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Ruled Surfaces with Bishop vectors via Smarandache geometry

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ABSTRACT

The objective of the present study is to examine the novel ruled surfaces that are generated by Bishop frame vectors through the conceptual framework of Smarandache geometry. The fundamental forms and the associated curvatures were determined for each ruled surface, thereby establishing its developability and minimality characteristics. Furthermore, the properties of the base curve and the corresponding striction curves of each surface were discussed through asymptoticity, geodesicity, and principal line. It has been observed that the characteristics of certain constructed ruled surfaces are directly influenced by a ruled surface designed by Bishop vectors of a slant helix-like curve.

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1. Introduction

Ruled surfaces are engaged to a broader range of areas such as engineering, computational constructions, architectural structures, computer graphics, works of art, textile, automobile industry, etc. Since they are mostly referred in computer aided geometric designs (CAGDs) to deal with real world problems and to model the real objects, introducing new ruled surfaces generated by different methods will lead new potentials to the related fields. Providing their characteristics may also enable easy adaptations for interested researchers. The basic theory on ruled surfaces is discussed in many differential geometry textbooks such as [1–4]. Generalization of ruled surfaces can however, be found in [5]. Moreover, some properties of the ruled surfaces with Frenet frame of a non-cylindrical ruled surface were investigated in [6]. The characteristics for the ruled surfaces according to Bishop frame ([7]) were examined in [8] and in [9], separately, whereas the main properties of ruled surfaces according to alternative frame were studied in [10]. Sannia frame based ruled surfaces were studied in [11] while quasi ruled surfaces were defined and examined in [12]

Recently, a new notion for generating new ruled surfaces has been given in [13] by taking the advantage of the idea of Smarandache geometry which was introduced in [14, 15]. By assigning the base curve as one of the Smarandache curves and assigning the other vector element of Frenet frame as ruling, the new ruled surfaces are named as the Smarandache ruled surfaces according to Frenet frame. The same method of generating such ruled surfaces is applied to the Darboux frame in [16], and according to the alternative frame in [17]. In [18–20], new ruled surfaces are examined and their characteristics are discussed by benefiting the similar techniques.

In this study, the novel concept of Smarandache ruled surfaces according to the Bishop frame was examined. Subsequently, certain characteristics were outlined, including developability and minimality, under the specified conditions. Furthermore, the constraints that a curve be asymptotic, geodesic, or principal line on each surface were investigated. Finally, the research was supported by the presentation of illustrative graphs of corresponding Smarandache ruled surfaces.

2. Preliminaries

This section reviews some fundamental concepts that are referenced throughout the paper.
Let $\gamma : R \to R^3$ be a regular *unit speed* curve in three dimensional Euclidean space and denote $\{T, N, B\}$ as the Frenet frame and $\{T, N_1, N_2\}$ as the Bishop frame of γ. Then, the corresponding Frenet and Bishop formulae are given as

$$\begin{aligned} T' &= \kappa N & T' &= k_1 N_1 + k_2 N_2 \\ N' &= -\kappa T + \tau B, & N_1{}' &= -k_1 T, \\ B' &= -\tau N & N_2{}' &= -k_2 T \end{aligned} \tag{2.1}$$

where $'$ stands for the derivative with respect to the arc length parameter s. The relations among the components and the curvatures of two frames are given as:

$$\begin{aligned} T &= \gamma', \\ N &= cos\theta N_1 + sin\theta N_2, \\ B &= -sin\theta N_1 + cos\theta N_2, \end{aligned} \tag{2.2}$$

and

$$k_1 = \kappa cos\theta, \qquad k_2 = \kappa sin\theta, \qquad \kappa = \sqrt{k_1^2 + k_2^2},$$
$$\theta = arctan\left(\frac{k_2}{k_1}\right), \qquad \tau = \theta'. \tag{2.3}$$

On the other hand, a surface is said to be ruled if it is formed with a straight line $X(s)$ that moves along the curve $\gamma(s)$. The parametric representation for a ruled surface is given by the following:

$$\chi(s, v) = \gamma(s) + vX(s), \tag{2.4}$$

where $\gamma(s)$ is the base curve, whereas $X(s)$ is the generator (ruling). The unit normal vector field of $\chi = \chi(s, v)$ is computed as

$$n_\chi = \frac{\chi_s \times \chi_v}{\|\chi_s \times \chi_v\|}, \tag{2.5}$$

where χ_s and χ_v are the partial derivatives of χ with respect to s and v, respectively. The striction curve of the ruled surface χ is defined to be as

$$\bar{\gamma} = \gamma - \frac{\langle \gamma', X' \rangle}{\|X'\|^2} X. \tag{2.6}$$

Moreover, the first and second fundamental forms are defined by

$$\begin{aligned} I &= Eds^2 + 2Fdsdv + Gdv^2, \\ II &= Lds^2 + 2Mdsdv + Ndv^2, \end{aligned} \tag{2.7}$$

where the corresponding coefficients are

$$\begin{aligned} E &= \langle \chi_s, \chi_s \rangle, & F &= \langle \chi_s, \chi_v \rangle, & G &= \langle \chi_v, \chi_v \rangle, \\ L &= \langle \chi_{ss}, n \rangle, & M &= \langle \chi_{sv}, n \rangle, & N &= \langle \chi_{vv}, n \rangle. \end{aligned} \tag{2.8}$$

