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Fixed Point Results for w-Distance Functions via Altering Distance Functions on Orthogonal Metric Spaces

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ABSTRACT

In 2018, Senapati et al. respectively defined the orthogonal lower semi continuity and presented the notion w-distance in orthogonal metric space. Also they gave a fixed point theorem which is the version of Banach fixed point theorem in orthogonal metric spaces thanks to the concept of w-distance. A fixed point theorem for w-distance functions on orthogonal metric spaces are presented in this work. This theorem is a generalization of the version of Banach fixed point theorem in orthogonal metric spaces owing to the concept of w-distance.

Keywords: w-distance, orthogonal metric space, fixed point theorems, orthogonal p-contraction

1. Introduction and Preliminaries

The foundation of Metric Fixed Point theory was laid by the famous Banach Contraction Principle [6] dated 1922. Subsequently, the concept of distance w in metric spaces was introduced by Kada, Suzuki and Takahashi [13] in 1996, and different famous results were obtained using this field.

Later, extensions of this work to Hilbert spaces and full metric spaces are also given in [4] and [18].

In 2017, orthogonal sets and orthogonal metric spaces are presented by Gordji et al.[10]. Later, extensions of this work to generalized orthogonal metric spaces and its effect on generalized convex contractions on orthogonal metric spaces were also examined in [9] and [16].

Some fixed point theorems that improve the result of Gordji et al. [10] are presented by Baghani et al. Proven by [5]. Later, a real generalization of Banach's fixed point theorem was presented by Ramezani and Baghani [17]. Some fixed point theorems on orthogonal metric spaces by changing distance functions were presented by Bilgili Güngör and Türkoğlu [7].In 2018 orthogonal lower semi continuity and concepts of distance w in orthogonal metric space were introduced by Senapati et al. [20].

Then, the concept of ϕ-Kannan orthogonal p-contraction conditions in orthogonal full metric spaces was presented by Bilgili Gungor [8].

Recently, some fixed point theorems on orthogonal metric spaces have been given (See [1-3,11,12,15,19]). In this paper, a fixed point theorem for w-distance functions on orthogonal metric spaces are presented. This theorem is a generalization of the version of Banach fixed point theorem in orthogonal metric spaces owing to the concept of w-distance.

Throughout the article, R^+, R, Z denote positive real numbers, real numbers and integers.

Definition 1. ([10]) Λ is a nonempty set and $\perp$ be a binary relation on Λ . If the following condition satisfies, then $(\Lambda, \perp)$ is called O-set.

$\exists t_0 \in X; (\forall s \in \Lambda, s \perp t_0) \vee (\forall s \in \Lambda, t_0 \perp s)$ (1.3)

And t_0 is called an orthogonal element.

Example 2. ([9]) $\Lambda = Z$. Define $t \perp s$ if there exists $p \in Z$ such that $t = ps$. $(\Lambda, \perp)$ is an O-set.

Indeed, $0 \perp s$ for all $s \in Z$.

Definition 4. ([10]) $(\Lambda, \perp)$ be an orthogonal set. For any two elements $s, t \in \Lambda$, $s \perp t$ or $t \perp s$ then these elements are said to be orthogonally related.

Definition 5. ([10]) For a sequence $\{t_n\}$, if

$(\forall n \in N; t_n \perp t_{n+1)} \vee (\forall n \in N; t_{n+1} \perp t_{n)}$ (1.4)

then $\{t_n\}$ is called orthogonal sequence (shortly O-sequence). And a Cauchy sequence $\{t_n\}$, if

$(\forall n \in N; t_n \perp t_{n+1)} \vee (\forall n \in N; t_{n+1} \perp t_{n)}$ (1.5)

then $\{t_n\}$ is said to be an orthogonally Cauchy sequence (O-Cauchy sequence).

Definition 6. ([10]) $(\Lambda, \perp)$ be an orthogonal set and μ be a usual metric on Λ. In this case,

$(\Lambda, \perp, \mu)$ is called an orthogonal metric space (O-metric space).

