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Research Article

On p-permutation equivalences between direct products of blocks

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Abstract

We extend the notion of a p-permutation equivalence to an equivalence between direct products of block algebras. We prove that a p-permutation equivalence between direct products of blocks gives a bijection between the factors and induces a p-permutation equivalence between corresponding blocks.

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1. Introduction

One of the main themes in representation theory of finite groups is to study equivalences between block algebras. Various authors have defined different notions of equivalences, such as Puig equivalence [6], splendid Rickard equivalence [5], derived equivalence, isotypy, perfect isometry [4], p-permutation equivalence [2],[1], and functorial equivalence [3]. Our aim in this paper is to extend the notion of a p-permutation equivalence to an equivalence between direct products of blocks.

Let G and H be finite groups. Let $p > 0$ be a prime and let $(\mathbb{K}, \mathcal{O}, k)$ denote a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring with residue field k of characteristic p and field of fractions $\mathbb{K}$ of characteristic 0. Suppose that $\mathcal{O}$ contains a root of unity whose order is equal to the exponent of $G \times H$.

Let A be a sum of blocks of $\mathcal{O}G$ and B a sum of blocks of $\mathcal{O}H$. Let $T^{\Delta}(A, B)$ denote the Grothendieck group with respect to split short exact sequences of p-permutation (A, B)-bimodules whose indecomposable summands have twisted diagonal vertices when regarded as $\mathcal{O}[G \times H]$-modules. In [1], Boltje and Perepelitsky define a p-permutation equivalence between A and B as an element $\gamma \in T^{\Delta}(A, B)$ such that

$$\gamma \cdot_H \gamma^\circ = [A] \in T^{\Delta}(A, A) \quad \text{and} \quad \gamma^\circ \cdot_G \gamma = [B] \in T^{\Delta}(B, B)$$

where γ° is the $\mathcal{O}$-dual of γ and where $\cdot_H$ is tensor product over $\mathcal{O}H$. Among many other interesting and important properties of p-permutation equivalences, they proved that if γ is a p-permutation equivalence between A and B, then there is a bijection between the block summands of A and B and γ induces a p-permutation equivalence between the corresponding blocks, see [1, Theorem 10.10]. We show that a similar phenomenon holds for p-permutation equivalences between direct products of blocks.

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Definition 1.1. Let $A_1 \times \cdots \times A_n$, $B_1 \times \cdots \times B_m$ and $C_1 \times \cdots \times C_l$ be direct products of block algebras of finite groups. Let $\gamma = (\gamma_{ij})$ and $\gamma' = (\gamma'_{jk})$ be matrices with entries $\gamma_{ij} \in T^\Delta(A_i, B_j)$ and $\gamma'_{jk} = T^\Delta(B_j, C_k)$. We denote by $\gamma \circ \gamma'$ the product of the matrices γ and γ'. More precisely,

$$(\gamma \circ \gamma')_{i,k} = \sum_{j=1}^{m} \gamma_{ij} \cdot_{H_j} \gamma'_{jk} \in T^\Delta(A_i, C_k)\,.$$

Definition 1.2. Let $G_1, \cdots, G_n$ and $H_1, \cdots, H_m$ be finite groups. Let $A_i \in \mathrm{Bl}(\mathcal{O}G_i)$ and $B_j \in \mathrm{Bl}(\mathcal{O}H_j)$ be block algebras for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. A *p-permutation equivalence* between the direct product algebras $A_1 \times \cdots \times A_n$ and $B_1 \times \cdots \times B_m$ is a matrix $\gamma = (\gamma_{ij})$ where $\gamma_{ij} \in T^\Delta(A_i, B_j)$ such that

$$\gamma \circ \gamma^\circ = \begin{pmatrix} [A_1] & 0 & \cdots & 0 \\ 0 & [A_2] & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0\cdots & [A_n] \end{pmatrix} \quad \text{and} \quad \gamma^\circ \circ \gamma = \begin{pmatrix} [B_1] & 0 & \cdots & 0 \\ 0 & [B_2] & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0\cdots & [B_m] \end{pmatrix}$$

where $\gamma^\circ = ((\gamma^\circ_{i,j})_{ij})^t$.