Regarding to the given coefficients, the Gaussian K and the mean H curvatures for a ruled surface are defined by

$$K = -\frac{M^2}{EG - F^2}, \qquad H = \frac{LG - 2MF}{2(EG - F^2)}, \tag{2.9}$$

respectively. In relation to the Gaussian and mean curvatures, the following proposition exists:

Proposition 2.1 ([1–3]) *A surface is developable (resp., minimal), if the Gaussian (resp., mean) curvature vanishes.*

Furthermore, the normal curvature, the geodesic curvature and the geodesic torsion of a ruled surface $\chi(s, v)$ are defined as:

$$\begin{aligned} \kappa_n &= \langle \gamma'', n_\chi \rangle, \qquad \kappa_g = \langle n_\chi \times T, T' \rangle, \\ \tau_g &= \langle n_\chi \times n_\chi{}', T' \rangle, \end{aligned} \tag{2.10}$$

respectively. According to the given relations above, the following propositions exist for the ruled surface $\chi(s, v)$:

Proposition 2.2 ([1–3])

- *The curve γ is an asymptotic line on the surface χ, if the normal curvature κ_n vanishes,*
- *The curve γ is a geodesic on the surface χ, if the geodesic curvature κ_g vanishes,*
- *The curve γ is a principal line on the surface χ, if the geodesic torsion τ_g vanishes.*

The following theorem is also needed since it is referred on characterization of the constructed surface in the next sections:

Theorem 2.1 ([8, 21]) *If N_1 has a constant angle with a fixed unit vector, then the curve γ is said to be a slant helix. Correspondingly, γ is a slant helix if and only if $\dfrac{k_1}{k_2} = constant$.*

3. Smarandache ruled surfaces according to Bishop frame

In this section, new ruled surfaces will be defined according to Bishop frame by referring to Smarandache geometry. The characteristics for each surface will also be outlined in the context.

3.1. The characteristics of TN_1 Smarandache ruled surface

Definition 3.1 *Let $\gamma(s): s \in I \subset R \to R^3$ be a unit speed curve in E^3, and denote $\{T(s),\ N_1(s),\ N_2(s)\}$ as the Bishop frame of γ. The ruled surface with base TN_1 Smarandache curve and with ruling N_2 is called a TN_1 Smarandache ruled surface which is defined by*

$$\xi(s,v) = \frac{T(s)+N_1(s)}{\sqrt{2}} + vN_2(s). \qquad (3.1)$$

Theorem 3.1 *The Gaussian and mean curvature of the TN_1 ruled surface ξ defined at* (3.1) *are given as*

$$K_\xi = -\frac{1}{2}\left(\frac{k_1k_2}{k_1^2+v^2k_2^2+vk_1k_2\sqrt{2}}\right)^2,$$

and

$$H_\xi = \frac{\begin{pmatrix} k_1k_2^2\left(1-2v^2\right)+vk_2\left(k_1'\sqrt{2}-2k_1^2\sqrt{2}\right) \\ -vk_1k_2'\sqrt{2}-2k_1^3 \end{pmatrix}}{4\left(k_1^2+v^2k_2^2+vk_1k_2\sqrt{2}\right)^{\frac{3}{2}}},$$

respectively.

Proof By considering the relations given at (2.1) and (2.2), the first and second order partial derivatives of ξ with respect to s and v, results

$$\xi_s = -\left(\frac{\sqrt{2}}{2}k_1+vk_2\right)T + \frac{\sqrt{2}}{2}k_1N_1 + \frac{\sqrt{2}}{2}k_2N_2,$$

$$\begin{aligned}\xi_{ss} =& \left(-\frac{\sqrt{2}}{2}\left(\kappa^2+k_1'\right)-vk_2'\right)T \\ &+\left(\frac{\sqrt{2}}{2}k_1'-\left(\frac{\sqrt{2}}{2}k_1+vk_2\right)k_1\right)N_1 \\ &+\left(\frac{\sqrt{2}}{2}k_2'-\left(\frac{\sqrt{2}}{2}k_1+v\,k_2\right)k_2\right)N_2,\end{aligned}$$

$$\xi_v = N_2, \quad \xi_{sv} = -k_2T, \quad \xi_{vv} = 0.$$

Thus, from (2.5), the normal vector field of the ruled surface ξ can be given

$$n_\xi = \frac{\sqrt{2}k_1T+(\sqrt{2}k_1+2vk_2)N_1}{2\sqrt{{k_1}^2+v^2{k_2}^2+vk_1k_2\sqrt{2}}}. \qquad (3.2)$$

Moreover, from (2.8),

$$E_\xi = \left(\frac{\sqrt{2}k_1}{2}+vk_2\right)^2+\frac{{k_1}^2+{k_2}^2}{2},$$

$$F_\xi = \frac{\sqrt{2}k_2}{2}, \qquad G_\xi = 1,$$

$$L_\xi = -\frac{\begin{pmatrix} 2k_1vk_2\left(\sqrt{2}k_1+k_2v\right)+v\sqrt{2}\left(k_1k_2'-k_1'k_2\right) \\ +k_1\left(2{k_1}^2+{k_2}^2\right)\end{pmatrix}}{2\sqrt{\sqrt{2}k_1vk_2+v^2{k_2}^2+{k_1}^2}},$$

$$M_\xi = \frac{-\sqrt{2}k_1k_2}{2\sqrt{\sqrt{2}k_1vk_2+v^2{k_2}^2+{k_1}^2}}, \qquad N_\xi = 0.$$

