Çanakkale Onsekiz Mart University

Journal of Advanced Research in Natural and Applied Sciences

dergipark.org.tr/tr/pub/jarnas

Open Access

e-ISSN: 2757-5195 | 2024, Vol. 10, Issue 2, Pages: 476-492 | doi.org/10.28979/jarnas.1440658

Transversal Lightlike Submersions

Esra Karataş[1], Cumali Yıldırım[2]

[1,2]Department of Mathematics, Faculty of Arts and Sciences, İnönü University, Malatya, Türkiye

Article History

Received: 21 Feb 2024

Accepted: 29 Apr 2024

Published: 25 Jun 2024

Research Article

Abstract − In this paper, we introduce the concept of transversal lightlike submersions from semi-Riemannian manifolds onto semi-Riemannian manifolds. Specifically, we present the concepts of transversal r-lightlike and isotropic transversal lightlike submersions and examine the geometry of foliations formed by these submersions through various examples. In this way, we demonstrate certain points where transversal r-lightlike submersions differ from semi-Riemannian submersions. Furthermore, we investigate O'Neill's tensors for transversal r-lightlike submersions and examine the integrability of certain distributions by employing these tensor fields. Thus, valuable information regarding such submersions' geometric structures and properties can be provided, paving the way for new research avenues. We finally discuss the need for further research.

Keywords − *Transversal submersion, Riemannian submersion, lightlike manifold, lightlike submersion*

1. Introduction

Riemannian submersions are foundational mappings within the realm of differential geometry, serving as potent instruments for unraveling the geometric properties of Riemannian manifolds. These submersions allow for methodically examining the interactions between several manifolds. Their importance spans several fields in pure mathematics and theoretical physics, providing a detailed framework for examining the complex interactions between different geometries and providing deep insights into the structure of the physical universe.

O'Neill [1] and Gray [2] introduced the theory of Riemannian submersion, which has subsequently become the subject of numerous studies [3-12]. Consequently, it has become a useful tool for clarifying the structure of Riemannian manifolds. It is well known that when M_1 and M_2 are Riemannian manifolds, the fibers become Riemannian manifolds; however, it has been noted that the fibers of f may not be semi-Riemannian when the manifolds are semi-Riemannian [13].

Şahin has recently introduced and studied the concept of submersion from lightlike manifolds onto semi-Riemannian manifolds in [14], along with the submersion from semi-Riemannian manifolds onto lightlike manifolds in [13], providing significant insights into the geometric relationship between these disparate manifold types.

Hereinafter, we will initially provide an overview of a lightlike manifold and subsequently introduce the concept of transversal submersion from semi-Riemannian manifolds to semi-Riemannian manifolds. We will investigate specific examples to assess the possibility of constructing such a submersion and draw conclusions based on our analysis. Thus, the concept of transversal lightlike submersion will pave the way for innovative

[1]esrakrts4422@gmail.com (Corresponding Author); [2]cumali.yildirim@inonu.edu.tr

geometric inquiries.

2. Lightlike Manifolds

Let V be a real vector space and g_1 be a bilinear form on V. If there exists a non-zero vector ξ in V such that $g_1(\xi, v) = 0$, for every $v \in V$, then g_1 is considered degenerate on V; otherwise, it is termed as non-degenerate. On the other hand, if $g_1(v, v) > 0$, then g_1 is said to be positively defined on V; conversely, $g_1(v, v) < 0$, then g_1 is said to be negatively defined on V. Consequently, a positive or negative defined g_1 is deemed to be non-degenerate [15].

Consider V as a vector space and suppose that there exists a symmetric bilinear form g_1 on V. In this case, there exist bases $\{e_i\}$ on V that

$$\begin{pmatrix} 1 \leq i \leq r & ; & g_1(e_i, e_i) & = & 0 \\ 1 \leq j \leq q & ; & g_1(e_j, e_j) & = & -1 \\ 1 \leq k \leq p & ; & g_1(e_k, e_k) & = & 1 \\ i \neq j & ; & g_1(e_i, e_j) & = & 0 \end{pmatrix}$$

These bases are referred to as orthonormal bases, and the triplet (r, q, p) is the type of the bilinear form g_1 [16]. Let (M_1, g_1) denote a real differentiable n dimensional manifold, where g_1 is a symmetric tensor field of type (0,2). Assume that M_1 is paracompact. The radical space of $T_x M_1$ denoted by Rad $T_x M_1$ and given as

$$\text{Rad } T_x M_1 = \{\xi \in T_x M_1 : g_1(\xi, X) = 0, X \in T_x M_1\}$$

The nullity degree of g_1 corresponds to the dimension of $T_x M_1$. Suppose Rad TM_1 corresponds to the radical subspace of Rad $T_x M_1$ for every $x \in M_1$. In this case, Rad TM_1 becomes the radical distribution of M_1, and if $0 < r \leq n$, this manifold M_1 is called a lightlike manifold [13]. The non-degenerate symmetric bilinear form g_1 on V is referred to as a semi-Euclidean metric, in this case, V is termed as a semi-Euclidean space [15].

We note that $g_1(v, v)$>0 or $v = 0$, then v is defined as spacelike. Similarly, if $g_1(v, v)$<0, then v is defined as timelike. Moreover, if $g_1(v, v)$=0 and $v \neq 0$, then v is defined as lightlike (null, isotropic), where $v \in V$, V is a semi-Euclidean space [15].

Afterward, we present the concepts of Riemannian and lightlike submersions, necessary for providing transversal lightlike submersions.

3. Riemannian Submersions

In this section, the definition of Riemann submersions is provided, along with some significant information concerning these submersions.

Let (M_1, g_1) and (M_2, g_2) be m and n dimensional Riemannian manifolds, respectively, and $f: M_1 \to M_2$ be a submersion. In this case, $\text{rank} f = \dim M_2 < \dim M_1$. For any $x \in M_2$, the fiber $F_x = f^{-1}(x)$ is a submanifold of M_1 with dimension $r = (m - n)$. Submanifolds $f^{-1}(x)$ are called submersion fibers.

The integrable distribution $\mathcal{V}$ of the submersion $f: M_1 \to M_2$ in (M_1, g_1) is defined by $\mathcal{V}_p = \ker f_{*\text{p}}$, where $p \in M_1$. $\mathcal{V}_p$ is called the vertical distribution of submersion. Furthermore, $\mathcal{H}_p = (\mathcal{V}_p)^\perp$ is orthogonal to and complements the vertical distribution. We refer to the $\mathcal{H}$ distribution as the horizontal distribution of the submersion [17, 18].

Consider (M_1, g_1) and (M_2, g_2) as Riemannian manifolds. A differentiable mapping f is referred to as a Riemannian submersion if it satisfies the following conditions:

i. f has maximal rank.

ii. For any $p \in M_1$, f_{*_p} preserves the length of $X_p \in \Gamma(\mathcal{H}_p)$, where $\mathcal{H}_p$ represents the horizontal vectors [1].