Our main result is the following.

Theorem 1.3. *Let $G_1, \cdots, G_n$ and $H_1, \cdots, H_m$ be finite groups. Let $A_i \in \mathrm{Bl}(\mathcal{O}G_i)$ and $B_j \in \mathrm{Bl}(\mathcal{O}H_j)$ be block algebras for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Assume that $\mathcal{O}$ contains a root of unity of order the exponent of G_i and H_j for each i and j. Let $\gamma = (\gamma_{ij})$ be a p-permutation equivalence between the direct products $A_1 \times \cdots \times A_n$ and $B_1 \times \cdots \times B_m$ of block algebras. Then $n = m$ and in each row and in each column of γ, there exists precisely one non-zero element. Moreover, if γ_{ij} is the non-zero element in the i-th row and j-th coloumn, then γ_{ij} is a p-permutation equivalence between A_i and B_j.*

2. The proof of the main theorem

Throughout $G, G_1, \cdots, G_n$, $H, H_1, \cdots, H_m$ denote finite groups. Also, $(\mathbb{K}, \mathcal{O}, k)$ denotes a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring with residue field k of characteristic p and field of fractions $\mathbb{K}$ of characteristic 0. We suppose that $\mathcal{O}$ contains a root of unity of order the exponent of G, G_i, H and H_j for all i and j. We follow the proof of [1, Theorem 10.10] closely.

2.1. We denote by $R(\mathbb{K}G)$ and $R(kG)$ the Grothendieck groups with respect to short exact sequences of $\mathbb{K}G$-modules and kG-modules, respectively, and by $T(\mathcal{O}G)$ and $T(kG)$ the Grothendieck groups with respect to split short exact sequences of p-permutation $\mathcal{O}G$-modules and p-permutation kG-modules, respectively.

We denote by $-^*$ the anti-involution $g \mapsto g^{-1}$ of any group algebra of a group G. If A is a block of $\mathcal{O}G$ and B is a block of $\mathcal{O}H$, then we can regard any (A, B)-bimodule M as an $A \otimes B^*$-module via the isomorphism $\mathcal{O}(G \times H) \cong \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}H$. We set $R(\mathbb{K}G, \mathbb{K}H) := R(\mathbb{K}[G \times H])$ and similarly define $R(A, B)$, $T(A, B)$ etc.

Let $P \leq G$ and $Q \leq H$ be subgroups and $\phi : Q \to P$ a group isomorphism. The subgroup $\Delta(P, \phi, Q) := \{(\phi(q), q) \mid q \in Q\} \leq G \times H$ is called *twisted diagonal*. We denote by $T^\Delta(A, B)$ the Grothendieck group with respect to split short exact sequences of p-permutation (A, B)-bimodules whose indecomposable summands have twisted diagonal vertices when regarded as $\mathcal{O}[G \times H]$-modules.

2.2. Let $\Delta(P, \phi, Q) \leq G \times H$ be a p-subgroup. Following the notation in [1, 10.1], for an element $\gamma \in T^\Delta(\mathcal{O}G, \mathcal{O}H)$, we write $\overline{\gamma}(P, \phi, Q)$ for the Brauer construction $\gamma(\Delta(P, \phi, Q)) \in T(kN_{G\times H}(\Delta(P, \phi, Q))$. Set $N := N_{G\times H}(\Delta(P, \phi, Q))$. The corresponding elements in the commutative diagram (see [1, 9.1(c)])

$$\begin{array}{ccc} T(\mathcal{O}N) & \xrightarrow{\kappa_N} & R(\mathbb{K}N) \\ \cong\big\downarrow & & \big\downarrow d_N \\ T(kN) & \xrightarrow[\eta_N]{} & R(kN) \end{array}$$

will be denoted by

$$\begin{array}{ccc} \gamma(P,\phi,Q) & \longrightarrow & \mu(P,\phi,Q) \\ \big\downarrow & & \big\downarrow \\ \overline{\gamma}(P,\phi,Q) & \longrightarrow & \nu(P,\phi,Q) \end{array}$$

where κ_N is induced by the scalar extension $\mathbb{K}\otimes_{\mathcal{O}} -$, d_N is the decomposition map and η_N is induced by the map $[M]\mapsto[M]$.

