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The Comparison Between Effects of Regular and Irregular Nonlinear Elastic Layers Overlying a Rigid Substratum on Bright Solitary-Like SH Waves

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Abstract

The influences of planar and corrugated free surfaces on bright solitary-like shear horizontal (SH) waves propagating on a nonlinear elastic layer over a rigid substratum are compared. The amplitude of irregularity of the free surface is considered as small compared to the average thickness of the layer. The layers consist of compressible, homogeneous and isotropic materials. A generalized nonlinear Schrödinger (GNLS) equation for the nonlinear modulation of SH waves is obtained via an asymptotic perturbation method. A comparative study between the influences of regular and irregular layers on the bright solitary-like SH waves is presented graphically.

Keywords: Corrugated free surface, irregular layer, Generalized nonlinear Schrödinger (GNLS) equation, Bright solitary-like SH waves

Introduction

Surface elastic waves propagating on various waveguides such as layers, plates, crusted semi-spaces, etc., become dispersive because of the repeated reflection between the boundaries. SH waves propagating in the layer of the Earth have been searched by many researchers on account of their important applications in various disciplines, such as materials engineering, geophysics and petroleum engineering [1-4]. It is well known that the Earth's crustal part is not always uniform, for example continental margins, mountain roots, ore deposits etc. There are important applications of SH waves on layered media with nonuniform boundaries in various disciplines, especially seismology, as it provides a model for seismic wave propagation along continental borders and other Earth regions with variable crustal thickness. Many researchers have investigated the effects of irregular boundary surfaces of the media on the linear surface elastic wave propagation for some particular types of irregularities [5-9].

The study of the effects of not only geometrical irregularity but also structural nonlinearity on surface SH waves is among the current studies due to the above mentioned application areas [10]. In this article, we research the effect of the corrugated free surface of a nonlinear compressible layer over a rigid substratum. It is assumed that the free surface varies sinusoidally in the direction of wave propagation. The amplitude of the sinusoidal change is taken to be small compared to the layer's average thickness. A generalized nonlinear Schrödinger (GNLS) equation with variable coefficients that depend on sinusoidal function related to the irregular free surface as well as material parameters of the compressible layer is derived via the multiple scales perturbation method. When the amplitude of the irregularity vanishes, the geometry of the problem is reduced to the one of plane free surface studied in [11]. The effects of planar and sinusoidal free surfaces on bright solitary-like SH waves are compared graphically.

Formulation

Let (x_1, x_2, x_3) and (X_1, X_2, X_3) be the spatial and material coordinates of a point referred to the same rectangular Cartesian system of axes. We consider a nonlinear, elastic, irregular layer of nonuniform thickness occupying the region $0 < X_2 < h + f(X_1)$ where h is the average thickness of the layer. $f(X_1) \in C^1$ represents the irregularity of the free surface. It is assumed that the free boundary $X_2 = h + f(X_1)$ is free of traction and the displacement vanishes at the rigid boundary $X_2 = 0$.

SH deformation of a particle is

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + u(X_1, X_2, t) \tag{1}$$

where t is the time and u is the displacement of a particle. The constituent material of the layer is taken to be compressible hyper-elastic, and thus, the strain potential function is of the form $\Sigma = \Sigma(I^{(1)}, I^{(2)}, I^{(3)})$ where $I^{(\mathrm{m})}, m = 1,2,3$ are the principal invariants of the Finger's deformation tensor $\boldsymbol{c}^{(-1)} = [x_{k,K}\, x_{l,K}]$ defined as follows

$$I^{(1)}=\mathrm{tr}\, \boldsymbol{c}^{(-1)}, \quad I^{(2)}= ((\mathrm{tr}\, \boldsymbol{c}^{(-1)})^2 - \mathrm{tr}\, \boldsymbol{c}^{(-2)})/2, \quad I^{(3)}=\det \boldsymbol{c}^{(-1)}. \tag{2}$$

