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RESEARCH ARTICLE

A Note on a Well-Defined Sectional Curvature of a Semi-Symmetric Non-Metric Connection

Adela Mihai* and Ion Mihai

(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this note we propose a new sectional curvature on a Riemannian manifold endowed with a semi-symmetric non-metric connection. A Chen-Ricci inequality is proven. Some possible applications in other fields are mentioned.

Keywords: *linear connection, semi-symmetric connection, metric connection, non-metric connection, sectional curvature.*
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1. Introduction

The notion of a linear connection is one of the most important in differential geometry. Remarkable contributions are due to Christoffel, Ricci, Levi-Civita, Cartan, Darboux, Koszul.

The simplest connections are symmetric, or torsion-free. On any differentiable manifold there exists a symmetric linear connection. On the other hand, Friedmann and Schouten (1924) initiated the study of semi-symmetric linear connections on a differentiable manifold. After that, Hayden (1932) introduced the notion of a metric connection with torsion on a Riemannian manifold. Semi-symmetric metric connections play an important role in the geometry of Riemannian manifolds. The semi-symmetric metric connections on a Riemannian manifold were investigated by Yano (1970). There are various physical problems involving semi-symmetric metric connection. More recently semi-symmetric non-metric connections were considered by Agashe (1992).

The sectional curvature of a Riemannian manifold is the natural generalization of the Gauss curvature of a surface. It is defined using the Levi-Civita connection, which is a symmetric and metric connection.

A similar construction can be done for a semi-symmetric metric connection on a Riemannian manifold. For a semi-symmetric non-metric connection the standard definition of the sectional curvature is not applicable.

The purpose of this note is to define a sectional curvature on a Riemannian manifold endowed with a semi-symmetric non-metric connection.

We compute the scalar curvature and the Ricci curvature for this new sectional curvature.

As application, we establish the Chen-Ricci inequality for submanifolds in a Riemannian space form endowed with a semi-symmetric non-metric connection.

Moreover, some possible applications in other fields are proposed.

2. Semi-symmetric non-metric connections

Let (M, g) be an n-dimensional ($n \geq 2$) Riemannian manifold and ∇ a linear connection on M.

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**** Corresponding author***

The torsion of ∇ is a $(1,2)$-tensor field T given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],\ \forall X,Y \in \Gamma(TM),$$

and the curvature of ∇ is a $(1,3)$-tensor field R defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\ \forall X,Y,Z \in \Gamma(TM).$$

The covariant derivative of g with respect to ∇ is expressed by

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

A distinguish connection on (M,g) is the Levi-Civita connection ∇^0, which is torsion-free (i.e., $T^0 = 0$) and g is parallel with respect to ∇^0 (i.e., $\nabla^0 g = 0$).

We consider a connection ∇ different from the Levi-Civita connection ∇^0.

The notion of a semi-symmetric connection on a Riemannian manifold was introduced by A. Friedmann and J.A. Schouten [5]. Submanifolds in Riemannian manifolds with semi-symmetric connections were studied by H.A. Hayden [6].

The connection ∇ is called a *semi-symmetric connection* if its torsion T is given by

$$T(X,Y) = \omega(Y)X - \omega(X)Y,\ \forall X,Y \in \Gamma(TM),$$

where ω is a 1-form on M. Denote by $P \in \Gamma(TM)$ the associated vector field to ω, i.e., $\omega(X) = g(X,P)$.

If g is parallel with respect to ∇, i.e., $\nabla g = 0$, then ∇ is called a *semi-symmetric metric connection*. If not, i.e., $\nabla g \neq 0$, then ∇ is called a *semi-symmetric non-metric connection*.

K. Yano [13] studied Riemannian manifolds endowed with semi-symmetric metric connections. Some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection were studied by T. Imai [7] and Z. Nakao [9] studied submanifolds of a Riemannian manifold endowed with a semi-symmetric metric connection. N.S. Agashe and M.R. Chafle [1], [2] introduced the notion of a semi-symmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection.