Definition 7. ([10]) An orthogonal metric space $(\Lambda, \perp, \mu)$ is said to be a complete O-metric space (O-complete) if every O-Cauchy sequence converges in Λ.

Definition 8. ([10]) $(\Lambda, \perp, \mu)$ be an orthogonal metric space. A function $h: \Lambda \to \Lambda$ is said

to be orthogonally continuous ($\perp$-continuous) at t if for each O-sequence $\{t_n\}$ converging to t implies $ht_n \to ht$ as $n \to \infty$. Also h is $\perp$-continuous on Λ if h is $\perp$-continuous in each $t \in$ A.

Definition 9. ([10]) $(\Lambda, \perp, \mu)$ be an orthogonal metric space and $k \in R, 0 < k < 1$. A function $h: \Lambda \to \Lambda$ is said to be orthogonal contraction ($\perp$-contraction) with Lipschitz constant k if

for all $t, s \in \Lambda$ whenever $t \perp s$. $\mu(ht, hs) \leq k\mu(t, s)$ (1.6)

Definition 10. ([10]) $(\Lambda, \perp, \mu)$ be an orthogonal metric space. A function $h: \Lambda \to \Lambda$ is called orthogonal preserving (⊥-preserving) if $ht \perp hs$ whenever $t \perp s$.

Theorem 12. ([10]) $(\Lambda, \perp, \mu)$ be an O-complete metric space and $0 < k < 1$. Let h: $\Lambda \to \Lambda$ be ⊥-continuous , ⊥-contraction (with Lipschitz constant k) and ⊥-preserving. Then h has a unique fixed point $t^* \in \Lambda$ and is a Picard operator, that is, $\lim_{n\to\infty} t_n = t^*$ for all t$\in \Lambda$.

And in [20], notable definitions and fixed point theorems on orthogonal metric spaces via the concept of w-distance are presented by Senapati et al.

Definition 20. ([14]) A function $\psi: [0,\infty) \to [0,\infty)$ satisfies the properties

(i) $\psi(m)$ is continuous and nondecreasing,

(ii) $\psi(m) = 0$ if and only if $m = 0$.

Then this function is called an altering distance function. The set of alterne distance functions ψ is denoted by ψ.

2. Main Results

Theorem 21. Let $(\Lambda, \perp, \mu)$ be an O-complete metric space with transitive relation ⊥ and a w-distance σ, $\beta : [0,\infty) - \{0\} \to [0,1)$ be decreasing function such that β(m) < 1 for every m> $0, \psi \in \psi$ be a $\text{sub} - \text{additive function}$ and $h : \Lambda \to \Lambda$ be a self map **.** Suppose that the inequality

$$\psi(\sigma(ht, hs)) \le \beta(\sigma(t, s))\psi(\sigma(t, s)) \quad (2.1)$$

Satisfies for all orthogonally related $t, s \in \Lambda$ whenever $t \neq s$ fort the orthogonally preserving self mapping h. Then, for any orthogonal element $t_0 \in \Lambda$, there exists a point $t^* \in \Lambda$ and the iteration sequence $\{h^n t_0\}$ converges to this point. Also, if h is orthogonal continuous at $t^* \in \Lambda$, then $t^* \in \Lambda$ is a unique fixed point of h.

Proof. Since $(\Lambda, \perp)$ is an O-set,

$$\exists t_0 \in X;\ (\forall s \in \Lambda, s \perp t_0) \vee (\forall s \in \Lambda, t_0 \perp s). \quad (2.2)$$

And since h is a self mapping on Λ, for any orthogonal element $t_0 \in \Lambda, t_1 \in \Lambda$ can be chosen as $t_1 = ht_0$. In this case,

$$t_0 \perp ht_0 \vee ht_0 \perp t_0 \Rightarrow t_0 \perp t_1 \vee t_1 \perp t_0. \quad (2.3)$$

If continued in a similar manner , $\{h^n t_0\}$ is an iteration sequence. If $t_n = t_{n+1}$ for any $n \in N$, then we get $t_n = ht_n$ and so h has a fixed point. Assume that $t_n \neq t_{n+1}$ for all $n \in N$.