Let $X = X_1,\ Y = X_2,\ Z = X_3$. Assuming that Σ is an analytic function of $I^{(1)}, I^{(2)}$ and $I^{(3)}$, we can obtain the following approximate governing equation and boundary conditions

$$\frac{\partial^2 u}{\partial t^2} - c_1^2\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right) = n\left\{\frac{\partial}{\partial X}\left(\frac{\partial u}{\partial X}\mathcal{N}(u)\right) + \frac{\partial}{\partial Y}\left(\frac{\partial u}{\partial Y}\mathcal{N}(u)\right)\right\}, \tag{3}$$

$$\frac{\partial u}{\partial Y} - \frac{\mathrm{d}f}{\mathrm{dX}}\frac{\partial u}{\partial X} = 0 \ \text{ on } \ y = h + f(X), \tag{4}$$

$$u = 0 \ \text{ on } \ y = 0 \tag{5}$$

where $\mathcal{N}(u) = \left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial u}{\partial Y}\right)^2$, c_1 is linear shear velocity such that $c_1^2 = \mu/\rho$. Here $\mu = \frac{d\Sigma}{dI^{(1)}}(3,3,1)$ is linear shear modulus, ρ is density. Nonlinear material function of the layer is given as $n = \left(4\frac{d^2\Sigma}{dI^{(1)^2}}(3,3,1) + 2\frac{d^2\Sigma}{dI^{(1)}I^{(2)}}(3,3,1)\right)/\rho$.

Asymptotic analysis

We analyse the effects of slowly varying free surface on the wave propagation by the method of multiple scales. It is assumed that the amplitude of irregularity of the free surface is small compared to the average layer thickness, and that the slow change in boundary surface is represented by the scale of a small parameter ε^2, i.e. $f = f(\varepsilon^2 X)$.

The new scales are introduced by

$$\xi = \varepsilon^2 X, \qquad \tau = \varepsilon\left(\frac{1}{\varepsilon^2}\int^{\xi}\frac{1}{V_g(X')}dX' - t\right), \qquad y = Y, \tag{6}$$

where $\varepsilon > 0$ is a measure of the weakly nonlinearity and the amplitude of the geometrical variation. (ξ, τ) are the slow variables to specify the slow variations of the amplitude, V_g is group velocity. We also define the phase variable θ satisfying $\theta_X = k, \theta_t = -\omega$.

Supposing that u is function of (θ, ξ, τ, y), we can write following asymptotic expansion:

$$u = \sum_{n=1}^{\infty}\varepsilon^n u_n(\theta, \xi, \tau, y) \tag{7}$$

Employing (6) and (7) into (3) together with the boundary conditions (4-5), we obtain the hierarchy of equations. Up to $O(\varepsilon^3)$ problems are written as follows:

$$O(\varepsilon): \quad \mathcal{L}_1(u_1) = 0 \tag{8}$$

$$\frac{\partial u_1}{\partial y} = 0 \text{ on } y = \mathrm{h} + f(\xi), \tag{9}$$

$$u_1 = 0 \text{ on } y = 0. \tag{10}$$

$$O(\varepsilon^2): \quad \mathcal{L}_1(u_2) = \mathcal{L}_2(u_1) \tag{11}$$

$$\frac{\partial u_2}{\partial y} = 0 \text{ on } y = \mathrm{h} + f(\xi), \tag{12}$$

$$u_2 = 0 \text{ on } y = 0. \tag{13}$$

$$O(\varepsilon^3): \quad \mathcal{L}_1(u_3) = \mathcal{L}_2(u_2) + \mathcal{L}_3(u_1) + \mathcal{M}(u_1) \tag{14}$$

$$\frac{\partial u_3}{\partial y} - k\frac{\mathrm{d}f}{\mathrm{d}\xi}\frac{\partial u_1}{\partial \theta} = 0 \text{ on } y = \mathrm{h} + f(\xi), \tag{15}$$

$$u_2 = 0 \text{ on } y = 0 \tag{16}$$

where

$$\mathcal{L}_1(\psi) = \frac{\partial^2\psi}{\partial\theta^2} - c_1^2\left(k^2\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial y^2}\right), \qquad \mathcal{L}_2(\psi) = 2\left(c_1^2\frac{k}{V_g} - \omega\right)\frac{\partial^2\psi}{\partial\theta\,\partial\tau},$$