By substituting these coefficients into (2.9), the proof is completed. □

Corollary 3.1 *The TN_1 Smarandache ruled surface is developable if and only if the main curve γ is a planar curve,*

Proof If TN_1 Smarandache ruled surface is developable, then $K_\xi = 0$, that is $k_1k_2 = 0$. If $k_1 = 0$, then from equations (2.3), $\theta = \frac{\pi}{2}k,\ k \in Z$. Similarly, if $k_2 = 0$, then $\theta = \pi k,\ k \in Z$. Since $\tau = \theta'$ and $\theta = constant$ for either case, this gives $\tau = 0$ meaning that γ is a planar curve. □

Corollary 3.2 *The TN_1 Smarandache ruled surface is either minimal or Constant-Mean-Curvature (CMC in short) surface if and only if the curve γ is a planar curve.*

Proof The proof follows the similar steps as of (ii), that is if $k_1 = 0$ then, $H_\xi = 0$, accordingly ξ is minimal, however if $k_2 = 0$, then $H_\xi = -\frac{1}{2}$ which means ξ is a (CMC) surface. □

Theorem 3.2 *The striction curve ς_ξ of the TN_1 Smarandache ruled surface is given as*

$$\zeta_\xi = \frac{T+N_1}{\sqrt{2}} - \frac{k_1k_2}{\sqrt{2}}N_2.$$

Proof The derivatives of the base and the ruling of TN_1 Smarandache ruled surface ξ are

$$\left(\frac{T+N_1}{\sqrt{2}}\right)' = \frac{1}{\sqrt{2}}\left(-k_1T+k_1N_1+k_2N_2\right),$$
$${N_2}' = -k_2T.$$

From relation (2.6), the proof is completed. □

Theorem 3.3 *The normal curvature, geodesic curvature and the geodesic torsion of the TN_1 Smarandache ruled surface are given in respective order as follows:*

$$\kappa_{n\xi} = \frac{k_1\left(k_2' + k_1^2 + k_2k_1 + \left(\sqrt{2}v+1\right)k_2^2\right) - k_1'\left(k_2\sqrt{2}v + k_1\right)}{2\sqrt{v^2k_2^2 + k_1k_2v\sqrt{2} + k_1^2}},$$
$$\kappa_{g\xi} = \frac{k_1^2\sqrt{2}\left(k_2^2v\sqrt{2} - k_2'\right) + k_1\sqrt{2}\left(2k_2^3 + k_1'k_2\right) + k_2\left(4k_2^3v + k_1^3\sqrt{2}\right)}{2\left(2k_2^2 + k_1^2\right)\sqrt{v^2k_2^2 + k_1k_2v\sqrt{2} + k_1^2}}, \tag{3.3}$$
$$\tau_{g\xi} = \frac{\eta_1\lambda_3\left(\sqrt{2}k_1 + 2k_2v\right) - \eta_3\lambda_1\left(k_1\sqrt{2} + 2k_2v\right) + k_1\sqrt{2}\left(\eta_3\lambda_2 - \lambda_3\eta_2\right)}{2\sqrt{v^2k_2^2 + k_1k_2v\sqrt{2} + k_1^2}}.$$

Proof

By referring the relations in (2.2), the tangent and the derivative of the tangent vector of TN_1- Smarandache curve are given as

$$T_{TN_1} = \frac{-k_1T + k_1N_1 + k_2N_2}{\sqrt{k_2^2 + 2k_1^2}}, \qquad T_{TN_1}' = \eta_1T + \eta_2N_1 + \eta_3N_2 \tag{3.4}$$

where

$$\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{bmatrix} = \frac{1}{\left(k_2^2 + 2k_1^2\right)^{\frac{3}{2}}}\begin{bmatrix} -k_2^2\left(k_2^2 + 3k_1^2\right) - 2k_1^4 + k_2\left(k_1k_2' - k_2k_1'\right)\\ -k_1^2\left(k_2^2 + 2k_1^2\right) + k_2\left(k_2k_1' - k_1k_2'\right)\\ k_1\left(-k_2^3 - 2\left(k_2k_1^2 - k_1k_2' + k_2k_1'\right)\right)\end{bmatrix}.$$

□

On the other hand, the second order derivative of TN_1 Smarandache curve results

$$\left(\frac{T+N_1}{\sqrt{2}}\right)'' = \frac{1}{\sqrt{2}}\begin{bmatrix} -k_1^2 - k_2^2 - k_1'\\ k_1' - k_1^2\\ k_2' - k_1k_2\end{bmatrix}\begin{bmatrix}T\\ N_1\\ N_2\end{bmatrix}.$$