It is shown in [1] that a semi-symmetric non-metric connection ∇ and the Levi-Civita connection ∇^0 are related by

$$\nabla_X Y = \nabla^0_X Y + \omega(Y)X.$$

We denote by R^0 the Riemannian curvature tensor. Then, we have (see [1]):

$$g(R(X,Y)Z,W) = g(R^0(X,Y)Z,W) + s(X,Z)g(Y,W) - s(Y,Z)g(X,W), \tag{2.1}$$

where $s(X,Y) = (\nabla^0_X \omega)Y - \omega(X)\omega(Y)$.

In particular, for orthonormal vectors $e_1, e_2 \in T_pM$, $p \in M$, we get

$$g(R(e_1,e_2)e_2,e_1) = g(R^0(e_1,e_2)e_2,e_1) - s(e_2,e_2).$$

Obviously $g(R^0(e_1,e_2)e_2,e_1)$ is the Riemannian sectional curvature of the plane section π spanned by e_1 and e_2. It follows that $g(R(e_1,e_2)e_2,e_1)$ is not independent of the orthonormal basis of π, therefore we cannot define the sectional curvature with respect to ∇ by $K(\pi) = g(R(e_1,e_2)e_2,e_1)$.

We establish the form of the Bianchi identity for the connection ∇.

$$\begin{aligned} &g(R(X,Y)Z,W) + g(R(Y,Z)X,W) + g(R(Z,X)Y,W) \\ &= g(R^0(X,Y)Z,W) + s(X,Z)g(Y,W) - s(Y,Z)g(X,W) \\ &+ g(R^0(Y,Z)X,W) + s(Y,X)g(Z,W) - s(Z,X)g(Y,W) \\ &+ g(R^0(Z,X)Y,W) + s(Z,Y)g(X,W) - s(X,Y)g(Z,W). \end{aligned}$$

By using the Bianchi identity for the Riemannian curvature tensor, we have

$$\begin{aligned} &R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\ &= [s(Z,Y) - s(Y,Z)]X + [s(X,Z) - s(Z,X)]Y + [s(Y,X) - s(X,Y)]Z. \end{aligned}$$

If we substitute the expression of s in the above equation, we get

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y$$
$$=[(\nabla^0_Z\omega)Y-(\nabla^0_Y\omega)Z]X+[(\nabla^0_X\omega)Z-(\nabla^0_Z\omega)X]Y+[(\nabla^0_Y\omega)X-(\nabla^0_X\omega)Y]Z.$$

Because

$$(\nabla^0_Z\omega)Y=Z\omega(Y)-\omega(\nabla^0_ZY)=Zg(Y,P)-g(\nabla^0_ZY,P)=g(Y,\nabla^0_ZP),$$

we obtain

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y$$
$$=[g(Y,\nabla^0_ZP)-g(Z,\nabla^0_YP)]X+[g(Z,\nabla^0_XP)-g(X,\nabla^0_ZP)]Y+[g(X,\nabla^0_YP)-g(Y,\nabla^0_XP)]Z.$$

3. A well-defined sectional curvature

Let (M,g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection ∇. Recall that

$$\nabla_XY=\nabla^0_XY+\omega(Y)X,$$

where ∇^0 is the Levi-Civita connection on (M,g).

We remarked in the previous section that one cannot define the sectional curvature of a plane section $\pi=\text{span}\,\{e_1,e_2\}\subset T_pM$, $p\in M$, by $g(R(e_1,e_2)e_2,e_1)$.

This is the reason for which a well-defined sectional curvature is necessary; the steps to get there are below.

First we consider the linear connection

$$\nabla'_XY=\nabla^0_XY-g(X,Y)P.$$

Then we prove the first result.

Proposition 3.1. *Let (M,g) be a Riemannian manifold, ∇ a semi-symmetric non-metric connection given by $\nabla_XY=\nabla^0_XY+\omega(Y)X$ and ∇' a linear connection defined by $\nabla'_XY=\nabla^0_XY-g(X,Y)P$. Then ∇ and ∇' are conjugate connections, i.e.,*

$$Zg(X,Y)=g(\nabla_ZX,Y)+g(X,\nabla'_ZY),\ \forall X,Y,Z\in\Gamma(TM).$$

Proof. Let $X,Y,Z\in\Gamma(TM)$. Then

$$g(\nabla_ZX,Y)+g(X,\nabla'_ZY)$$
$$=g(\nabla^0_ZX+\omega(X)Z,Y)+g(X,\nabla^0_ZY-g(Z,Y)P)=Zg(X,Y)+\omega(X)g(Z,Y)-g(Z,Y)g(X,P)=Zg(X,Y).$$

□

The basic properties of the connection ∇' are given in the following.