Since h is ⊥-preserving, $\{h^n t_0\}$ is an O-sequence and by using inequality (2.1)

$$\begin{aligned}\psi(\sigma(t_{n+1},t_n)) &= \psi\big(\sigma(ht_n,ht_{n-1})\big)\\ &\leq \beta\big(\sigma(t_n,t_{n-1})\big)\psi\big(\sigma(t_n,t_{n-1})\big)\\ &< \psi(\sigma(t_n,t_{n-1})). \qquad (2.4)\end{aligned}$$

Since $\psi \in \psi, \{\sigma(t_{n+1},t_n)\}$ is a sequence of decreasing nonnegative real numbers. So there is a $w \geq 0$ and $\lim_{n\to\infty} \sigma(t_{n+1},t_n) = w$. We will show that $w = 0$. Suppose, on the contrary, that $w > 0$. In this case, by taking the limit $n \to \infty$ in inequality (2.4) and the continuoty of ψ, we obtain

$$\psi(w) < \psi(w). \qquad (2.5)$$

This is a contradiction. Therefore we get $w = 0$. Next, we will prove that $\{t_n\}$ is an Cauchy sequence. If $\{t_n\}$ is not an O-Cauchy sequence, by using Lemma 16 (L3), there exists a sequence $\{r_n\}$ of positive real numbers converging to 0 and the corresponding

subsequences $\{p(n)\}$ and $\{s(n)\}$ of N satisfying $p(n) > s(n)$ for which

$$\sigma(t_{p(n)},t_{s(n)}) > r_{p(n)}. \qquad (2.6)$$

Thus, there exists $\partial > 0$ which satisfies

$$\sigma(t_{p(n)},t_{s(n)}) > r_{p(n)} \geq \partial. \qquad (2.7)$$

If $p(n)$ is chosen as the smallest integer satisfying (2.6), that is

$$\sigma(t_{p(n)-1}, t_{s(n)}) < \partial. \qquad (2.8)$$

By (2.6),(2.8) and triangular inequality of σ, we easily derive that

$$\partial \leq \sigma(t_{p(n)},t_{s(n)}) \leq \sigma(t_{p(n)},t_{s(n)-1}) + \sigma(t_{p(n)-1},t_{s(n)}) < \sigma(t_{p(n)},t_{p(n)-1}) + \partial. \qquad (2.9)$$

Letting $n \to \infty$, by using $\lim_{n\to\infty} \sigma(t_{n+1},t_n) = 0$ we get

$$\lim_{n\to\infty} \sigma(t_{p(n)},t_{s(n)}) = \partial. \qquad (2.10)$$

Also, for each $n \in N$, by using the triangular inequality of σ,

$$\begin{aligned}\sigma(t_{p(n)},t_{s(n)}) - \sigma(t_{p(n)},t_{p(n)+1}) - \sigma(t_{s(n)+1},t_{s(n)}) &\leq \sigma(t_{p(n)+1},t_{s(n)+1})\\ &\leq \sigma(t_{p(n)},t_{p(n)+1}) + \sigma(t_{p(n)},t_{s(n)}) + \sigma(t_{s(n)+1},t_{s(n)}). \qquad (2.11)\end{aligned}$$

Taking the limit as the $n \to \infty$ in the last inequality we obtain

$$\sigma(t_{p(n)+1},t_{s(n)+1}) = \partial. \qquad (2.12)$$

Using the inequality (2.1), transitivity of orthogonality relation and the triangular inequality of σ,

$$\psi(\sigma(t_{p(n)+1},t_{s(n)+1})) = \psi\Big(\sigma\big(ht_{p(n)},ht_{s(n)}\big)\Big)$$

$$\leq \beta(\sigma(t_{p(n)}, t_{s(n)}))\,\psi(\sigma(t_{p(n)}, t_{s(n)}))$$

$$\leq \beta(\partial)\,\psi(\sigma(t_{p(n)}, t_{s(n)})). \tag{2.13}$$

Taking the limit as the $n \to \infty$ in the last inequality we obtain

$$\psi(\partial) \leq \beta(\partial)\,\psi(\partial) < \psi(\partial). \tag{2.14}$$