Moreover, the derivative of the normal vector field of TN_1 Smarandache ruled surface is:

$$\left(n_\xi\right)' = \lambda_1T + \lambda_2N_1 + \lambda_3N_2,$$

where

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \frac{\begin{bmatrix} k_1'k_2v\left(k_2v\sqrt{2} + k_1\right) - k_1k_2'v\left(k_2\sqrt{2}v + k_1\right) - k_1k_2^2v\left(3k_1v\sqrt{2} + 2k_2v^2\right) - k_1^3\left(k_1\sqrt{2} + 4k_2v\right)\\ k_1\left(k_1k_2^2\sqrt{2}v^2 + 2k_2k_1^2v - k_2k_1'v + k_1^3\sqrt{2} + k_1k_2'v\right)\\ k_1k_2\sqrt{2}\left(v^2k_2^2 + k_1k_2v\sqrt{2} + k_1^2\right)\end{bmatrix}}{2\left(v^2k_2^2 + k_1k_2v\sqrt{2} + k_1^2\right)^{\frac{3}{2}}}.$$

Upon substituting these into (2.10), the proof is completed. According to the Theorem 3.3, the following two corollaries can ve given without the need for proof.

Corollary 3.3

(i) *The TN_1 Smarandache curve is asymptotic on TN_1 Smarandache ruled surface if $k_1 = 0$ that is $\theta = \frac{\pi}{2}k,\ k \in Z$.*

(ii) *The TN_1 Smarandache curve is geodesic on TN_1 Smarandache ruled surface if $k_2 = 0$ that is $\theta = \pi k,\ k \in Z$.*

3.2. The characteristics of TN_2 Smarandache ruled surface

Definition 3.2 *Let $\gamma(s) : s \in I \subset R \to R^3$ be a unit speed curve in E^3, and denote $\{T(s),\ N_1(s),\ N_2(s)\}$ as the Bishop frame of γ. The ruled surface with a base TN_2 Smarandache curve and with ruling N_1 is called a TN_2 Smarandache ruled surface which is defined by*

$$\delta(s,v) = \frac{T(s) + N_2(s)}{\sqrt{2}} + vN_1(s). \tag{3.5}$$

Theorem 3.4 *The Gaussian and mean curvature of the TN_2 ruled surface δ defined at* (3.5) *are given as*

$$K_\delta = -\frac{1}{2}\left(\frac{k_1k_2}{k_2^2 + v^2k_1^2 + vk_1k_2\sqrt{2}}\right)^2,$$

$$H_\delta = \frac{\left(\begin{array}{c} k_1^2k_2\left(1 - 2v^2\right) + vk_1\left(k_2'\sqrt{2} - 2k_2^2\sqrt{2}\right) \\ -vk_1'k_2\sqrt{2} - 2k_2^3 \end{array}\right)}{4\left(k_2^2 + v^2k_1^2 + vk_1k_2\sqrt{2}\right)^{\frac{3}{2}}}.$$

Proof By using (2.1) and (2.2), the first and second order partial derivatives of δ with respect to s and v is computed as follows:

$$\delta_s = -\left(\frac{\sqrt{2}}{2}k_2 + vk_1\right)T + \frac{\sqrt{2}}{2}k_1N_1 + \frac{\sqrt{2}}{2}k_2N_2,$$

$$\delta_{ss} = \begin{bmatrix} -\frac{\sqrt{2}}{2}\left(\kappa^2 + k_2'\right) - vk_1' \\ \frac{\sqrt{2}}{2}k_1' - \left(\frac{\sqrt{2}}{2}k_2 + vk_1\right)k_1 \\ \frac{\sqrt{2}}{2}k_2' - \left(\frac{\sqrt{2}}{2}k_2 + vk_1\right)k_2 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

$$\delta_v = N_1, \qquad \delta_{sv} = -k_1T, \qquad \delta_{vv} = 0.$$

Thus, from (2.5), the normal vector field of the ruled surface δ can be given

$$n_\delta = -\frac{\sqrt{2}k_2T + (\sqrt{2}k_2 + 2vk_1)N_2}{2\sqrt{k_2^2 + v^2k_1^2 + vk_1k_2\sqrt{2}}}. \tag{3.6}$$

Moreover, from (2.8),

$$E_\delta = \left(\frac{\sqrt{2}k_2}{2} + vk_1\right)^2 + \frac{k_1^2 + k_2^2}{2},$$

$$F_\delta = \frac{\sqrt{2}k_1}{2}, \qquad G_\delta = 1,$$

$$L_\delta = \frac{\left(\begin{array}{c} 2k_2vk_1\left(k_2\sqrt{2} + vk_1\right) - v\sqrt{2}\left(k_2'k_1 - k_1'k_2\right) \\ +k_2\left(2k_2^2 + k_1^2\right) \end{array}\right)}{2\sqrt{\sqrt{2}k_2vk_1 + v^2k_1^2 + k_2^2}},$$

$$M_\delta = \frac{\sqrt{2}k_2k_1}{2\sqrt{\sqrt{2}k_2vk_1 + v^2k_1^2 + k_2^2}}, \qquad N_\delta = 0.$$

By substituting these coefficients into (2.9), the proof is completed. □

From Proposition 2.1 and Theorem 3.4, similar corollaries can be obtained as like below:

Corollary 3.4 *The TN_2 Smarandache ruled surface is developable if and only if the main curve γ is a planar curve,*