Proposition 3.2. *Let (M,g) be a Riemannian manifold and ∇' the connection defined by $\nabla'_XY=\nabla^0_XY-g(X,Y)P$, where ∇^0 is the Levi-Civita connection. Then:*
i) ∇' is symmetric, i.e., its torsion $T'=0$.
ii) ∇' is non-metric.

Proof. Let $X,Y,Z\in\Gamma(TM)$. We have:
i)

$$T'(X,Y)=\nabla'_XY-\nabla'_YX-[X,Y]$$
$$=\nabla^0_XY-g(X,Y)P-\nabla^0_YX+g(X,Y)P-[X,Y]=0.$$

ii)

$$(\nabla'_Xg)(Y,Z)=Xg(Y,Z)-g(\nabla^0_XY-g(X,Y)P,Z)-g(Y,\nabla^0_XZ-g(X,Z)P)$$
$$=-g(X,Y)\omega(Z)+g(X,Z)\omega(Y)\neq0.$$

□

Next, we prove an important relation between the curvatures of the conjugate connections ∇ and ∇'.

Theorem 3.1. *Let (M, g) be a Riemannian manifold, ∇ a semi-symmetric non-metric connection and ∇' its conjugate connection defined by $\nabla_X Y = \nabla^0_X Y + \omega(Y)X$, $\nabla'_X Y = \nabla^0_X Y - g(X, Y)P$. Then*

$$g(R'(X, Y)Z, W) = -g(R(X, Y)W, Z).$$

Proof. Let $X, Y, Z, W \in \Gamma(TM)$. Then:

$$\begin{aligned} g(R(X, Y)Z, W) &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W) \\ &= Xg(\nabla_Y Z, W) - g(\nabla_Y Z, \nabla'_X W) - Y g(\nabla_X Z, W) + g(\nabla_X Z, \nabla'_Y W) \\ &- [X, Y]g(Z, W) + g(Z, \nabla'_{[X,Y]} W) \\ &= XY g(Z, W) - Xg(Z, \nabla'_Y W) - Y g(Z, \nabla'_X W) + g(Z, \nabla'_Y \nabla'_X W) \\ &- Y Xg(Z, W) + Y g(Z, \nabla'_X W) + Xg(Z, \nabla'_Y W) - g(Z, \nabla'_X \nabla'_Y W) \\ &- [X, Y]g(Z, W) + g(Z, \nabla'_{[X,Y]} W) \\ &= -g(Z, R'(X, Y)W). \end{aligned}$$

□

Inspired by an idea of B. Opozda [10], we then define a (0,4)-tensor field S by

$$S(X, Y, Z, W) = \frac{1}{2}[g(R(X, Y)W, Z) + g(R'(X, Y)W, Z)].$$

Theorem 3.1 implies

$$S(X, Y, Z, W) = -S(X, Y, W, Z)$$

and

$$S(X, Y, Z, W) = \frac{1}{2}[g(R(X, Y)W, Z) - g(R(X, Y)Z, W)].$$

Let $p \in M$ and $\pi \subset T_pM$ a plane section. For an orthonormal basis $\{e_1, e_2\}$ of π, we derive

$$S(e_1, e_2, e_1, e_2) = \frac{1}{2}[g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_2)e_1, e_2)].$$

By the formula (2.1) of the curvature tensor of a semi-symmetric non-metric connection, it follows that

$$S(e_1, e_2, e_1, e_2) = R^0(e_1, e_2, e_1, e_2) - \frac{1}{2}[s(e_2, e_2) + s(e_1, e_1)],$$

which does not depend on the orthonormal basis $\{e_1, e_2\}$ of π. Therefore, we are now able to introduce the following definition of a sectional curvature of the semi-symmetric non-metric connection ∇.