It is a contradiction. Therefore $\{t_n\}$ is a O-Cauchy sequence. By the O-completeness of Λ, there exists $t^* \in \Lambda$ such that $\{t_n\} = \{h^n t_0\}$ converges to this point.

Now we show that t^* is a fixed point of h when h is ⊥-continuous at $t^* \in \Lambda$. Assume that h is ⊥-continuous at $t^* \in \Lambda$. Thus,

$$t^* = \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} h(t_n) = ht^* \tag{2.15}$$

So $t^* \in \Lambda$ is a fixed point of h.

Now we can show the uniqueness of the fixed point. Suppose that there exist two distinct fixed points t^* and s^*. Then,

(i) If $t^* \perp s^* \vee s^* \perp t^*$, by using the inequality (2.1)

$$\psi(\sigma(t^*, s^*)) = (p(ht^*, hys^*))$$

$$\leq \beta\big(\sigma(t^*, s^*)\big)\,\psi\big(\sigma(t^*, s^*)\big)$$

$$< \psi(\sigma(t^*, s^*)) \tag{2.16}$$

which is a contradiction so that $t^* \in \Lambda$ is a unique fixed point of h.

(ii) If not, for the chosen orthogonal element $t_0 \in \Lambda$,

$$[(t_0 \perp t^*) \wedge (t_0 \perp s^*)] \vee [(t^* \perp t_0) \wedge (s^* \perp t_0)] \tag{2.17}$$

and since h is ⊥- preserving.

$$[(ht_n \perp t^*) \wedge (ht_n \perp s^*)] \vee [(t^* \perp ht_n) \wedge (s^* \perp ht_n)] \tag{2.18}$$

is obtained. Now, by using the triangular inequality of σ, ψ is nondecreasing sub-additive function and the inequality (2.1)

$$\psi(\sigma(t^*, s^*)) = \psi(\sigma(ht^*, hs^*))$$

$$\leq \psi(\sigma(ht^*, ht_{n+1}) + \sigma(ht_{n+1}, hs^*))$$

$$\leq \psi(\sigma(ht^*, h(ht_n))) + \psi(\sigma(h(ht_n), hs^*))$$

$$\leq \beta(\sigma(t^*, ht_n))\,\psi(\sigma(t^*, ht_n)) + \beta(\sigma(ht_n, s^*))\,\psi(\sigma(ht_n, s^* \tag{2.19}$$

and taking limit $n \to \infty$, we get that $t^* = s^*$. Thus, $t^* \in \Lambda$ is a unique fixed point of h.

Example 22. Let $\Lambda = [0,1)$ be a set and define $\mu: \Lambda \times \Lambda \to R$ such that $\mu(t,s) = |t-s|$. Let binary relation $\perp$ on Λ such that $t \perp s \leftrightarrow ts \leq \max\left\{\frac{t}{5}, \frac{s}{5}\right\}$. Then $(\Lambda, \perp)$ is an orthogonal set and μ is a metric on Λ. So $(\Lambda, \perp, \mu)$ is an orthogonal metric space. In this space, any orthogonal Cauchy sequence is convergent. Indeed, any $\{t_n\}$ is an arbitrary orthogonal Cauchy sequence in Λ, then there exists a subsequence $\{t_{n_m}\}$ of $\{t_n\}$, for all $n \in N$ $t_{n_m} = 0$ or a subsequence $\{t_{n_m}\}$ of $\{t_n\}$, for all $n \in N$ $t_{n_m} \leq \frac{1}{5}$. So this subsequence is convergent in Λ. Every Cauchy sequence with a convergent subsequence is convergent, so $\{t_n\}$ is convergent in Λ. So, $(\Lambda, \perp, \mu)$ is an orthogonal complete metric space. Consider $\sigma: \Lambda \times \Lambda \to [0,\infty)$, $\sigma(t,s) = s$ which is a w-distance on X. Let $h: \Lambda \to \Lambda$, if $0 \leq t \leq \frac{1}{5}$ then $h(t)=\frac{t}{5}$

and if $\frac{1}{5} < t < 1$ then $h(t)$=0.