Proof The proof is similar as of the proof for Corollary 3.1. □

Corollary 3.5 *The TN_2 Smarandache ruled surface is either minimal or constant-mean-curvature (CMC) surface if and only if the curve γ is a planar curve.*

Proof The proof is slightly different from the proof for Corollary 3.2, that is if $k_2 = 0$, then $H_\delta = 0$, and if $k_1 = 0$, then $H_\delta = \frac{1}{2}$. □

Theorem 3.5 *The striction curve of the TN_2 Smarandache ruled surface is given as*

$$\zeta_\delta = \frac{T + N_2}{\sqrt{2}} - \frac{k_1k_2}{\sqrt{2}}N_1.$$

Proof The derivatives of the base and the ruling of TN_2 Smarandache ruled surface δ are

$$\left(\frac{T + N_2}{\sqrt{2}}\right)' = \frac{\sqrt{2}}{2}\left(-k_2T + k_1N_1 + k_2N_2\right),$$

$$N_1' = -k_1T.$$

By considering relation (2.6), the proof is completed. □

Theorem 3.6 *The normal curvature, geodesic curvature and the geodesic torsion of the TN_2 Smarandache ruled surface are given in respective order as follows:*

$$\kappa_{n\,\delta} = \frac{k_1 v\sqrt{2}\left(k_2{}^2 - k_2'\right) + k_2\left(k_1{}^2 + 2k_2{}^2\right)}{2\sqrt{v^2 k_1{}^2 + k_2 k_1 v\sqrt{2} + k_2{}^2}},$$

$$\kappa_{g\,\delta} = \frac{2\left(k_1' k_2 - k_1 k_2'\right)\left(\sqrt{2}k_2 + k_1 v\right) - k_1 k_2\sqrt{2}\left(2k_2{}^2 + k_1{}^2\right) - 2k_1{}^2 v\left(k_1{}^2 + 4k_2{}^2\right)}{2\left(2k_2{}^2 + k_1{}^2\right)\sqrt{k_1 k_2 v\sqrt{2} + v^2 k_1{}^2 + k_2{}^2}},$$

$$\tau_{g\,\delta} = \frac{\left(\alpha_2\omega_1 - \alpha_1\omega_2\right)\left(k_2\sqrt{2} + 2k_1 v\right) + k_2\sqrt{2}\left(\alpha_3\omega_2 - \alpha_2\omega_3\right)}{2\sqrt{k_1 k_2 v\sqrt{2} + v^2 k_1{}^2 + k_2{}^2}},$$

Proof By using the relations in 2.2, the tangent and the derivative of the tangent vector of TN_2 Smarandache curve are given as

$$T_{TN_2} = \frac{-k_2 T + k_1 N_1 + k_2 N_2}{\sqrt{k_1{}^2 + 2k_2{}^2}}, \quad T_{TN_2}{}' = \omega_1 T + \omega_2 N_1 + \omega_3 N_2, \tag{3.7}$$

where □

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \frac{1}{\left(2k_2{}^2 + k_1{}^2\right)^{\frac{3}{2}}}\begin{bmatrix} k_1\left(k_1' k_2 - k_2' k_1\right) - 2k_2{}^4 - 3k_1{}^2 k_2{}^2 - k_1{}^4\\ k_2\left(2\left(k_1' k_2 - k_2' k_1\right) - 2k_2{}^2 k_1 - k_1{}^3\right)\\ k_1\left(k_2' k_1 - k_1' k_2\right) - 2k_2{}^4 - k_1{}^2 k_2{}^2\end{bmatrix}.$$

On the other hand, the second order derivative of TN_2 Smarandache curve is

$$\left(\frac{T + N_2}{\sqrt{2}}\right)'' = \frac{1}{\sqrt{2}}\begin{bmatrix} -\left(k_1{}^2 + k_2{}^2 + k_2'\right)\\ \left(k_1' - k_1 k_2\right)\\ \left(k_2' - k_2{}^2\right)\end{bmatrix}\begin{bmatrix}T\\ N_1\\ N_2\end{bmatrix}.$$

Lastly, the derivative of the normal vector field of the TN_2 Smarandache ruled surface is given as follows:

$$(n_\delta)' = \alpha_1 T + \alpha_2 N_1 + \alpha_3 N_2,$$

where

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \frac{\begin{bmatrix}\left(\sqrt{2}k_1 v + k_2\right)\left(k_1' k_2 - k_1 k_2\right) v + k_1{}^2 k_2 v^2\left(3k_2\sqrt{2} + 2k_1 v\right) + k_2{}^3\left(k_2\sqrt{2} + 4k_1 v\right)\\ -k_1 k_2\sqrt{2}\left(k_1 k_2 v\sqrt{2} + v^2 k_1{}^2 + k_2{}^2\right)\\ -k_2{}^2\left(\sqrt{2}\left(k_1{}^2 v^2 + k_2{}^2\right) + v\left(2k_1 k_2 - k_1 + k_1'\right)\right)\end{bmatrix}}{2\left(k_1 k_2 v\sqrt{2} + v^2 k_1{}^2 + k_2{}^2\right)^{\frac{3}{2}}}$$

Upon substituted the given relations into (2.10), the proof is completed.