Definition 3.1. The *sectional curvature* of the plane section $\pi \subset T_pM$ spanned by the orthonormal basis $\{e_1, e_2\}$ is defined by

$$K(\pi) = \frac{1}{2}[g(R(e_1, e_2)e_2, e_1) + g(R(e_2, e_1)e_1, e_2)].$$

Using the above definition, we can compute the scalar curvature and the Ricci curvature of a Riemannian space form admitting a semi-symmetric non-metric connection.

Let $M(c)$ be an n-dimensional Riemannian space form (the sectional curvature associated to the Levi-Civita connection is a constant c) admitting a semi-symmetric non-metric connection ∇. Let $p \in M(c)$ and $\{e_1, ..., e_n\}$ an orthonormal basis of T_pM.

The *scalar curvature* with respect to ∇ is

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where $e_i \wedge e_j$ is the plane section spanned by e_i and e_j.

By using the definition of the sectional curvature K, we have

$$\tau = \frac{1}{2}\sum_{1\le i<j\le n}[g(R(e_i,e_j)e_j,e_i)+g(R(e_j,e_i)e_i,e_j)] = \frac{1}{2}\sum_{1\le i,j\le n} g(R(e_i,e_j)e_j,e_i).$$

By the formula of the curvature tensor of a semi-symmetric non-metric connection, it follows that

$$\tau = \frac{1}{2}n(n-1)c + \frac{1}{2}(n-1)\text{trace } s.$$

Let $p \in M(c)$, $X \in T_pM$ unit and $\{e_1 = X, e_2, ..., e_n\}$ an orthonormal basis of T_pM. It is known that

$$\begin{aligned}\text{Ric } (X) &= \sum_{j=2}^{n} K(X \wedge e_j)\\ &= \frac{1}{2}\sum_{j=2}^{n}[g(R(X,e_j)e_j,X)+g(R(e_j,X)X,e_j)] = (n-1)c + \frac{1}{2}[(n-2)s(X,X)+\text{trace } s].\end{aligned}$$

4. An application in the submanifolds theory: Chen-Ricci inequality

B.-Y. Chen [3] established an estimate of the mean curvature in terms of the Ricci curvature for any Riemannian submanifold of dimension n in a Riemannian space form $\tilde{M}(c)$ of constant sectional curvature c:

$$\text{Ric } (X) \le (n-1)c + \frac{n^2}{4}\|H\|^2.$$

It is known as the *Chen-Ricci inequality*.

In this section we establish the Chen-Ricci inequality for submanifolds in a Riemannian space form admitting a semi-symmetric non-metric connection by using the sectional curvature defined in the previous section.

Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form, $\tilde{\nabla}$ a semi-symmetric non-metric connection on $\tilde{M}(c)$ and M an n-dimensional $(n \ge 2)$ submanifold of $\tilde{M}(c)$.

The Gauss formulae for the semi-symmetric connection $\tilde{\nabla}$ and the Levi-Civita connection $\tilde{\nabla}^0$, respectively, are written as

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}^0_X Y = \nabla^0_X Y + h^0(X,Y),$$

for all vector fields X, Y on the submanifold M. In the above formulae, h^0 is the second fundamental form of M and h is a $(0,2)$-tensor on M. In [2], it is proven that $h^0 = h$.

We decompose the vector field P on M uniquely into its tangent and normal components $P^\top$ and $P^\perp$, respectively; we have $P = P^\top + P^\perp$.

The Gauss equation with respect to the semi-symmetric non-metric connection is given by (see [2])

$$\begin{aligned}g(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z))\\ &+ g(P^\perp,h(Y,Z))g(X,W) - g(P^\perp,h(X,Z))g(Y,W),\end{aligned} \tag{4.1}$$

for any vector fields X, Y, Z and W on M.

Theorem 4.1. *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form, $\tilde{\nabla}$ a semi-symmetric non-metric connection on it and M an n-dimensional $(n \ge 2)$ submanifold of $\tilde{M}(c)$. Then we have the following:*

(1) For each unit vector $X \in T_pM$,

$$\begin{aligned}\text{Ric}(X) \le \frac{n^2}{4}\|H\|^2 + (n-1)c - \frac{1}{2}[\text{trace}\, s + (n-2)s(X,X)]\\ - \frac{1}{2}[n\omega(H) + (n-2)g(P^\perp, h(X,X))].\end{aligned} \tag{4.2}$$

(2) If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (4.2) if and only if $X \in N_p$, where $N_p = \{X \in T_pM | h(X, Y) = 0, \forall Y \in T_pM\}$.
(3) The equality case of (4.2) holds identically for all unit tangent vectors at p if and only if either:

(i) p is a totally geodesic point, or
(ii) $n = 2$ and p is a totally umbilical point.