2.3. Let A be a block of kG and B a block of kH. Let (P,e) be an A-Brauer pair. We denote by Λ_H the set of pairs $(\phi,(Q,f))$ where (Q,f) is a kH-Brauer pair and $\phi: Q\to P$ is an isomorphism. The group $N_G(P,e)\times H$ acts on Λ_H via $(g,h)\cdot(\phi,(Q,f)) = (c_g\phi c_h^{-1}, {}^h(Q,f))$.

We also set $\Lambda_B\subseteq\Lambda_H$ to be the subset consisting of the pairs $(\phi,(Q,f))$ where (Q,f) is a B-Brauer pair. Note that Λ_B is still an $N_G(P,e)\times H$-set via the above action. We denote by $\tilde{\Lambda}_H$ a set of representatives of the H-orbits of Λ_H and set $\tilde{\Lambda}_B := \tilde{\Lambda}_H\cap\Lambda_B$.

The crucial point in the proof of Theorem 1.3 is to observe that Lemma 10.3 in [1] can be generalized as follows.

Proposition 2.1. *Let $A\in\mathrm{Bl}(\mathcal{O}G)$ be a block algebra and let $B=B_1\times\cdots\times B_m$ be a direct product of block algebras where $B_j\in\mathrm{Bl}(\mathcal{O}H_j)$. For each $j\in\{1,\cdots,m\}$, let $\gamma_j\in T^\Delta(A,B_j)$ be such that*

$$\gamma_1\cdot_{H_1}\gamma_1^\circ+\cdots+\gamma_m\cdot_{H_m}\gamma_m^\circ=[A]\in T^\Delta(A,A)\,. \tag{2.1}$$

Let also (P,e) be an A-Brauer pair. Consider the set of pairs $\Lambda_{B_j}\subseteq\Lambda_{H_j}$ as in 2.3. Then there exists a unique $j\in\{1,\cdots,m\}$ and a unique H_j-orbit of pairs $(\phi_j,(Q_j,f_j))\in\Lambda_{B_j}$ such that

$$e\mu_j(P,\phi_j,Q_j)f_j\neq 0 \quad in \quad R(\mathbb{K}C_G(P)e,\mathbb{K}C_{H_j}(Q_j)f_j)\,.$$

Moreover, $e\mu_j(P,\phi_j,Q_j)f_j$ is a perfect isometry between $\mathbb{K}C_G(P)e$ and $\mathbb{K}C_{H_j}(Q_j)f_j$ and $e\nu_j(P,\phi,Q_j)f_j\neq 0$ in $R(kC_G(P)e,kC_{H_j}(Q_j)f_j)$.

Proof. The proof of this lemma is similar to the proof of [1, Lemma 10.3]. The key point is to observe that Corollary 8.8 in [1] is still applicable in this case. We add a sketch of the proof for the convenience of the reader.

Apply the Brauer construction with respect to $\Delta(P)$ to Equation (2.1). The equality

$$\begin{aligned}[][kC_G(P)e] &= [eA\,(\Delta(P))\,e] = e\left(\sum_{j=1}^m\left(\gamma_j\cdot_{H_j}\gamma_j^\circ\right)(\Delta(P))\right)e \\ &= \sum_{j=1}^m\sum_{(\phi_j,(Q_j,f_j))\in\tilde{\Lambda}_{H_j}} e\overline{\gamma}_j\,(P,\phi_j,Q_j)\,f_j\cdot_{C_{H_j}(Q_j)} f_j\overline{\gamma^\circ}(Q_j,\phi_j^{-1},P)e\end{aligned}$$

holds in $T^\Delta\,(kC_G(P)e,kC_G(P)e)$. Lifting this equation from k to $\mathcal{O}$ and extending the scalars to $\mathbb{K}$, we get

$$[\mathbb{K}C_G(P)e]=\sum_{j=1}^m\sum_{(\phi_j,(Q_j,f_j))\in\tilde{\Lambda}_{H_j}} e\mu_j\,(P,\phi_j,Q_j)\,f_j\cdot_{C_{H_j}(Q_j)}\left(e\mu_j\,(P,\phi_j,Q_j)\,f_j\right)^\circ$$

in $R(\mathbb{K}C_G(P)e, \mathbb{K}C_G(P)e)$. The statement follows now from Corollary 8.8 and Lemma 8.11 in [1]. □

Now we can prove a weaker version of Theorem 1.3.