As a result of this theorem, two corollaries can be easily given without the need for proof as follows:

Corollary 3.6

(i) The TN_2 Smarandache curve is asymptotic on TN_2 Smarandache ruled surface if $k_2 = 0$ that is $\theta = \pi k,\ k \in Z$.

(ii) The TN_2 Smarandache curve is geodesic on TN_2 Smaran-

dache ruled surface if $k_1 = 0$ *that is* $\theta = \frac{\pi}{2}k,\ k \in Z.$

3.3. The N_1N_2 Smarandache ruled surface

Definition 3.3 *Let $\gamma(s) : s \in I \subset R \to R^3$ be a unit speed curve in E^3, and denote $\{T(s),\ N_1(s),\ N_2(s)\}$ as the Bishop frame of γ. The ruled surface with a base N_1N_2 Smarandache curve and with ruling T is called a N_1N_2 Smarandache ruled surface which is defined by*

$$\varepsilon(s,v) = \frac{N_1(s) + N_2(s)}{\sqrt{2}} + vT(s). \qquad (3.8)$$

Theorem 3.7 *The Gaussian curvature of ε defined at* (3.8) *vanishes, whereas its mean curvature is given by*

$$H_\varepsilon = \frac{k_1'k_2 - k_1k_2'}{2v\kappa^3}.$$

Proof The first and second order partial derivatives of ε with respect to s and v, from the relations given at (2.1) and (2.2), it is clear to have

$$\varepsilon_s = -\frac{\sqrt{2}}{2}(k_1 + k_2)T + vk_1N_1 + vk_2N_2, \qquad \varepsilon_v = T,$$

$$\varepsilon_{ss} = \begin{bmatrix} -\left(v\kappa^2 + \frac{\sqrt{2}}{2}(k_1' + k_2')\right) \\ \left(vk_1' - \frac{\sqrt{2}}{2}k_1(k_1 + k_2)\right) \\ \left(vk_2' - \frac{\sqrt{2}}{2}k_2(k_1 + k_2)\right) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

$$\varepsilon_{sv} = k_1N_1 + k_2N_2, \qquad \varepsilon_{vv} = 0.$$

Thus, from (2.5), the normal vector field of the ruled surface ε can be given

$$n_\varepsilon = \frac{k_2N_1 - k_1N_2}{\kappa}. \qquad (3.9)$$

Moreover, from (2.8),

$$E_\varepsilon = \frac{(k_1 + k_2)^2}{2} + v^2\left(k_1{}^2 + k_2{}^2\right), \quad F_\varepsilon = -\frac{\sqrt{2}(k_1 + k_2)}{2},$$
$$G_\varepsilon = 1, \quad L_\varepsilon = \frac{v(k_2k_1' - k_1k_2')}{\sqrt{k_2{}^2 + k_1{}^2}}, \quad M_\varepsilon = N_\varepsilon = 0.$$

By substituting these coefficients into (2.9), the proof is completed. □

Remark 3.1 *From the given proposition 2.1, the N_1N_2 Smarandache ruled surface is always developable.*

Proof The proof is clear by given the Proposition 2.1. □

Corollary 3.7 *The N_1N_2 Smarandache ruled surface is minimal if and only if the curve γ is a slant helix.*

Proof Let us recall the Theorem 2.1 that γ is a slant helix if and only if $\left(\frac{k_1}{k_2}\right)' = 0$. From Theorem 3.7, this corresponds to that $H_\varepsilon = 0$, which means the ruled surface ε is minimal.
Conversely, if ε is minimal ($H_\varepsilon = 0$), then by Theorem 3.7, $k_1'k_2 - k_1k_2' = 0$. Thus, $\frac{k_1}{k_2} = constant$ meaning that γ is a slant helix. □

Theorem 3.8 *The striction curve of the N_1N_2 Smarandache ruled surface is given as*

$$\zeta_\varepsilon = \frac{N_1 + N_2}{\sqrt{2}}.$$

The derivatives of the base and the ruling of N_1N_2 Smarandache ruled surface ε are

$$\left(\frac{N_1 + N_2}{\sqrt{2}}\right)' = -\left(\frac{k_1 + k_2}{\sqrt{2}}\right)T,$$
$$T' = k_1N_1 + k_2N_2.$$

By considering relation (2.6), the proof is completed.

Remark 3.2 *Note that the striction curve coincides with the base curve for N_1N_2 Smarandache ruled surface.*

Theorem 3.9 *The normal curvature, geodesic curvature and the geodesic torsion of the N_1N_2 Smarandache ruled surface is*

$$\kappa_{n\,\varepsilon} = 0, \qquad \kappa_{g\,\varepsilon} = -\kappa, \qquad \tau_{g\,\varepsilon} = 0, \qquad (3.10)$$

respectively.