The first condition in the definition ensures that the mapping is a submersion. The second condition states that the f_* derivative transformation at the point $p \in M_1$ is a linear isometry from the horizontal space $\mathcal{H}_\mathrm{p}$ to the tangent space $T_{f(p)}M_2$. Therefore, $g_{1_p}(u, v) = g_{2f(p)}(f_{*p}u, f_{*p}v)$, holds for $u, v \in \mathcal{H}_p$, $p \in M_1$ [17]. Furthermore, given that X is horizontal and f–related to a vector field $\tilde{X}$ on M_2, that is, $f_*(X) = \tilde{X}_f(p)$ for any $p \in M_1$ a vector field X on M_1 is considered basic [1].

Proposition 3.1 Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, where $f: (M_1, g_1) \to (M_2, g_2)$ is a Riemannian submersion, and let ∇ and ∇' denote the Levi-Civita connections of M_1 and M_2, respectively. Suppose the basic vector fields X and Y on M_1 are f −related to the vector fields X' and Y'. In this case, the following equations are obtained [18]:

1. $g_1(X, Y) = g_2(X', Y') \circ f$

2. The basic vector field $h[X, Y]$ corresponds to $[\tilde{X}, \tilde{Y}]$

3. The basic vector field $h(\nabla_X Y)$ corresponds to $\nabla_{X'} Y'$

4. Lightlike Submersions

Sahin and Gündüzalp previously introduced several concepts related to lightlike submersions in [13]. Let (M_1, g_1) be a semi-Riemannian manifold, (M_2, g_2) be an r-lightlike manifold. Consider a differentiable submersion $f: M_1 \to M_2$, where f_* denotes the derivative transformation. The kernel of f_* at the point $p \in M_2$, denotes as ker f_* , is defined as [13]:

$$\ker f_* = \{X \in T_p(M_1):\ f_*(X) = 0\}$$

Case 4.1 $0 < \dim \Delta < \min\{\dim(\ker f_*), \dim(\ker f_*)^\perp\}$: In this case, Δ is the radical subspace of T_pM_1. Thus, a quasi-orthonormal basis of M_1 along ker f_* is constructed as described in [15]. Since ker f_* is a real lightlike vector space, there exists a non-degenerate subspace that complements Δ [15]. Then,

$$\ker f_* = \Delta \perp S(\ker f_*)$$

and similarly,

$$(\ker f_*)^\perp = \Delta \perp S(\ker f_*)^\perp$$

where $S(\ker f_*)^\perp$ denotes the complementary subspace of Δ in $(\ker f_*)^\perp$. Given the expression $T_pM_1 = S(\ker f_*) \perp (S(\ker f_*))^\perp$, since $S(\ker f_*)$ is non-degenerate in T_pM_1, it can be observed that $(S(\ker f_*))^\perp$ is the complementary subspace of $S(\ker f_*)$ in T_pM_1. Additionally, since $S(\ker f_*)$ and $(S(\ker f_*))^\perp$ are non-degenerate, we can observe that

$$(S(\ker f_*))^\perp = S(\ker f_*)^\perp \perp (S(\ker f_*)^\perp)^\perp$$

Then, according to Proposition 2.4 in [15], it is known that “There exists a quasi-orthonormal basis of ker f_*”. Therefore, we have the following expressions:

$$\begin{pmatrix} g_1(\xi_i,\xi_j) = g_1(N_i,N_j) = 0 & ; & g_1(\xi_i,N_j) = \delta_{ij} \\ g_1(W_\alpha,\xi_j) = g_1(W_\alpha,N_j) = 0 & ; & g_1(W_\alpha,W_\alpha) = \varepsilon_\alpha\delta_{\alpha\beta} \end{pmatrix}$$

where $i,j \in \{1,\ldots,r\}$ and $\alpha,\beta \in \{1,\ldots,t\}$. Here $\{N_i\}$ represents differentiable null vector fields of $(S(\ker f_*)^\perp)^\perp$, $\{\xi_i\}$ is the basis of Δ, and $\{W_\alpha\}$ is the basis of $S(\ker f_*)^\perp$. The set of vector fields $\{N_i\}$ is denoted by $\mathrm{ltr}(\ker f_*)$, and consider the following subspace:

$$\mathrm{tr}(\ker f_*) = \mathrm{ltr}(\ker f_*) \perp S(\ker f_*)^\perp$$

It should be noted that $\mathrm{ltr}(\ker f_*)$ and $(\ker f_*)$ are not orthogonal to each other. The space $\ker f_*$, denoted as $\mathcal{V}$, is referred to as the vertical space of T_pM_1, while $\mathrm{tr}(\ker f_*)$, denoted as $\mathcal{H}$, is called the horizontal space of T_pM_1, as is usual in the theory of Riemannian submersions. Thus, we have the decomposition:

$$T_pM_1 = \mathcal{V}_p \oplus \mathcal{H}_p$$

We notice that $\mathcal{V}$ and $\mathcal{H}$ are not orthogonal [13].

Definition 4.2 [13] Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) be an r-lightlike manifold. Consider a submersion $f: M_1 \to M_2$ satisfying the following conditions:

i. $\dim \Delta = \dim\{(\ker f_*) \cap (\ker f_*)^\perp\} = r$, $0 < r < \min\{\dim(\ker f_*), \dim(\ker f_*)^\perp\}$.

ii. f_* preserves the length of horizontal vectors, i.e. $g_1(X,Y) = g_2(f_*X, f_*Y)$ *for* $X, Y \in \Gamma(\mathcal{H})$. In this case, we can say that f is an r-lightlike submersion.

Case 4.3. [13] $\dim \Delta = \dim(\ker f_*) < \dim(\ker f_*)^\perp$. Then, $\mathcal{V} = \Delta$ and $\mathcal{H} = \mathcal{S}(\ker f_*)^\perp \perp \mathrm{ltr}(\ker f_*)$. Thus, we name f an isotropic submersion.

Case 4.4. [13] $\dim \Delta = \dim(\ker f_*)^\perp < \dim(\ker f_*)$. Then, $\mathcal{V} = \mathcal{S}(\ker f_*) \perp \Delta$ and $\mathcal{H} = \mathrm{ltr}(\ker f_*)$. Thus, we name f co-isotropic submersion.

Case 4.5. [13] $\dim \Delta = \dim(\ker f_*)^\perp = \dim(\ker f_*)$. Then, $\mathcal{V} = \Delta$ and $\mathcal{H} = \mathrm{ltr}(\ker f_*)$. Thus, we name f totally lightlike submersion.

Therefore, in conjunction with this information, we present a new concept.

5. Transversal Lightlike Submersions

In this section, we will introduce the concept of transversal submersion and provide four related examples. Through these examples, we will explore the existence of various types of submersions. Additionally, we will introduce O'Neill tensors for transversal submersions, which will lead to different results regarding their overall properties.