Corollary 2.2. *Let $A = A_1 \times \cdots \times A_n$ and $B = B_1 \times \cdots \times B_m$ be direct products of block algebras where $A_i \in \mathrm{Bl}(\mathcal{O}G_i)$ and $B_j \in \mathrm{Bl}(\mathcal{O}H_j)$ with a_i and b_j their respective identity elements. Assume that there exists a p-permutation equivalence $\gamma = (\gamma_{ij})$ between A and B. Then for each $i \in \{1, \cdots, n\}$ there exists a unique $j \in \{1, \cdots, m\}$ such that*

$$\mu_{ij} \neq 0 \quad in \quad R(\mathbb{K}G_i a_i, \mathbb{K}H_j b_j)\,.$$

This defines a bijection between the sets $\{1, \cdots, n\}$ and $\{1, \cdots, m\}$. In particular, we have $n = m$ and if A_i and B_j are corresponding blocks via the bijection above, then μ_{ij} is a perfect isometry between $\mathbb{K}G_i a_i$ and $\mathbb{K}H_j b_j$.

Proof. Let $i \in \{1, \cdots, n\}$. Since γ is a p-permutation equivalence between A and B, we have

$$\gamma_{i1} \cdot_{H_1} \gamma_{i1}^{\circ} + \cdots + \gamma_{im} \cdot_{H_m} \gamma_{im}^{\circ} = [A_i] \in T^{\Delta}(A_i, A_i)\,.$$

Proposition 2.1 applied to the A_i-Brauer pair $(\{1\}, a_i)$ implies that there exists a unique $j \in \{1, \cdots, m\}$ such that

$$\mu_{ij} \neq 0 \quad \text{in} \quad R(\mathbb{K}G_i a_i, \mathbb{K}H_j b_j)\,.$$

Since by symmetry, a similar statement holds for every element $j \in \{1, \cdots, m\}$ it follows that γ is a square matrix and in each row and in each column of γ there exists a unique entry with a nonzero image in the corresponding character ring. The last statement also follows from Proposition 2.1. □

The following is essentially Lemma 10.4 in [1]. One can easily follow the proof of Lemma 10.4 in [1] and make the necessary changes as we did in the proof of Proposition 2.1 to prove it.

Proposition 2.3. *Let $A \in \mathrm{Bl}(\mathcal{O}G)$ be a block algebra and let $B = B_1 \times \cdots \times B_m$ be a direct product of block algebras where $B_j \in \mathrm{Bl}(\mathcal{O}H_j)$. For each $j \in \{1, \cdots, m\}$, let $\gamma_j \in T^{\Delta}(A, B_j)$ be such that*

$$\gamma_1 \cdot_{H_1} \gamma_1^{\circ} + \cdots + \gamma_m \cdot_{H_m} \gamma_m^{\circ} = [A] \in T^{\Delta}(A, A)\,. \tag{2.2}$$

Let (P, e) be an A-Brauer pair and set $I = N_G(P, e)$ and $X = N_{I \times I}(\Delta(P))$. For each $j \in \{1, \cdots, m\}$ consider the set Λ_{B_j} together with its $I \times H_j$-action from 2.3. For $\lambda_j = (\phi_j, (Q_j, f_j)) \in \Lambda_{B_j}$ we set

$$J(\lambda_j) := N_{H_j}(Q_j, f_j)\,, \quad I(\lambda_j) := N_{(I, \phi_j, J(\lambda_j))} \leq I, \quad \textit{and} \quad X(\lambda_j) := N_{I \times J(\lambda_j)}(\Delta(P, \phi_j, Q_j))\,.$$