Proof By considering both (2.2) and (2.3), the tangent and the derivative of the tangent vector of N_1N_2 Smarandache curve are given as

$$T_{N_1N_2} = -T, \qquad (3.11)$$
$$T_{N_1N_2}{}' = -k_1N_1 - k_2N_2.$$

Moreover, the second order derivative of N_1N_2 Smarandache curve is

$$\left(\frac{N_1 + N_2}{\sqrt{2}}\right)'' = -\frac{(k_1' + k_2')T + (k_1^2 + k_1k_2)N_1 + (k_1k_2 + k_2^2)N_2}{\sqrt{2}}.$$

Lastly, the derivative of the normal vector field of the TN_2 Smarandache ruled surface is given as follows:

$$(n_\varepsilon)' = \frac{k_1(k_1k_2' - k_1'k_2)N_1 + k_2(k_1k_2 - k_1'k_2)N_2}{\kappa^3}.$$

When the given relations substituted into (2.10), the proof is completed. □

The following two corollaries can be expressed as a result of Theorem 3.9 without the need for proof.

Corollary 3.8

(i) The N_1N_2 Smarandache curve is always asymptotic and principal line on N_1N_2 Smarandache ruled surface.

(ii) The geodesic curvature of N_1N_2 Smarandache ruled surface is negative of the curvature of the main curve γ.

Example 3.1 *Let us consider the standard unit helix curve parameterized as*

$$\gamma(s) = \frac{\sqrt{2}}{2}(cos(s), sin(s), s),$$

then, the Frenet curvatures of γ are $\kappa = \tau = \frac{\sqrt{2}}{2}$. Since $\tau = \theta'$, this results $\theta = \int \tau ds = \frac{s\sqrt{2}}{2}$. Thus the Bishop curvatures can be established as $k_1 = \frac{\sqrt{2}}{2}cos\left(\frac{s\sqrt{2}}{2}\right)$, and $k_2 = \frac{\sqrt{2}}{2}sin\left(\frac{s\sqrt{2}}{2}\right)$. Thus the vectors of Bishop frame can be provided as follows:

$$T(s) = \frac{\sqrt{2}}{2}\Big(-sin(s), cos(s), 1\Big),$$

$$N_1(s) = \begin{pmatrix} -\cos\left(\frac{s\sqrt{2}}{2}\right)\cos(s) - \frac{\sqrt{2}}{2}\sin\left(\frac{s\sqrt{2}}{2}\right)\sin(s), \\ -\cos\left(\frac{s\sqrt{2}}{2}\right)\sin(s) + \frac{\sqrt{2}}{2}\sin\left(\frac{s\sqrt{2}}{2}\right)\cos(s), \\ \frac{\sqrt{2}}{2}\sin\left(\frac{s\sqrt{2}}{2}\right), \end{pmatrix},$$

$$N_2(s) = \begin{pmatrix} -\sin\left(\frac{s\sqrt{2}}{2}\right)\cos(s) + \frac{\sqrt{2}}{2}\cos\left(\frac{s\sqrt{2}}{2}\right)\sin(s), \\ -\sin\left(\frac{s\sqrt{2}}{2}\right)\sin(s) - \frac{\sqrt{2}}{2}\cos\left(\frac{s\sqrt{2}}{2}\right)\cos(s), \\ \frac{\sqrt{2}}{2}\cos\left(\frac{s\sqrt{2}}{2}\right) \end{pmatrix}.$$

By referring to the definitions for TN_1, TN_2 and N_1N_2 Smarandache ruled surfaces, the graphs are provided in Fig. 1, Fig. 2 and Fig.3 3 where $s \in [-\pi, \pi]$ and $v \in [-1, 1]$.

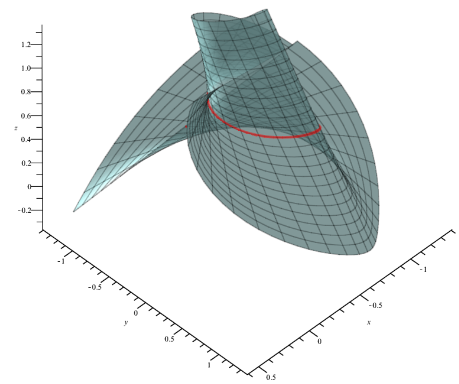

Figure 1: The ruled surface $\xi(s, v)$

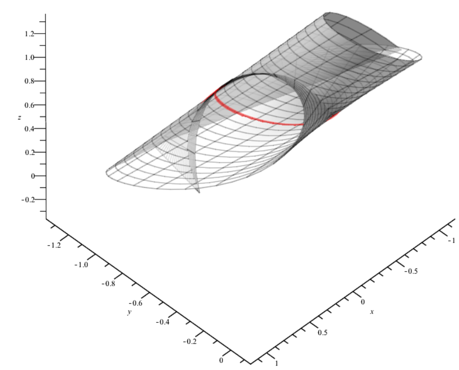

Figure 2: The ruled surface $\delta(s, v)$

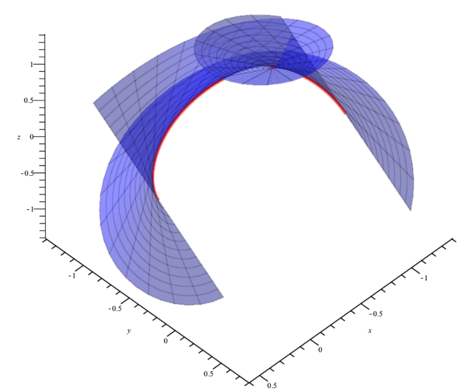

Figure 3: The ruled surface $\varepsilon(s, v)$

4. Conclusion

The paper introduced new ruled surfaces via Smarandache geometry using Bishop frame vectors. The characteristics of each surface, such as developability and minimality, were discussed. Furthermore, the characteristic curves on these surfaces were determined by providing the required conditions. It is shown that the special slant helix curve defines the minimality condition for the ruled surface $\varepsilon(s,v)$. Finally, regardless of the choice of main curve, if the base curve of the ruled surface is considered to be N_1N_2 Smarandache curve, the developability characteristic remains valid This is analogous to the fact that tangent ruled surfaces are always developable.