$$\mathcal{L}_3(\psi) = c_1^2\left(\frac{1}{{V_g}^2}\frac{\partial^2\psi}{\partial\tau^2} + 2k\frac{\partial^2\psi}{\partial\theta\,\partial\xi} + \frac{\mathrm{d}k}{\mathrm{d}\xi}\frac{\partial\psi}{\partial\theta}\right) - \frac{\partial^2\psi}{\partial\tau^2}$$

$$\mathcal{M}(\psi) = n\left(k\frac{\partial}{\partial\theta}\left(k\frac{\partial\psi}{\partial\theta}\mathrm{K}(\psi)\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\mathrm{K}(\psi)\right)\right)$$

$$K(\psi) = \left(k\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2.$$

The solution of the equation in the problem of $O(\varepsilon)$ is

$$u_1 = \sum_{l=1}^{\infty}\left\{A_1^{(l)}(\xi,\tau)e^{ilkpy} + B_1^{(l)}(\xi,\tau)e^{-ilkpy}\right\}e^{il\theta} + c.c. \tag{17}$$

Here $p = (\omega^2/(k^2c_1^2) - 1)^{\frac{1}{2}}$. $A_1^{(l)}$, $B_1^{(l)}$ are the first order amplitudes, k and ω are the wave number and frequency, respectively, "c.c." stands for the complex conjugate. Substitution of (17) in (9-10) yields

$$\boldsymbol{W}_l\boldsymbol{U}_1^{(l)} = \boldsymbol{0}, \text{ such that } \boldsymbol{U}_1^{(l)} = (A_1^{(l)}, B_1^{(l)})^T,\ l = 1,2,.. \tag{18}$$

$\boldsymbol{W}_l$, dispersion matrix, is given by

$$\boldsymbol{W}_l = \begin{pmatrix} ikple^{ilkp(h+f)} & -ikple^{-ilkp(h+f)} \\ 1 & 1 \end{pmatrix}$$

For the nontrivial solutions of (18), det $\boldsymbol{W}_1$ must be zero, it gives the following dispersion relation

$$cos[kp(h+f)] = 0 \tag{19}$$

When $f = 0$, (19) reduces to the dispersion relation for uniform layer deriven in [11].

The solution of (18) for $l = 1$ is $\boldsymbol{U}_1^{(1)} = A_1\boldsymbol{R}$, when $l \neq 1$ $\boldsymbol{U}_1^{(l)} = \boldsymbol{0}$ where the complex function of the slow variables A_1 represents the first order slowly varying amplitude of the fundamental wave. **R** satisfies $\boldsymbol{W}_1\boldsymbol{R} = 0$ and its components are

$$R_1 = 1,\ R_2 = e^{2ikp(h+f)}. \tag{20}$$

Thus, the first order solution can be expressed by

$$u_1 = A_1(R_1 e^{ikpy} + R_2 e^{-ikpy})e^{i\theta} + c.c., \tag{21}$$

To find the first order solution completely, A_1 must be determined by analysing the higher-order perturbation problem. Substituting (21) in equation (11), u_2 can be found by the method of undetermined coefficients, for comprehensive analysis see [10].

To obtain the third order solutions, u_1 and u_2 are introduced into the equation of $O(\varepsilon^3)$ given by (14). u_3 is categorized into two groups $u_3 = \overline{u_3} + \widehat{u_3}$. Here $\overline{u_3}$ is the particular solution of the equation (14) that can be found by the method of undetermined coefficients. $\widehat{u_3}$ represents the solution of the corresponding homogeneous equation and can be written in the form of the first order solution in (17) by replacing $U_1^{(l)}$ by $U_3^{(l)} = \left(A_3^{(l)},\ B_3^{(l)}\right)$. Then Eqs. (15-16) give

$$\boldsymbol{W}_l \boldsymbol{U}_3^{(l)} = \boldsymbol{b}_3^{(l)}, \tag{22}$$

where $\boldsymbol{b}_3^{(l)}$ is zero vector for $l \neq 1,3$ and

$$\boldsymbol{b}_3^{(1)} = \left[i\frac{\partial \boldsymbol{W}_1}{\partial k}\frac{\partial A_1}{\partial \xi} + \frac{1}{2}\left(\frac{\partial^2 \boldsymbol{W}_1}{\partial \omega^2} + \frac{2}{V_g}\frac{\partial^2 \boldsymbol{W}_1}{\partial k \partial \omega} + \frac{1}{{V_g}^2}\frac{\partial^2 \boldsymbol{W}_1}{\partial k^2}\right)\frac{\partial^2 A_1}{\partial \tau^2}\right]\boldsymbol{R}$$