Firstly we note that a basic vector field on M_1 is a horizontal vector field X that is f-related to vector field $\tilde{X}$ on M_2, meaning that $f_*(X_p) = \tilde{X}_{f(p)}$ for all $p \in M_1$ (Where f_* is a derivative map). Every vector field $\tilde{X}$ on M_2 has a unique horizontal lift X to M_1, and X is basic. Therefore, the correspondence $X \leftrightarrow \tilde{X}$ establishes a one-to-one relationship between fundamental vector fields on M_1 and arbitrary vector fields on M_2 [13]. Thus, we can give the following definition.

Definition 5.1 Consider (M_1, g_1) and (M_2, g_2) be a semi-Riemannian manifold and let $f: M_1 \to M_2$ be a submersion. If the condition

$$g_1(X,Y) = g_2(f_*(X), f_*(Y)) \tag{5.1}$$

holds for all $X, Y \in \Gamma(S(\ker f_*)^\perp)$, we call the mapping f as a transversal submersion.

Therefore,

- f has maximal rank,
- At each point p in M_1, the f_{*p} mapping preserves the lengths of horizontal vectors; that is, $g_{1p}(X,Y) = g_{2f(p)}\big(f_{*p}X, f_{*p}Y\big)$. This implies that at a point p in M_1, the f_* derivative transformation states a linear isometry from $\Gamma(S(\ker f_*)^\perp)$ space onto $T_{f(p)}M_2$.

Note that for $p \in M_2$, $f^{-1}(p)$ is a submanifold with dim M_1-dim M_2.

Definition 5.2 Consider (M_1, g_1) and (M_2, g_2) as semi-Riemannian manifolds and let $f: M_1 \to M_2$ be a transversal submersion. According to Definition 4.2 in Case 4.1, f is characterized as a transversal r-lightlike submersion. Furthermore, as per Definition 4.2 in Case 4.3, f is denoted as an isotropic transversal lightlike submersion.

We will give examples of transversal r-lightlike and isotropic transversal lightlike submersions.

Example 5.3 Consider $\mathbb{R}^6_1$ and $\mathbb{R}^3_1$ to be $\mathbb{R}^6$ and $\mathbb{R}^3$endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2$$

and

$$g_2 = -(dy_1)^2 + \frac{1}{2}(dy_2)^2 + \frac{1}{2}(dy_3)^2$$

where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{y_1, y_2, y_3\}$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^3$, respectively. Moreover, we define the following map:

$$\begin{array}{cccc} f : & \mathbb{R}^6_1 & \to & \mathbb{R}^3_1 \\ & (x_1, x_2, x_3, x_4, x_5, x_6) & \to & (x_1 - x_5, x_2 + x_6, x_3 + x_4) \end{array}$$

The kernel of f_* is then given by

$$\ker f_* = \operatorname{Span}\left\{W_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5},\ W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, W_3 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\}$$

Thus,

$$(\ker f_*)^\perp = \operatorname{Span}\left\{T_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, T_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, T_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\}$$

Therefore,

$$W_1 = T_1, \qquad \Delta = \ker f_* \cap (\ker f_*)^\perp = \operatorname{Span}\{W_1\}$$

Then,

$$\operatorname{ltr}(\ker f_*) = \operatorname{Span}\left\{N = \frac{1}{2}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}\right)\right\}$$

Using $N = \frac{1}{g_1(\xi,V)}\left\{V - \frac{g_1(V,V)}{2g_1(\xi,V)}\xi\right\}$ from Equation (1.5) in [15], it is easy to check that $g_1(N,W_1) = 1$, $g_1(N,W_2) = 0$, and $g_1(N,W_3) = 0$. Thus, we give the vertical and horizontal spaces as:

$$\mathcal{V} = \mathrm{Span}\{W_1, W_2, W_3\}, \mathcal{H} = \mathrm{Span}\{T_2, T_3, N\}$$

Furthermore, since $f_*(T_2) = 2\frac{\partial}{\partial y_2}$, $f_*(T_3) = 2\frac{\partial}{\partial y_3}$, and $f_*(N) = -\frac{\partial}{\partial y_1}$, we obtain that

$$g_1(T_2,T_2) = g_2\big(f_*(T_2), f_*(T_2)\big) = 2, g_1(T_3,T_3) = g_2(f_*(T_3), f_*(T_3)) = 2$$

$$g_1(N,N) = 0, g_2(f_*(N), f_*(N)) = -1$$

Here, we state that the lengths of the vectors in $S(\ker f_*)^{\perp}$ are conserved, but we cannot say the same for $\mathrm{ltr}(\ker f_*)$. In this case, the mapping f is a transversal $1-$lightlike submersion.

Example 5.4 Consider $\mathbb{R}^6_2$ and $\mathbb{R}^3_2$ be $\mathbb{R}^6$ and $\mathbb{R}^3$endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2$$

and

$$g_2 = -(dy_1)^2 - (dy_2)^2 + \frac{1}{2}(dy_3)^2$$

where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{y_1, y_2, y_3\}$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^3$, respectively. We define the following map:

$$\begin{array}{cccc} f: & \mathbb{R}^6_2 & \to & \mathbb{R}^3_2 \\ & (x_1,x_2,x_3,x_4,x_5,x_6) & \to & (x_1+x_4, x_2+x_5, x_3+x_6) \end{array}$$

The kernel of f_* is then

$$\ker f_* = \mathrm{Span}\left\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, W_3 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6}\right\}$$

Thus,

$$(\ker f_*)^{\perp} = \mathrm{Span}\left\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, T_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, T_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6}\right\}$$

Therefore, we have $W_1 = T_1$ and $W_2 = T_2$,

$$\Delta = (\ker f_*) \cap (\ker f_*)^{\perp} = \mathrm{Span}\{W_1 = T_1,\ W_2 = T_2\}$$

Then,

$$\mathrm{ltr}(\ker f_*) = \mathrm{Span}\left\{N_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}\right), N_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}\right)\right\}$$

Moreover, we have

$$f_*(T_3) = 2\frac{\partial}{\partial y_3}, f_*(N_1) = \frac{\partial}{\partial y_1}, f_*(N_2) = \frac{\partial}{\partial y_2}$$

Then, we obtain that

$$g_1(T_3,T_3) = g_2(f_*(T_3), f_*(T_3)) = 2$$

$$g_1(N_1,N_1) = 0, g_2(f_*(N_1), f_*(N_1)) = -1, g_1(N_2,N_2) = 0, g_2(f_*(N_2), f_*(N_2)) = -1$$

Thus, f is a transversal 2-lightlike submersion.