In this case, h is orthogonal preserving mapping. Indeed, suppose that $t \perp s$. Without loss of generality, $ts \leq \frac{t}{5}$ can be chosen. So, $t \geq 0$ and $0 \leq s \leq \frac{1}{5}$. Thus, two cases are obtained:

Case I: $0 \leq t \leq \frac{1}{5}$ and $s \leq \frac{1}{5}$; then $h(t) = \frac{t}{5}, h(s) = \frac{s}{5}$.

Case II: $1 > t > \frac{1}{5}$ and $s \leq \frac{1}{5}$; then $h(t) = 0, h(s) = \frac{s}{5}$.

These cases imply that $h(t) \perp h(s)$.

Consider $\psi: [0,\infty) \to [0,\infty)$, $\psi(m) = \frac{m}{2}$ and $\beta: [0,\infty) - \{0\} \to [0,1)$, $\beta(m) = \frac{k}{2}$, $0 < k < 1$.

In this case, h satisfies inequality (2.1). Indeed, for any orthogonally related $t,s \in \Lambda$, $ts \leq \max\left\{\frac{t}{5}, \frac{s}{5}\right\}$ is obtained. Then, there are two cases:

Case I: Suppose that $\frac{t}{5} \geq \frac{s}{5}$, and so $ts \leq \frac{t}{5}$. Then, $(s \leq \frac{1}{5}) \Lambda (t \leq \frac{1}{5})$ or $(s \leq \frac{1}{5}) \Lambda (t > \frac{1}{5})$; in both cases, inequality (2.1) satisfied.

Case II: Suppose that $\frac{t}{5} < \frac{s}{5}$, and so $ts \leq \frac{s}{5}$. Then, $(t \leq \frac{1}{5}) \Lambda (s \leq \frac{1}{5})$ or $(t \leq \frac{1}{5}) \Lambda (s > \frac{1}{5})$; in both cases, inequality (2.1) satisfied.

Therefore, all hypotheses of Theorem 21 are satisfied. For any orthogonal element $t_0 \in \Lambda$, iteration sequence $\{h^n t_0\}$ converges to $t^* = 0 \in \Lambda$. T is $\perp$-continuous at $t^* \in \Lambda$, so this point is the unique fixed point of h.

If assumed to be ψ is an identity function and $\beta(m) = k < 1$ for every m > 0 is a constant function in Theorem 21, the following Corollary is obtained. Also, it is clear that the transitivity of orthogonality relation is not necessary in the following corollary therefore it is omitting from the hypothesis.

Corollary 23. Let $(\Lambda, \perp, \mu)$ be an O-complete metric space with a w-distance $\sigma, h : \Lambda \to \Lambda$ be a self map. Suppose that there exists a $k \in [0,1)$ and h is $\perp$-preserving self mapping satisfying the inequality $\sigma(ht, hs)) \leq k\psi(\sigma(t,s))$ (2.20)

for all orthogonally related $t, s \in \Lambda$ whenever $t \neq s$. In this case, there exists a point $t^* \in \Lambda$ such that for any orthogonal element $t_0 \in \Lambda$, the iteration sequence $\{h^n t_0\}$ converges to this point. Also, if h is $\perp$-continuous at $t^* \in X$, then $t^* \in \Lambda$ is a unique fixed point of h.

Thus, one can see that Theorem 21 is a generalization of the Theorem 19 given in [20].

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