Proof.

(1) Let $p \in M$ and $X \in T_pM$ a unit tangent vector. Consider an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ in $T_p\tilde{M}(c)$, with $e_1 = X, e_2, ..., e_n$ tangent to M at p.

As usual, one denotes by $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j \in \{1, ..., n\}$, $r \in \{n + 1, ..., m\}$. We have

$$\text{Ric}\,(X) = \sum_{j=2}^{n} K(e_1 \wedge e_j). \tag{4.3}$$

If we take $X = W = e_1$ and $Y = Z = e_j$ in the Gauss equation, we have

$$g(R(e_1, e_j)e_j, e_1) = c - s(e_j, e_j) + \sum_{r=n+1}^{m} [h^r_{11}h^r_{jj} - (h^r_{1j})^2] - g(P^{\perp}, h(e_j, e_j)), \tag{4.4}$$

respectively, from the Gauss equation if we put $X = Z = e_1$, $Y = W = e_j$, we obtain

$$g(R(e_j, e_1)e_1, e_j) = c - s(e_1, e_1) + \sum_{r=n+1}^{m} [h^r_{11}h^r_{jj} - (h^r_{1j})^2] - g(P^{\perp}, h(e_1, e_1)). \tag{4.5}$$

Because

$$K(e_1 \wedge e_j) = \frac{1}{2}[g(R(e_1, e_j)e_j, e_1) + g(R(e_j, e_1)e_1, e_j)], \tag{4.6}$$

from the equations (4.4) and (4.5), we have

$$\begin{aligned} K(e_1 \wedge e_j) = c - \frac{1}{2}[s(e_j, e_j) + s(e_1, e_1)] + \sum_{r=n+1}^{m} [h^r_{11}h^r_{jj} - (h^r_{1j})^2] \\ - \frac{1}{2}[g(P^{\perp}, h(e_j, e_j)) + g(P^{\perp}, h(e_1, e_1))]. \end{aligned} \tag{4.7}$$

By substituting the equation (4.7) in (4.3), we find

$$\begin{aligned} \text{Ric}\,(X) = (n - 1)c - \frac{1}{2}[\text{trace}\, s + (n - 2)s(X, X)] \\ - \frac{1}{2}[n\omega(H) + (n - 2)g(P^{\perp}, h(X, X))] \\ + \sum_{j=2}^{n} \sum_{r=n+1}^{m} [h^r_{11}h^r_{jj} - (h^r_{1j})^2]. \end{aligned}$$

The last equation implies

$$\begin{aligned} \text{Ric}\,(X) \leq (n - 1)c - \frac{1}{2}[\text{trace}\, s + (n - 2)s(X, X)] \\ - \frac{1}{2}[n\omega(H) + (n - 2)g(P^{\perp}, h(X, X))] \\ + \sum_{j=2}^{n} \sum_{r=n+1}^{m} h^r_{11}h^r_{jj}. \end{aligned} \tag{4.8}$$

Obviously one has

$$h_{11}^r\left(\sum_{j=2}^n h_{jj}^r\right) \le \frac{1}{4}\left(\sum_{i=1}^n h_{ii}^r\right)^2,$$

with equality if and only if

$$h_{11}^r = h_{22}^r + \cdots + h_{nn}^r.$$

From the equation (4.8), it follows that

$$\begin{aligned}\text{Ric}\,(X) \le\ & \frac{n^2}{4}\|H\|^2 + (n-1)c - \frac{1}{2}[\text{trace}\ s + (n-2)s(X,X)]\\ & -\frac{1}{2}[n\omega(H) + (n-2)g(P^{\perp}, h(X,X))].\end{aligned} \tag{4.9}$$

(2) If a unit vector X at p satisfies the equality case of (4.2), we get

$$\begin{cases} h_{1i}^r = 0,\ 2 \le i \le n,\ \forall r \in \{n+1,...,m\},\\ h_{11}^r = h_{22}^r + ... + h_{nn}^r,\ \forall r \in \{n+1,...,m\}.\end{cases}$$

Therefore, because $H(p) = 0$, we have $h_{1j}^r = 0$, for all $j \in \{1,...,n\}$, $r \in \{n+1,...,m\}$, that is $X \in N_p$.