$$+\left(\frac{1}{V_g}\frac{\partial \boldsymbol{W}_1}{\partial k} + \frac{\partial \boldsymbol{W}_1}{\partial \omega}\right)\left(\frac{1}{V_g}\frac{\partial \boldsymbol{R}}{\partial k} + \frac{\partial \boldsymbol{R}}{\partial \omega}\right)\frac{\partial^2 A_1}{\partial \tau^2} + \mathbf{F}|A_1|^2 A_1 + i\mathbf{G}A_1. \tag{23}$$

Here $\mathbf{F} = (F_1, F_2)^{\mathrm{T}}$, $\mathbf{G} = (G_1, G_2)^{\mathrm{T}}$ such that

$$F_1 = -\frac{e^{i(h+f)kp}\, n(h+f)k^4(9+2p^2+9p^4)}{c^2},\ F_2 = -\frac{\left(e^{-2i(h+f)kp} + e^{4i(h+f)kp}\right)nk^2(-3-2p^2+9p^4)}{8c^2p^2},$$

$$G_1 = e^{i(h+f)kp}k(1+p^2)f', \quad G_2 = 0.$$

Notice that $det\boldsymbol{W}_1 = 0$ and $\boldsymbol{b}_3^{(1)} \neq \mathbf{0}$. For the solution of equations (22) to exist for $\boldsymbol{U}_3^{(1)}$, the following compatibility condition must be satisfied

$$\boldsymbol{L}.\boldsymbol{b}_3^{(1)} = 0. \tag{24}$$

Here, $\boldsymbol{L}$ is a row vector satisfying $\boldsymbol{L}\boldsymbol{W}_1 = 0$. Its components are

$$L_1 = 1,\ \ L_2 = -ikpe^{i(h+f)kp}\ . \tag{25}$$

(24) gives the following GNLS equation with nondimensional variables $A = A_1/h$, $\tilde{\xi} = \xi/h$, $\tilde{\tau} = \tau\omega$.

$$i\frac{\partial A}{\partial \xi} + \Gamma\frac{\partial^2 A}{\partial \tau^2} + \Delta|A|^2 A = i\Lambda\, \mathrm{A} \tag{26}$$

where tildes on variables are omitted. The dimensionless coefficients Γ, Δ and Λ can be written as

$$\Gamma(\xi) = \frac{\omega^2 h}{2{V_g}^3}\frac{d^2\omega}{dk^2}\ ,\ \Delta(\xi) = h^3(\mathbf{L}.\mathbf{F})/\left(\mathbf{L}\frac{\partial W_1}{\partial k}\mathbf{R}\right), \qquad \Lambda(\xi) = -h(\mathbf{L}.\mathbf{G})/\left(\mathbf{L}\frac{\partial W_1}{\partial k}\mathbf{R}\right). \tag{27}$$

Now, we seek the soliton-like solutions of the GNLS equation (26) through the following ansatz

$$A(\xi, \tau) = g(\xi, \tau)e^{ir(\xi)}. \tag{28}$$

To obtain bright soliton-like solution, we define $g(\xi, \tau)$ as $g(\xi, \tau) = g_1(\xi)\, sech(\tau)$. Therefore substitution of (28) in the GNLS equation (26) gives

$$g_1(\xi) = \sqrt{\frac{2\Gamma}{\Delta}} \quad , \qquad r(\xi) = \int^{\xi} \Gamma(\dot{x})d\dot{x} \tag{29}$$

with the following integrability condition [12]

$$\Lambda = \frac{\Delta\Gamma_{\xi} - \Gamma\Delta_{\xi}}{2\Gamma\Delta}. \tag{30}$$

Hence, we obtain following bright soliton-like solution [14]

$$A(\xi, \tau) = \sqrt{\frac{2\Gamma}{\Delta}}\, sech(\tau) e^{i\int^{\xi} \Gamma(\grave{x})d\grave{x}}, \text{ for } \Gamma\Delta > 0. \tag{31}$$

Note that in cases where the integrability condition (30) is not satisfied by the variable coefficients of the GNLS equation, bright soliton-like solutions can be found numerically by means of the pseudo-spectral method [13].