Example 5.5 Let $\mathbb{R}_1^4$ and $\mathbb{R}_1^3$ be $\mathbb{R}^4$ and $\mathbb{R}^3$endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2$$

and

$$g_2 = -(dy_1)^2 + (dy_2)^2 + (dy_3)^2$$

where $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3\}$ are the canonical coordinates on $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. We define the following map:

$$\begin{array}{cccc} f : & \mathbb{R}_1^4 & \rightarrow & \mathbb{R}_1^3 \\ & (x_1, x_2, x_3, x_4) & \rightarrow & (x_1 + x_2, x_3, x_4) \end{array}$$

The kernel of f_* is then

$$\ker f_* = Span\left\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right\}$$

Thus,

$$(\ker f_*)^\perp = \mathrm{Span}\left\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, T_2 = \frac{\partial}{\partial x_3}, T_3 = \frac{\partial}{\partial x_4}\right\}$$

Hence, we have

$$\Delta = \ker f_* \cap (\ker f_*)^\perp = \mathrm{Span}\{W_1\}$$

Then,

$$\mathrm{ltr}(\ker f_*) = \mathrm{Span}\left\{N = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\right\}$$

Moreover,

$$f_*(N) = \frac{\partial}{\partial y_1}, f_*(T_2) = \frac{\partial}{\partial y_2}, f_*(T_3) = \frac{\partial}{\partial y_3}$$

Thus, we obtain that

$$g_1(T_2, T_2) = 1, g_2\big(f_*(T_2), f_*(T_2)\big) = 1, g_1(T_3, T_3) = 1, g_2(f_*(T_3), f_*(T_3)) = 1$$

$$g_1(N, N) = 0, g_2(f_*(N), f_*(N))) = -1$$

Hence, f is isotropic transversal 1-lightlike submersion.

Example 5.6 Let $\mathbb{R}_2^6$ and $\mathbb{R}_2^4$ be $\mathbb{R}^6$ and $\mathbb{R}^4$endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2$$

and

$$g_2 = -(dy_1)^2 - (dy_2)^2 + (dy_3)^2 + (dy_4)^2$$

where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{y_1, y_2, y_3, y_4\}$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^4$, respectively. We define the following map:

$$f \ : \quad \mathbb{R}_2^6 \quad \rightarrow \quad \mathbb{R}_2^4$$
$$(x_1, x_2, x_3, x_4, x_5, x_6) \ \rightarrow \ \left(\frac{x_1 + x_5}{\sqrt{2}}, \frac{x_2 + x_6}{\sqrt{2}}, x_3, x_4\right)$$

The kernel of f_* is then

$$\ker f_* = \operatorname{Span}\left\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}\right\}$$

Thus,

$$(\ker f_*)^{\perp} = \operatorname{Span}\left\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, T_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, T_3 = \frac{\partial}{\partial x_3}, T_4 = \frac{\partial}{\partial x_4}\right\}$$

Hence, we have $W_1 = T_1$, $W_2 = T_2$,and

$$\Delta = \ker f_* \cap (\ker f_*)^{\perp} = \operatorname{Span}\{W_1 = T_1, W_2 = T_2\}$$

Then,

$$\operatorname{ltr}(\ker f_*) = \operatorname{Span}\left\{N_1 = \frac{1}{2}\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_5}, N_2 = \frac{1}{2}\frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial}{\partial x_6}\right\}$$

Moreover, we have

$$f_*(N_1) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1}, f_*(N_2) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_2}, f_*(T_3) = \frac{\partial}{\partial y_3}, f_*(T_4) = \frac{\partial}{\partial y_4}$$

Then,

$$g_1(T_3, T_3) = g_2(f_*(T_3), f_*(T_3)) = 1, g_1(T_4, T_4) = g_2(f_*(T_4), f_*(T_4)) = 1$$

$$g_1(N_1, N_1) = 0, g_2\big(f_*(N_1), f_*(N_1)\big) = -\frac{1}{2}, g_1(N_2, N_2) = 0, g_2\big(f_*(N_2), f_*(N_2)\big) = -\frac{1}{2}$$

Thus, f is isotropic transversal 2 −lightlike submersion.

Corollary 5.7 From Definition 5.1 and the examples provided above for a transversal submersion with degenerate fibers, we can introduce the concepts of transversal r-lightlike and isotropic transversal submersion. However, it is important to note that the notions of co-isotropic and totally lightlike submersion are not applicable in this context.

Lemma 5.8 Let $f: (M_1, g_1) \rightarrow (M_2, g_2)$ be a transversal r −lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then, for all $X, Y \in \Gamma(S(\ker f_*)^{\perp})$

$$g_1(X, Y) = g_2(\tilde{X}, \tilde{Y}) \circ f$$

Proof. The proof can be made easily from the isometry condition in Definition 5.1 using (5.1).

Theorem 5.9 Let $f: (M_1, g_1) \rightarrow (M_2, g_2)$ be a transversal r −lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then $h\nabla_X Y$ is the fundamental vector field corresponding to $\overset{M_2}{\nabla}_{\tilde{X}}\tilde{Y}$, for X, $Y \in \Gamma(S(\ker f_*)^{\perp})$.

Proof. Since M_1 is a semi-Riemannian manifold with the Levi-Civita connection, the Koszul equality holds, leading to

$$2g_1(\nabla_X Y, Z) = X\big(g_1(Y, Z)\big) + Y\big(g_1(X, Z)\big) - Z\big(g_1(X, Y)\big) + g_1([X, Y], Z) + g_1([Z, X], Y) - g_1([Y, Z], X)$$

where $X, Y, Z \in \Gamma(S(\ker f_*)^{\perp})$. By utilizing Lemma 5.8, we obtain $(g_1(Y,Z)) = \tilde{X} g_2(\tilde{Y},\tilde{Z}) \circ f$. Similarly, if we generalize this equality, we have

$$2g_1(\nabla_X Y, Z) = \tilde{X} g_2(\tilde{Y},\tilde{Z}) \circ f + \tilde{Y} g_2(\tilde{Z},\tilde{X}) \circ f - \tilde{Z} g_2(\tilde{Y},\tilde{X}) \circ f + g_2([\tilde{X},\tilde{Y}],\tilde{Z}) \circ f + g_2([\tilde{Z},\tilde{X}],\tilde{Y}) \circ f - g_2([\tilde{Y},\tilde{Z}],\tilde{X}) \circ f$$

Considering that M_2 is a semi-Riemannian manifold, it has a Levi-Civita connection. From here, we can state that $\overset{M_2}{\nabla}$ satisfies Koszul's equality. Then,

$$g_1(\nabla_X Y, Z) = g_2(\overset{M_2}{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) \circ f$$

Therefore, we deduce that $h\nabla_X Y$ represents the fundamental vector field associated with $\overset{M_2}{\nabla}_{\tilde{X}} \tilde{Y}$.

Remark 5.10 *Let* $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2), in this case, the expression in Theorem 5.9 does not apply to N_1, $N_2 \in \Gamma(\mathrm{ltr}(\ker f_*))$.