(3) The equality case of inequality (4.2) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h_{ij}^r = 0, \quad 1 \le i \ne j \le n, \quad r \in \{n+1,...,m\},\\ h_{11}^r + ... + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1,...,n\}, \quad r \in \{n+1,...,m\},\end{cases}$$

which imply

$$\begin{cases} h(e_i,e_j) = 0, \quad 1 \le i \ne j \le n,\\ (n-2)H(p) = 0.\end{cases}$$

We distinguish two cases:

- (i) $n \ne 2$; $h(e_i,e_j) = 0, \forall i,j \in \{1,...,n\}$, i.e., h_p vanishes on T_pM.
- (ii) $n = 2$; then $h(e_i,e_j) = g(e_i,e_j)H(p)$, for any $i,j \in \{1,2\}$, i.e., p is a totally umbilical point.

□

5. Applications in other fields

In [11] two nice models of semi-symmetric connections are given (see also [8]):

1-st. *If a man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North Pole, then this displacement is semi-symmetric and metric.*

2-nd. *During the mathematical congress in Moscow in 1934 one evening mathematicians invented the Moscow displacement. The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the street always facing the Kremlin, then this displacement is semi-symmetric and metric.*

To our knowledge, similar models of semi-symmetric non-metric connections are not given.

Then, we expect to develop models and applications of semi-symmetric non-metric connections in some other fields, e.g., in quantum and theoretical chemistry, directions in which the first author of this paper and her co-workers took preliminary steps, using the tools of differential geometry (see [4], [12]).

For instance, one may devise applications in the topology of multidimensional surfaces representing the energy dependence of chemical edifices as function of symmetry-classified changes in molecular geometries, with relevance in modeling physical properties and chemical transformations of molecules and materials. In this sense, a special situation, which deserves further attention, with the help of analytic tools outlined here, would be the case of spontaneous breaking of molecular symmetry in systems with energy surfaces containing the so-called conical intersections (see [4]). We foresee that the use conformal changes and semi-symmetric connections could open a new geometric perspective on such problems, pertaining to molecular symmetry and chemical dynamics.

Another particular application may occur in a class of problems recently approached in [12], in preliminary manner. Namely, we envisage analyzing through the eyes of analytic geometry descriptors the effective Hamiltonian models describing special properties, such as luminescence due to inter-shell quantum transitions in lanthanide-based materials. The involved quantum operators admit the formulation as 3D or multi-dimensional surfaces, the related wave functions behaving as irreducible representations in the group of sphere (spherical harmonic functions). The effective Hamiltonians can be thought as (topological) invariants, the semi-symmetric connection approach representing a new analytic perspective.

Such a formalization, aiming to reach very concrete promises in the property engineering of new materials (e.g. lanthanide luminescence, used in energy-saving lighting devices) represents a challenge for the potential extension of devised methodologies into multi-disciplinary problems, in a line of rationales involving mathematics, quantum calculations and materials sciences.

Of course, for all models briefly presented above the curvatures associated to the semi-symmetric metric or non-metric connection play a very important role. Then, to fix the background, to define well types of curvatures becomes essential not only as a theoretical aspects, but also to be used in applications.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Affiliations

ADELA MIHAI
ADDRESS: Technical University of Civil Engineering Bucharest, Dept. of Mathematics and Computer Science, 020396, Bucharest-Romania, and
Transilvania University of Brasov, Interdisciplinary Doctoral School, 500036, Brasov-Romania.
E-MAIL: adela.mihai@utcb.ro; adela.mihai@unitbv.ro
ORCID ID: 0000-0003-2033-8394

ION MIHAI
ADDRESS: University of Bucharest, Dept. of Mathematics, 010014, Bucharest-Romania.
E-MAIL: imihai@fmi.unibuc.ro
ORCID ID: 0000-0003-3782-2983