Conflict of interests statement

The authors declare no conflict of interests.

Data availability statement

The manuscript has no associated data.

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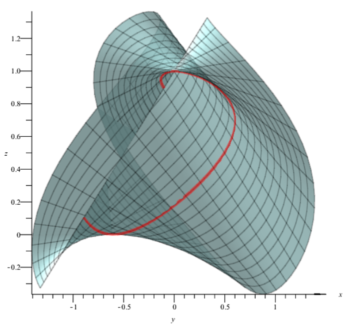
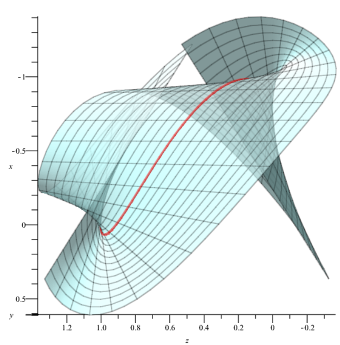
Appendix

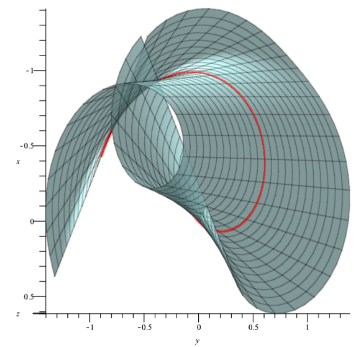
The following figures Fig. 4, Fig. 5 and Fig. 6 are also presented to examine the view of each surface from different angles. The orientations are fixed to the x, y and z axis, respectively.



(a) oriented to x axis

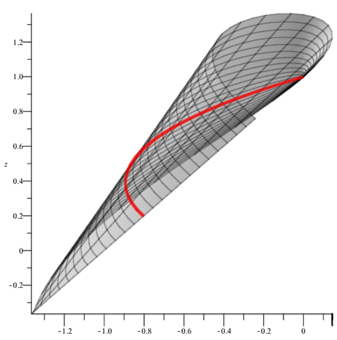


(b) oriented to y axis

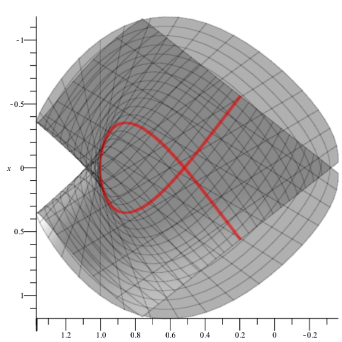


(c) oriented to z axis

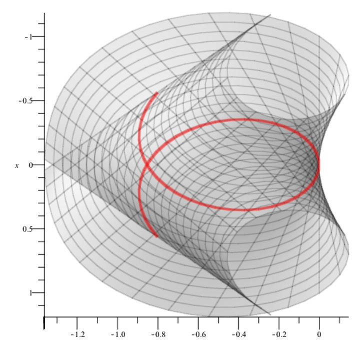
Figure 4: The TN_1 Smarandache ruled surface $\xi(s, v)$



(a) oriented to x axis

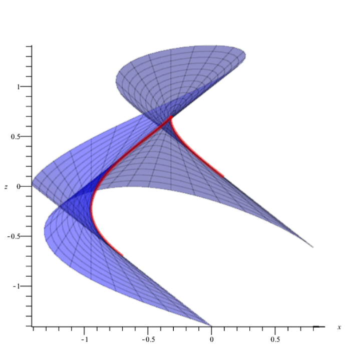


(b) oriented to y axis

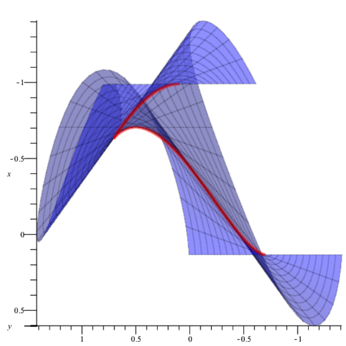


(c) oriented to z axis

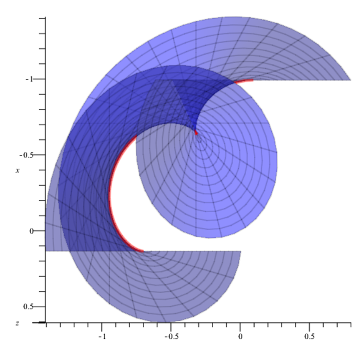
Figure 5: The TN_2 Smarandache ruled surface $\delta(s, v)$



(a) oriented to x axis



(b) oriented to y axis



(c) oriented to z axis

Figure 6: The N_1N_2 Smarandache Ruled surface $\varepsilon(s, v)$