Remark 5.11 Let $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). In this case, for any $U \in \Gamma(S(\ker f_*))$ and $X \in \Gamma(S(\ker f_*)^{\perp})$, $[X, U]$ is a vertical vector field.

Theorem 5.12 Consider the semi-Riemannian manifolds, M_1 and M_2, equipped with the metrics g_1 and g_2, respectively. Let $f: M_1 \to M_2$ be a transversal r-lightlike submersion. In this case, we have $g_1(\nabla_{N_1} N_2, N_3) = -g_1(N_2, \nabla_{N_1} N_3)$ for all N_1, N_2, N_3 in $\Gamma(\mathrm{ltr}(\ker f_*))$, where ∇ represents the Levi-Civita connection.

Proof. Since ∇ is the Levi-Civita connection for N_1, N_2, N_3 in $\Gamma(\mathrm{ltr}(\ker f_*))$, we have

$$(\nabla_{N_1} g_1)(N_2, N_3) = N_1(g_1(N_2, N_3)) - g_1(\nabla_{N_1} N_2, N_3) - g_1(N_2, \nabla_{N_1} N_3)$$

$$g_1(\nabla_{N_1} N_2, N_3) = -g_1(N_2, \nabla_{N_1} N_3)$$

□

Let M_1 and M_2 be semi-Riemannian manifolds, $f: M_1 \to M_2$ be transversal submersion and E, F arbitrary vector fields on M_1. Also, let the projections $h: TM_1 \to \mathcal{H}$ and $v: TM_1 \to \mathcal{V}$ denote the natural projections associated with the decomposition of $TM_1 = \mathcal{H} \oplus \mathcal{V}$. Moreover, ∇ represents the Levi-Civita connection of (M_1, g_1). We define the fundamental tensor field T of type (1,2);

$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{5.2}$$

has the following properties:

i. T exchances the role of horizontal and vertical subspaces

ii. T is vertical: $T_E = T_{vE}$

The tensor field A;

$$A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{5.3}$$

has the following properties:

i. A exchances the role of horizontal and vertical subspaces

ii. A is horizontal: $A_X = A_{hX}$

Lemma 5.13 Let $f:(M_1,g_1)\to(M_2,g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1,g_1) and (M_2,g_2). In this case, we obtain the followings:

i. $T_UV = h\nabla_UV$ (5.4)

ii. $T_U\xi = h\nabla_U\xi$ (5.5)

iii. $T_\xi V = h\nabla_\xi V$ (5.6)

iv. $T_{\xi_1}\xi_2 = h\nabla_{\xi_1}\xi_2$ (5.7)

where $U, V \in \Gamma(S(\ker f_*))$, $\xi, \xi_1, \xi_2 \in \Gamma(\Delta)$.

Proof. Here, we will consider two situations:

i. If we use (5.2) for transversal r-lightlike submersion, we can express it as

$$T_UV = h\nabla_UV \tag{5.8}$$

ii. Considering elements for which the multiplication equations by T_UV is non-zero, examine the following equations:

$$Ug_1(V,\xi) = g_1(\nabla_UV,\xi) + g_1(V,\nabla_U\xi)$$
$$g_1(\nabla_UV,\xi) = -g_1(V,\nabla_U\xi)$$
$$g_1(h\nabla_UV,\xi) = -g_1(V,v\nabla_U\xi) - g_1(V,h\nabla_U\xi)$$
$$g_1(T_UV,\xi) = \underbrace{-g_1(V,v\nabla_U\xi)}_{\neq 0} \tag{5.9}$$

where $U, V \in \Gamma(S(\ker f_*))$, $\xi \in \Gamma(\Delta)$. In this case, $T_UV \neq 0$.

$$Ug_1(V,X) = g_1(\nabla_UV,X) + g_1(V,\nabla_UX)$$
$$g_1(\nabla_UV,X) = -g_1(V,\nabla_UX)$$
$$g_1(h\nabla_UV,X) = -g_1(V,v\nabla_UX) - g_1(V,h\nabla_UX)$$
$$g_1(T_UV,X) = \underbrace{-g_1(V,v\nabla_UX)}_{\neq 0} \tag{5.10}$$

In this case, $T_UV \neq 0$ where $U,V \in \Gamma(S(\ker f_*))$, $X \in \Gamma(S(\ker f_*)^\perp)$. Consequently, from (5.9) and (5.10), we derive the non-zero equality expressed as $T_UV = h\nabla_UV$. Similarly, we can establish the proofs for (5.5)-(5.7).

Corollary 5.14 Let $f:(M_1,g_1)\to(M_2,g_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds (M_1,g_1) and (M_2,g_2). In this case, utilizing Lemma 5.13, we can deduce the expression:

$$T_{W_1}W_2 = h\nabla_{W_1}W_2 \tag{5.11}$$

where $W_1, W_2 \in \Gamma(\ker f_*)$.

Lemma 5.15 Let $f:(M_1,g_1)\to(M_2,g_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds (M_1,g_1) and (M_2,g_2). In this case, we have the following equations:

i. $T_UX = v\nabla_UX$

ii. $T_UN = v\nabla_UN$

iii. $T_\xi X = v\nabla_\xi X$

iv. $T_\xi N = v\nabla_\xi N$

where $U \in \Gamma(S(\ker f_*))$, $\xi \in \Gamma(\Delta)$, $X \in \Gamma(S(\ker f_*)^\perp)$, $N \in \Gamma(\mathrm{ltr}(\ker f_*))$.

Proof. The proof of the first equation is done in a similar way to the proof of Lemma 5.13, using (5.2) and the Levi-Civita connection. Other equations can be obtained similarly easily.

Corollary 5.16 Let $f:(M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). In this case, by using Lemma 5.15, we obtain

$$T_W F = v\nabla_W F$$

where $W \in \Gamma(\ker f_*)$, $F \in \Gamma(\mathrm{tr}(\ker f_*))$.

Lemma 5.17 Let $f:(M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). In this case, we have

i. $A_X U = h\nabla_X U$

ii. $A_X \xi = h\nabla_X \xi$

iii. $A_N U = h\nabla_N U$

iv. $A_N \xi = h\nabla_N \xi$

where $U \in \Gamma(S(\ker f_*))$, $\xi \in \Gamma(\Delta)$, $X \in \Gamma(S(\ker f_*)^\perp)$, $N \in \Gamma(\mathrm{ltr}(\ker f_*))$.

Proof. The proof of the first equation is done in a similar way to the proof of Lemma 5.13, using (5.3) and the Levi-Civita connection. Other equations can be obtained similarly easily.

Corollary 5.18 Let (M_1, g_1), (M_2, g_2) be semi-Riemannian manifold and $f:(M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion. In this case, based on Lemma 5.17, we obtain $A_F W = h\nabla_F W$, where $F \in \Gamma(\mathrm{tr}(\ker f_*))$, $W \in \Gamma(\ker f_*)$.

