin2t \le 0.2 = \tau_N,\\
f(x) &= 0.4x+\frac{x}{10+|x|}, g(x) = 0.9x+\frac{4x}{10+|x|}, h(x) = 0.7x+\frac{2x}{10+|x|}.
\end{aligned}$$

It is clear that $f(0) = g(0) = h(0) = 0$. Additionally, $\forall x \in R, 0 \le \frac{4}{10+|x|} \le 1$, we can deduce

$$\forall x \in R-\{0\}, \frac{f(x)}{x} \ge 0.4 = f_1, \frac{g(x)}{x} \ge 0.9 = g_1, \frac{h(x)}{x} \ge 0.7 = h_1.$$

Furthermore, we can get

$$\begin{aligned}
|f'(x)| &= |0.4+\frac{10}{(10+|x|)^2}| \le 0.5 = f_2,\\
|g'(x)| &= |0.9+\frac{40}{(10+|x|)^2}| \le 1.3 = g_2,\\
|h'(x)| &= |0.7+\frac{20}{(10+|x|)^2}| \le 0.9 = h_2,
\end{aligned}$$

With the help of a simple mathematical calculation, the following conclusion is reached.

$$-2c_1f_1+d_2+e_2+(c_2f_2^2+c_2+d_2+e_2)(a+b)+(\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N})(1+a+b) = -0.95.$$

From the solutions explained above, it can be seen that all criteria of Theorem 3.2 are met. Thus, the zero solution of fractional neutral differential equation (4.2) is asymptotically stable. Also, the graph showing the orbital behavior of the fractional neutral differential equation (4.2) is as follows.

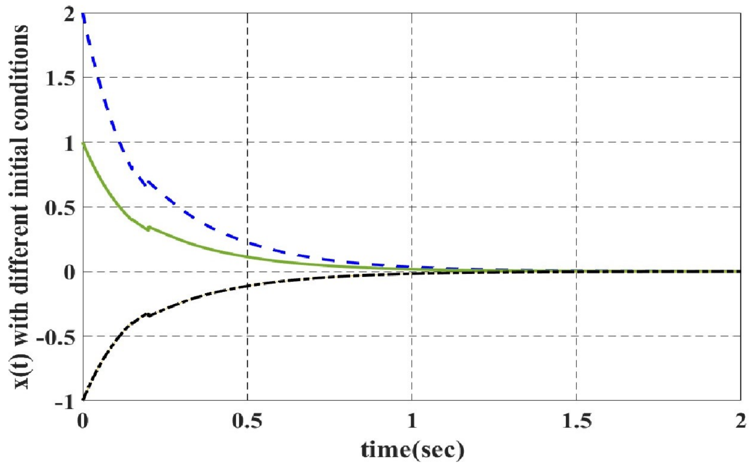


**Figure 2.** Orbital behavior of the fractional neutral differential equation (4.2).

**Example 4.3.** Let us define the following fractional delay differential equation, which is the special case of fractional neutral differential equation (3.5).

$$ {}_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma)] = -c(t)f(x(t)) - d(t)g(x(t-\tau(t))). \tag{4.3}$$

The values in this equation are as follows,

$$\begin{aligned} c_1 &= 1 \le c(t) = 1+\frac{2}{5+t^2} \le 1.4 = c_2, a = \frac{1}{50} < 1, \alpha \in (0,1), \\ d_1 &= 0.2 \le d(t) = 0.2 + \frac{3}{10+t^2} \le 0.5 = d_2, \\ 0 &\le \tau(t) = 0.15sin^2t \le 0.15 = \tau_k, \tau'(t) = 0.15sin2t \le 0.15 = \tau_K, \\ f(x) &= 0.8x + \frac{4x}{10+|x|}, g(x) = 0.6x + \frac{4x}{10+|x|}. \end{aligned}$$

It is clear that $f(0) = g(0) = 0$. Additionally, $\forall x \in R, 0 \le \frac{4}{10+|x|} \le 1$, we can deduce

$$\forall x \in R - \{0\}, \frac{f(x)}{x} \ge 0.8 = f_1, \frac{g(x)}{x} \ge 0.6 = g_1.$$

Furthermore, we can get

$$\begin{aligned} |f'(x)| = |0.8 + \frac{40}{(10+|x|)^2}| \le 1.2 = f_2, \\ |g'(x)| = |0.6 + \frac{40}{(10+|x|)^2}| \le 1 = g_2, \end{aligned}$$

With the help of a simple mathematical calculation, the following conclusion is reached.

$$-2c_1f_1 + d_2 + a(c_2f_2^2 + c_2 + d_2) + \frac{d_2g_2^2(1+a)}{1-\tau_K} = -0.42.$$

From the solutions explained above, it can be seen that all criteria of Theorem 3.3 are met. Thus, the zero solution of fractional neutral differential equation (4.3) is asymptotically stable. Also, the graph showing the orbital behavior of the fractional neutral differential equation (4.3) is as follows.

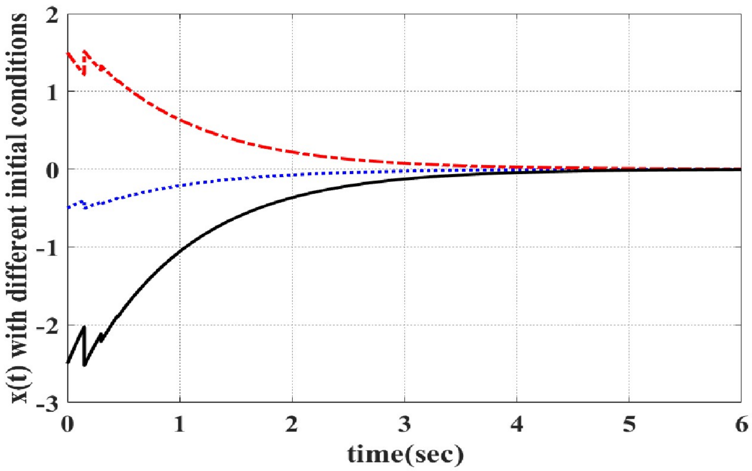


**Figure 3.** Orbital behavior of the fractional neutral differential equation (4.3).

*Remark* 4.1. When the solutions of the examples (examples 4.1, 4.2 and 4.3) given in this section are examined, the conditions that ensure stability of the zero solution of the equations discussed in a certain time interval and under different initial conditions can be easily seen. Graphs (figures 1, 2, 3) expressing these stability states are shown for different initial conditions.

In addition, it can be easily seen that the results of this study are more general when compared to the results of similar studies in the literature, especially the study we based on [11]. In this study, the time delay was taken as constant and examples showing the practical applicability of theoretical results were not supported by graphics. However, some delay terms of the equations in our study were taken as variable dependent and our examples showing the practical applicability of theoretical results were supported with graphs.

## 5. Conclusion

In this note, we have investigated the asymptotic stability of some fractional delay neutral differential equations of a certain type by applying three different Lyapunov functions. Also, we have obtained a new lemma of Riemann-Liouville derivative order of quadratic function. Based on the Lyapunov functions, some sufficient asymptotic stability conditions for these fractional delay neutral differential equations have been proved. Compared to the stability criteria in the relevant literature, our criteria are simple and applicable. To demonstrate the effectiveness of these criteria, we have given some examples with simulations (Figure1, Figure 2 and Figure 3). Theoretical findings, complemented by examples and graphical representations, provide meaningful insights into the orbital behavior of these equations. As a result, the obtained conditions extend and improve some criteria found in the relevant literature.

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