Comparisons and Conclusion

We compare the influences of planar and corrugated free surfaces on bright solitary shear horizontal (SH) waves for the following two cases. In the numerical calculations, the dimensionless phase velocity $C = c/c_1$ is fixed as $C = 2$. The dimensionless wave number $K = kh$ depending on (C, ξ) is evaluated for the first branch of dispersion relation (19). We determine the variable coefficients of the NLS equation by symbolic computation in Mathematica. On the other hand, the Matlab codes of the pseudo spectral method given in [13] have been developed to obtain the bright soliton-like solutions of the GNLS equation with variable coefficients and the graphs have been plotted in Matlab.

Case 1: Plane free surface of the layer with constant thickness

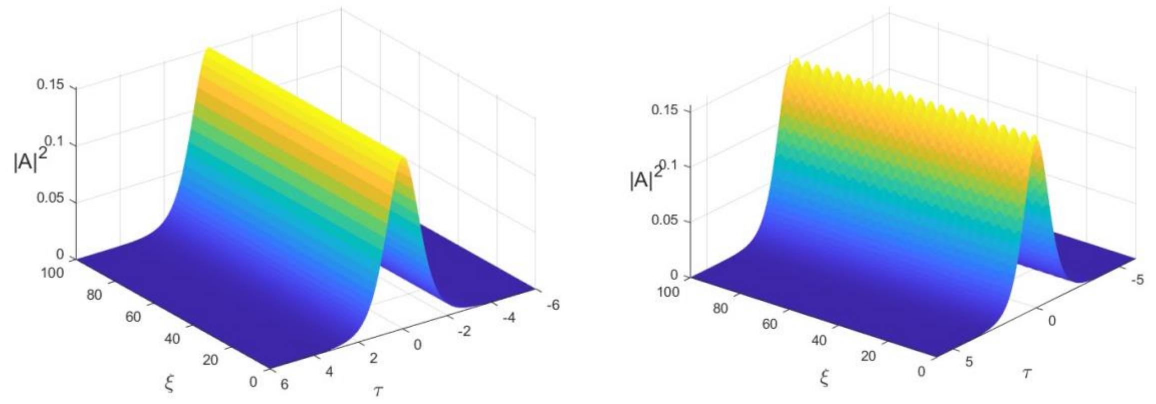
When $f = 0$, i.e. the layer has the plane free surface $y = 0$, the GNLS equation (26) is reduced to the NLS equation with constant coefficients

$$i\frac{\partial A}{\partial \xi} + \Gamma\frac{\partial^2 A}{\partial \tau^2} + \Delta|A|^2 A = 0 \tag{32}$$

which arises in the propagation of nonlinear Love waves on a layer of constant thickness overlying a rigid substratum (see e.g. [11]). Note that, $\Lambda = 0$ and the integrability condition (30) is satisfied for all constant coefficients Γ and Δ. In this case, the solution (31) represents bright soliton waves. The nonlinear evolution of the solitons, view from top and maximum amplitude of these soliton waves are shown in Figures 1(a), 2 (a), 3(a) respectively, for the nonlinear material parameter $\mathrm{n} = -2$ and hence $\Gamma\Delta > 0$.