Lemma 5.19 Let $f:(M_1, g_1) \to (M_2, g_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). In this case, we have

i. $A_X Y = v\nabla_X Y$ (5.12)

ii. $A_X N = v\nabla_X N$ (5.13)

iii. $A_N X = v\nabla_N X$ (5.14)

iv. $A_{N_1} N_2 = v\nabla_{N_1} N_2$ (5.15)

where $X, Y \in \Gamma(S(\ker f_*)^\perp)$, $N, N_1, N_2 \in \Gamma(\mathrm{ltr}(\ker f_*))$.

Proof: The proof of (5.12) is done in a similar way to the proof of Lemma 5.13, using (5.3) and the Levi-Civita connection. Other equations can be obtained similarly easily.

Corollary 5.20 Let $f:(M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). In this case, utilizing Lemma 5.19, we derive $A_{F_1} F_2 = v\nabla_{F_1} F_2$, where $F_1, F_2 \in \Gamma(\mathrm{tr}(\ker f_*))$.

Lemma 5.21 Let (M_1, g_1), (M_2, g_2) be semi-Riemannian manifolds, ∇ be Levi-Civita connection in M_1, T and A be tensor fields, $f:(M_1, g_1) \to (M_2, g_2)$ be a transversal r-lightlike submersion. In this case, *i.* $\nabla_{W_1} W_2 = T_{W_1} W_2 + v\nabla_{W_1} W_2$ (5.16)

ii. $\nabla_W \xi = T_W \xi + v\nabla_W \xi$ (5.17)

iii. $\nabla_U W = T_U W + v\nabla_U W$ (5.18)

iv. $\nabla_U V = T_U V + v\nabla_U V$ (5.19)

v. $\nabla_U \xi = T_U \xi + v\nabla_U \xi$ (5.20)

vi. $\nabla_\xi W = T_\xi W + v\nabla_\xi W$ (5.21)

vii. $\nabla_{\xi_1} \xi_2 = T_{\xi_1} \xi_2 + v\nabla_{\xi_1} \xi_2$ (5.22)

viii. $\nabla_\xi V = T_\xi V + v\nabla_\xi V$ (5.23)

where $W, W_1, W_2, \in \Gamma(\ker f_*)$, $\xi, \xi_1, \xi_2 \in \Gamma(\nabla)$, $U, V \in \Gamma(S(\ker f_*))$.

Proof. Here, we will prove only (5.16). For any vector fields $W_1, W_2 \in \Gamma(\ker f_*)$, we can establish the equation $\nabla_{W_1} W_2 = v\nabla_{W_1} W_2 + h\nabla_{W_1} W_2$. Using Lemma 5.13 and Corollary 5.14, we obtain the equation $\nabla_{W_1} W_2 = v\nabla_{W_1} W_2 + T_{W_1} W_2$. The proof of the remaining equations can also be carried out similarly. □

Lemma 5.22 Let (M_1, g_1), (M_2, g_2) be semi-Riemannian manifolds, ∇ be Levi-Civita connection in M_1, T and A be tensor fields, $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal r −lightlike submersion. In this case, we obtain the following equations:

i. $\nabla_W N = T_W N + h\nabla_W N$

ii. $\nabla_U N = T_U N + h\nabla_U N$

iii. $\nabla_\xi N = T_\xi N + h\nabla_\xi N$

iv. $\nabla_U F = T_U F + h\nabla_U F$

v. $\nabla_\xi F = T_\xi F + h\nabla_\xi F$

vi. $\nabla_W F = T_W F + h\nabla_W F$

where $W \in \Gamma(\ker f_*)$, $U, V \in \Gamma(S(\ker f_*))$, $N \in \Gamma(\text{ltr}(\ker f_*))$, $F \in \Gamma(\text{tr}(\ker f_*))$.

Proof. Here, we will prove only second equation. For any vector fields $U \in \Gamma(S(\ker f_*))$, and $N \in \Gamma(\text{ltr}(\ker f_*))$, we can establish the equation $\nabla_U N = v\nabla_U N + h\nabla_U N$. Using Lemma 5.15, we can establish $\nabla_U N = T_U N + h\nabla_U N$. Other equations can be obtained in a similar way.

Lemma 5.23 Let (M_1, g_1), (M_2, g_2) be semi-Riemannian manifolds, ∇ be Levi-Civita connection in M_1, T and A be tensor fields, $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal r-lightlike submersion. In this case, we obtain the following equations:

i. $\nabla_X W = A_X W + v\nabla_X W$

ii. $\nabla_X \xi = A_X \xi + v\nabla_X \xi$

iii. $\nabla_N W = A_N W + v\nabla_N W$

iv. $\nabla_N U = A_N U + v\nabla_N U$

v. $\nabla_N \xi = A_N \xi + v\nabla_N \xi$

vi. $\nabla_F W = A_F W + v\nabla_F W$

vii. $\nabla_F U = A_F U + v\nabla_F U$

viii. $\nabla_F \xi = A_F \xi + v\nabla_F \xi$

where $U \in \Gamma(S(\ker f_*))$, $\xi \in \Gamma(\Delta)$, $W \in \Gamma(\ker f_*)$, $X \in \Gamma(S(\ker f_*)^{\perp})$, $N \in \Gamma(\mathrm{ltr}(\ker f_*))$, $F \in \Gamma(\mathrm{tr}(\ker f_*))$.

Proof. Here we will prove only fourth equation. For any vector fields $N \in \Gamma(\mathrm{ltr}(\ker f_*))$ and $U \in \Gamma(S(\ker f_*))$, we can establish the equation $\nabla_N U = v\nabla_N U + h\nabla_N U$. Using Lemma 5.17, we can establish the equation $\nabla_N U = A_N U + v\nabla_N U$. Other equations can be obtained in a similar way.

Lemma 5.24 Let (M_1, g_1), (M_2, g_2) be semi-Riemannian manifolds, ∇ be Levi-Civita connection in M_1, T and A be basic tensor fields, $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal r-lightlike submersion. In this instance, we obtain the following equations:

i. $\nabla_X N = A_X N + h\nabla_X N$ (5.24)

ii. $\nabla_{N_1} N_2 = A_{N_1} N_2 + h\nabla_{N_1} N_2$ (5.25)

iii. $\nabla_{F_1} F_2 = A_{F_1} F_2 + h\nabla_{F_1} F_2$ (5.26)

iv. $\nabla_X F = A_X F + h\nabla_X F$ (5.27)

v. $\nabla_N F = A_N F + h\nabla_N F$ (5.28)

vi. $\nabla_F N = A_F N + h\nabla_F N$ (5.29)

vii. $\nabla_X Y = A_X Y + h\nabla_X Y$ (5.30)

where $X, Y \in \Gamma(S(\ker f_*)^{\perp})$, $N, N_1, N_2 \in \Gamma(\mathrm{ltr}(\ker f_*))$, $F, F_1, F_2 \in \Gamma(\mathrm{tr}(\ker f_*))$.