Then, $X * X(\lambda_j) = X(\lambda_j)$, and for each $\chi \in \mathrm{Irr}(\mathbb{K}X(e \otimes e^*))$, there exists a unique $j \in \{1, \cdots, m\}$ and a unique $I \times H_j$-orbit of pairs $\lambda_j = (\phi_j, (Q_j, f_j)) \in \Lambda_{B_j}$ such that

$$\chi \cdot_G^{X, X(\lambda_j)} e\mu_j(P, \phi_j, Q_j) f_j \neq 0 \quad in \quad R(\mathbb{K}[X(\lambda_j)](e \otimes f_j^*)\,.$$

Moreover, for each $\lambda_j = (\phi_j, (Q_j, f_j)) \in \Lambda_{B_j}$ satisfying this condition, one has

$$\chi \cdot_G^{X, X(\lambda_j)} e\mu_j(P, \phi_j, Q_j) f_j \in \pm \mathrm{Irr}(\mathbb{K}[X(\lambda_j)](e \otimes f_j^*)\,.$$

Remark 2.4. Suppose that we have

$$\gamma_1 \cdot_{H_1} \gamma_1^{\circ} + \cdots + \gamma_m \cdot_{H_m} \gamma_m^{\circ} = [A] \in T^{\Delta}(A, A) \tag{2.3}$$

as in Proposition 2.1. Since by Proposition 2.3, the results of Lemma 10.4 in [1] hold, it follows that Corollaries 10.5 and 10.6 in [1] are still valid in this case as well.

Corollary 2.5. *Let $A \in \mathrm{Bl}(\mathcal{O}G)$ be a block algebra with identity element a and let $B = B_1 \times \cdots \times B_m$ be a direct product of block algebras where $B_j \in \mathrm{Bl}(\mathcal{O}H_j)$ with identity element b_j. For each $j \in \{1, \cdots, m\}$, let $\gamma_j \in T^{\Delta}(A, B_j)$ be such that*

$$\gamma_1 \cdot_{H_1} \gamma_1^{\circ} + \cdots + \gamma_m \cdot_{H_m} \gamma_m^{\circ} = [A] \in T^{\Delta}(A, A)\,.$$

Then there exists a unique $j \in \{1, \cdots, m\}$ such that $\gamma_j \neq 0$ in $T^{\Delta}(A, B_j)$.

Proof. By Corollary 2.2, there exists a unique $j \in \{1, \cdots, m\}$ such that

$$\mu_j \neq 0 \quad \text{in} \quad R(\mathbb{K}Ga, \mathbb{K}H_j b_j)\,.$$

This means that for any $j' \in \{1, \cdots, m\}$ with $j' \neq j$, one has $\mu_{j'} = 0$ in $R(\mathbb{K}Ga, \mathbb{K}H_{j'} b_{j'})$. For every $A \otimes B_{j'}^*$-Brauer pair $(\Delta(P, \phi, Q), (e \otimes f^*))$, since $(\{1\}, (a \otimes b_{j'})) \leq (\Delta(P, \phi, Q), (e \otimes f^*))$ holds, [1, Corollary 10.6] implies that

$$e\mu_{j'}(P, \phi, Q)f = 0 \quad \text{in} \quad R(\mathbb{K}[C_G(P)]e, \mathbb{K}[C_{H_j}(Q)]f)\,.$$

Therefore, by [1, Corollary 10.5] one has

$$e\mu_{j'}(P, \phi, Q)f = 0 \quad \text{in} \quad R(\mathbb{K}[N_{G\times H_j}(\Delta(P, \phi_j, Q_j))](e \otimes f^*)\,.$$

This shows that the element $\gamma_{j'}$ is in the kernel of the injective map in [1, Proposition 9.2(b)] and hence equals to zero. □

Proof of Theorem 1.3: The fact that $n = m$ follows from Corollary 2.2. For each $i \in \{1, \cdots, n\}$, by Corollary 2.5, there exists a unique $j \in \{1, \cdots, m\}$ such that $\gamma_{ij} \neq 0$ in $T^{\Delta}(A_i, B_j)$. This proves the theorem. □

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