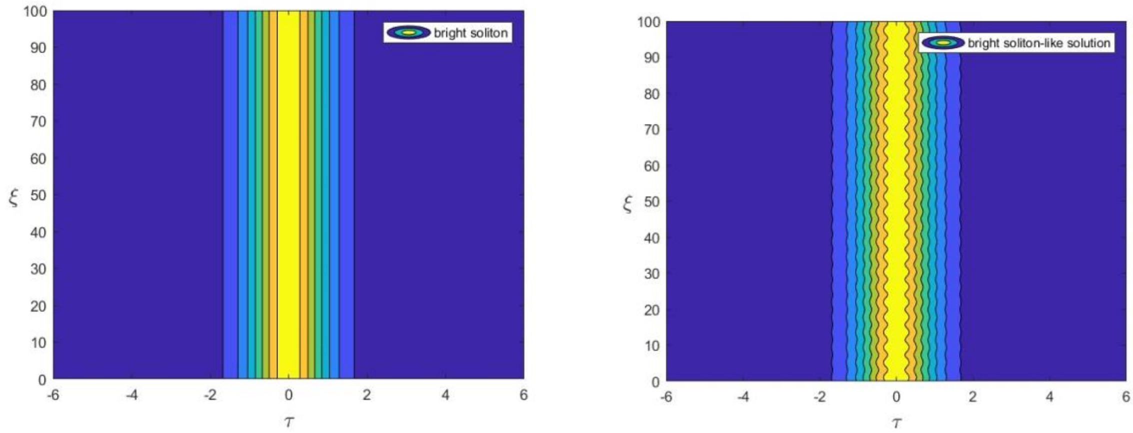
Case 2: Periodic free surface of the layer with variable thickness

For this case, we consider sinusoidally varying free surface with the choice of $f(\xi) = u \sin(k\xi)$ where u is amplitude, k is wave number and ξ is position parameter of the periodic free surface $y = \mathrm{h} + f(\xi)$. When $f(\xi) \neq 0$, integrability condition (30) does not satisfied. Hence, the bright soliton-like solution is searched numerically via pseudo-spectral method. The effect of corrugated free surface on the nonlinear evolution of the solitons, view from top and maximum amplitude as a function of ξ are observed in Figures 1(b) , 2(b), 3(b), respectively for $\mathrm{n} = -2$ and hence $\Gamma\Delta > 0$. In the calculations the dimensionless flatness parameter $U = \frac{u}{h}$, and corrugation parameter $s = k/h$ are chosen as $U = 0.03$ and $s = 1.5$.



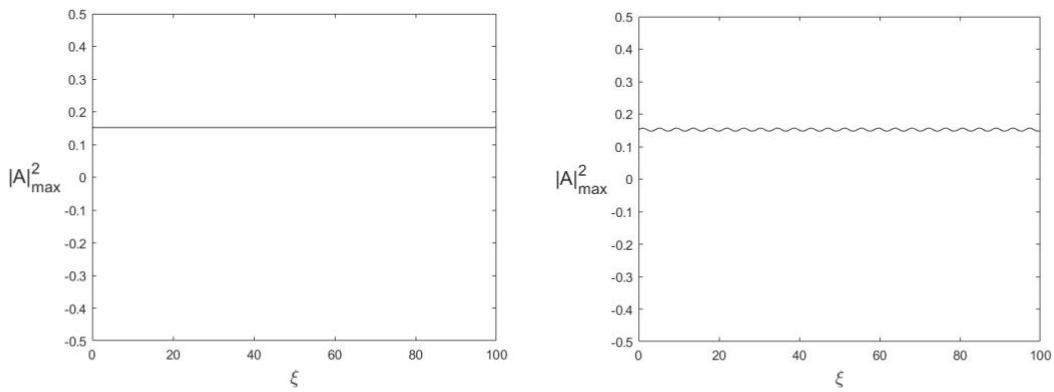
(a) (b)

Figure 1. a) Nonlinear evolution of the bright soliton for plane free surface (b) Nonlinear evolution of the bright soliton-like solution for sinusoidally varying free surface.



(a) (b)

Figure 2. View from top of (a) the bright soliton for plane free surface (b) the bright soliton-like solution for sinusoidally varying free surface.



(a) (b)

Figure 2. Maximum amplitude as a function of the propagation distance of (a) the bright soliton for plane free surface (b) the bright soliton-like solution for sinusoidally varying free surface.

We confirm by means of Fig. 1(a) that the bright soliton-like solution (31) reduces to the bright soliton solution of the NLS equation with constant coefficients (32) in the case of the plane free surface of the layer of constant thickness overlying a rigid substratum. It is observed that bright solitons preserve their profile for planar free surface, while the free surface varies sinusoidally, small variations on the free surface cause small oscillations on the bright soliton-like waves without distorting the wave profile. It is also seen that bright solitons preserve their maximum amplitude for planar free surface, while sinusoidally varying free surface cause relatively small oscillations on the maximum amplitude of the bright soliton-like wave.

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