Proof. Here we will prove only first equation. For any vector fields $X \in \Gamma(S(\ker f_*)^{\perp})$ and $N \in \Gamma(\mathrm{ltr}(\ker f_*))$, we can establish the equation $\nabla_X N = v\nabla_X N + h\nabla_X N$. Using Lemma 5.19, we can establish $\nabla_X N = A_X N + h\nabla_X N$. Other equations can be obtained in a similar way.

Corollary 5.25. Let $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then,

i. $g_1(T_U N_1, N_2) = -g_1(N_1, T_U N_2)$

ii. $g_1(T_\xi N_1, N_2) = -g_1(N_1, T_\xi N_2)$

iii. $g_1(A_N \xi_1, \xi_2) = -g_1(\xi_1, A_N \xi_2)$

iv. $g_1(A_X N_1, N_2) = -g_1(N_1, A_X N_2)$

where $U \in \Gamma(S(\ker f_*))$, $\xi_1, \xi_2 \in \Gamma(\Delta)$, $X \in \Gamma(S(\ker f_*)^{\perp})$, $N_1, N_2 \in \Gamma(\mathrm{ltr}(\ker f_*))$.

Proof. Provide the proof solely for the first equality.

Since (M_1, g_1) is a semi-Riemannian manifold the torsion-free metric connection used here is the Levi-Civita connection. For $U \in \Gamma(S(\ker f_*))$, $N_1, N_2 \in \Gamma(\mathrm{ltr}(\ker f_*))$, we have

$$U g_1(N_1, N_2) = g_1(\nabla_U N_1, N_2) + g_1(N_1, \nabla_U N_2)$$

$$g_1(v\nabla_U N_1, N_2) = -g_1(N_1, v\nabla_U N_2)$$

By using Lemma 5.15 and Lemma 5.22, we can derive the expression $g_1(T_U N_1, N_2) = -g_1(N_1, T_U N_2)$. Similarly, by using Lemma 5.23 and Lemma 5.24, we can obtain other equations.

Theorem 5.26. Let $f:(M_1,g_1)\to(M_2,g_2)$ be a transversal $r-$lightlike submersion between semi-Riemannian manifolds (M_1,g_1) and (M_2,g_2). In this case, ker f_* integrable for W_1, $W_2\in\Gamma(\ker f_*)$.

Proof. Since W_{1p}, W_{2p} are elements of $\Gamma(\ker f_{*p})$, we have $f_*(W_1)=\widetilde{W_1}=0$ and $f_*(W_2)=\widetilde{W_2}=0$. From equation in Definition 20 [1], we obtain $[\widetilde{W}_1,\widetilde{W}_2]g_2=f_*([W_1,W_2])\circ g_2$. Therefore, $[W_1,W_2]$ belongs to $\Gamma(\ker f_*)$, then $\ker f_*$ is integrable. □

Remark 5.27 Let $f:(\widetilde{M}_1,\tilde{g}_1)\to(\widetilde{M}_2,\tilde{g}_2)$ be a transversal $r-$ lightlike submersion between semi-Riemannian manifolds $(\widetilde{M}_1,\tilde{g}_1)$ and $(\widetilde{M}_2,\tilde{g}_2)$, where ∇ is the Levi-Civita connection corresponding to $\tilde{g}_1$ on the manifold M_1. $S(\ker f_*)$ and $\text{tr}(\ker f_*)$ denote the corresponding screen distribution and transversal lightlike vector bundle of M_1, respectively. By utilizing the expression $T\widetilde{M}_1=\ker f_*\oplus\text{tr}(\ker f_*)$, we can derive $\nabla_UV=v\nabla_UV+h\nabla_UV$, where $U,V\in\Gamma(\ker f_*)$. Furthermore, using (5.4), we obtain $\tilde{\nabla}_UV=\hat{\nabla}_UV+T_UV$, where T_UV is associated with $\Gamma(\text{tr}(\ker f_*))$ and $\tilde{\nabla}_UV$ is associated with $\Gamma(\ker f_*)$.

Remark 5.28 Let P denote the projection morphism of $(\ker f_*)$ onto $S(\ker f_*)$ based on the decomposition of V. By utilizing the equation $\nabla_UPV=v\overset{*}{\nabla}_UPV+h\overset{*}{\nabla}_UPV$, we get the following equation:

$$\nabla_UPV=\widehat{\overset{*}{\nabla}}_UPV+\overset{*}{T}_UPV \tag{5.31}$$

where $\overset{*}{T}_UPV$ is associated with $\Gamma(\Delta)$, while $\widehat{\overset{*}{\nabla}}_UPV$ is associated with $\Gamma(\ker f_*)$.

Theorem 5.29 Let $f:(\widetilde{M}_1,\tilde{g}_1)\to(\widetilde{M}_2,\tilde{g}_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds $(\widetilde{M}_1,\tilde{g}_1)$ and $(\widetilde{M}_2,\tilde{g}_2)$. In this case, the necessary and sufficient condition for the integrability $S(\ker f_*)$ is that $\overset{*}{T}_UPV=\overset{*}{T}_VPU$.

Proof. Let U, $V\in\Gamma(\ker f_*)$, $N\in\Gamma(\text{ltr}(\ker f_*))$. Since ∇ is torsion-free, we have $\tilde{g}_1([U,V],N)=\tilde{g}_1(\nabla_UV,N)-\tilde{g}_1(\nabla_VU,N)$. From (5.31), it is easy to see that $\tilde{g}_1([U,V],N)=\tilde{g}_1(\widehat{\overset{*}{\nabla}}_UPV+\overset{*}{T}_UPV,N)-\tilde{g}_1(\widehat{\overset{*}{\nabla}}_VPU+\overset{*}{T}_VPU,N)$. Then, since $\tilde{g}_1(\widehat{\overset{*}{\nabla}}_UPV,N)=0$ and $\tilde{g}_1(\widehat{\overset{*}{\nabla}}_VPU,N)=0$, we have

$$\tilde{g}_1([U,V],N)=\tilde{g}_1\left(\overset{*}{T}_UPV,N\right)-\tilde{g}_1\left(\overset{*}{T}_VPU,N\right)$$

□

Theorem 5.30 Let $f:(M_1,g_1)\to(M_2,g_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds (M_1,g_1) and (M_2,g_2). In this case, for $W_1,W_2\in\Gamma(\ker f_*)$, we have,

$$T_{W_1}W_2=T_{W_2}W_1$$

Proof. Utilizing equation (5.11) for $W_1,W_2\in\Gamma(\ker f_*)$, we observe $T_{W_1}W_2-T_{W_2}W_1=h[W_1,W_2]$. By employing Theorem 5.26, as $[W_1,W_2]\in\Gamma(\ker f_*)$, we deduce $h[W_1,W_2]=0$. Consequently, this completes proof.

Theorem 5.31 Let $f:(M_1,g_1)\to(M_2,g_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds (M_1,g_1) and (M_2,g_2). In this case, if $S(\ker f_*)^\perp$ is integrable, we obtain $A_XY=A_YX$ for $X,Y\in\Gamma(S(\ker f_*)^\perp)$. Conversely, if $A_XY=A_YX$, we have $[X,Y]\in\mathcal{H}$.

Proof. We will prove this theorem by considering two situations together.

i. Since ∇ is the Levi-Civita connection, the following equation holds:

$$g_1([X,Y],U) = g_1(\nabla_X Y, U) - g_1(\nabla_Y X, U)$$

where $X, Y \in \Gamma(S(\ker f_*)^\perp)$, $U \in \Gamma(S(\ker f_*))$. By using (5.30), we can derive the following expression.

$$g_1([X,Y],U) = g_1(A_X Y, U) - g_1(A_Y X, U)$$

If $S(\ker f_*)^\perp$ is integrable, we can further simplify the equation and obtain:

$$g_1(A_X Y, U) = g_1(A_Y X, U) \tag{5.32}$$

ii. From the equation $g_1([X,Y],N) = g_1(\nabla_X Y, N) - g_1(\nabla_Y X, N)$ and the integrability of $S(\ker f_*)^\perp$, we obtain the following equation:

$$g_1(A_X Y, N) = g_1(A_Y X, N) \tag{5.33}$$

If we consider (5.32) and (5.33) together, it follows that $A_X Y = A_Y X$. We can easily show that if $A_X Y = A_Y X$, then $[X,Y] \in \mathcal{H}$.

We also note that A has the alternation property $A_X Y = -A_Y X$ for a Riemannian submersion. However, this situation differs for transversal r-lightlike submersions.

Theorem 5.32 Let $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal r-lightlike submersion between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). In this case, if the $\mathrm{ltr}(\ker f_*)$ distribution is parallel in the direction of the $S(\ker f_*)^\perp$ distribution, we obtain the equality $A_X Y = -A_Y X$, where $X, Y \in \Gamma(S(\ker f_*)^\perp)$, $N \in \Gamma(\mathrm{ltr}(\ker f_*))$.

Proof. We first establish $A_X X = 0$ for any $\mathrm{X} \in \Gamma(S(\ker f_*)^\perp)$. Let $\mathrm{X}, Y \in \Gamma(S(\ker f_*)^\perp)$, then we derive $V g_1(X,X) = 2g_1(\nabla_V X, X)$, where $V \in \Gamma(S(\ker f_*))$. By utilizing Remark 5.11, we then have $2g_1(\nabla_V X, X) = 2g_1(\nabla_X V, X)$. Subsequently, it becomes apparent that $2g_1(\nabla_X V, X) = -2g_1(\nabla_X X, V)$. Furthermore, in accordance with (5.30), we conclude that

$$2g_1(\nabla_X V, X) = -2g_1(A_X X, V) \tag{5.34}$$

On the other hand, since M_1 is a semi-Riemannian manifold, $g_1(X,X)$ is constant on each fiber, and thus $V g_1(X,X) = 0$. From this, we conclude that $g_1(A_X X, V) = 0$. However, the condition for the result to be zero relies on two possibilities: either $A_X X \in \Gamma(\Delta)$ or $A_X X = 0$. It can be observed that if $A_X X = 0$, then $A_X Y = -A_Y X$. If we consider the expression $g_1(A_X X, N)$ for $N \in \Gamma(\mathrm{ltr}(\ker f_*))$, using (5.12), we obtain

$$g_1(A_X X, N) = g_1(\nabla_X X, N) = X g_1(X, N) - g_1(\nabla_X N, X)$$

$$g_1(A_X X, N) = -g_1(\nabla_X N, X)$$

Then, if the $\mathrm{ltr}(\ker f_*)$ distribution for $N \in \Gamma(\mathrm{ltr}(\ker f_*))$ is parallel in the direction of the $S(\ker f_*)^\perp$ distribution, then, $\nabla_X N \in \Gamma(\mathrm{ltr}(\ker f_*))$, we have

$$g_1(A_X X, N) = 0 \tag{5.35}$$

Thus, from (5.34) and (5.35), we obtain the expression

$$A_X Y = -A_Y X$$

6. Conclusion

In this study, we introduced the concept of transversal lightlike submersions, and to illustrate the existence of such a structure, we offer illustrative examples. Our research delves into significant geometric analyses by examining O'Neill tensors for submersion, which we have defined as transversal lightlike submersions. In this way, various connections were obtained according to vector fields selected from certain fibers by utilizing these tensor fields, and meaningful results were obtained by investigating the integrability of certain distributions.

Thus, meaningful outcomes can be derived by computing various curvatures on the structure established for transversal submersions. Moreover, examining these submersions from two perspectives, transversal r-lightlike and isotropic transversal lightlike submersions, facilitates a geometric comparison. These investigations offer valuable insights into the intrinsic geometric properties of such mappings, potentially paving the way for new avenues of research.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

References

[1] B. O'Neill, *The fundamental equations of a submersion*, Michigan Mathematical Journal 13 (4) (1966) 459–469.

[2] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, Journal of Mathematics and Mechanics 16 (7) (1967) 715–737.

[3] R. H. Escobales Jr, *Riemannian submersions with totally geodesic fibers*, Journal of Differential Geometry 10 (2) (1975) 253–276.

[4] S. Ianuş, R. Mazzocco, G. E. Vilcu, *Riemannian submersions from quaternionic manifolds*, Acta Applicandae Mathematicae 104 (2008) 83–89.

[5] V. N. Beretovskii, L. Guijarro, *A metric characterization of Riemannian submersions*, Annals of Global Analysis and Geometry 18 (6) (2000) 577–588.

[6] Ş. Eken Meriç, M. Gülbahar, E. Kılıç, *Some inequalities for Riemannian submersions*, Analele Stiintifice Ale Universitatii Al I Cuza Din Iasi-Serie Noua-Matematica 63 (3) (2017) 1–12.

[7] B. Şahin, *Riemannian submersions from almost Hermitian manifolds*, Taiwanese Journal of Mathematics 17 (2) (2013) 629–659.

[8] F. Narita, *Riemannian submersion with isometric reflections with respect to the fibers*, Kodai Mathematical Journal 16 (3) (1993) 416–427.

[9] M. A. Akyol, S. Beyendi, *Riemannian submersions endowed with a semi-symmetric non-metric connection*, Konuralp Journal of Mathematics 6 (1) (2018) 188–193.

[10] H. Demir, R. Sarı, *Riemannian submersions with quarter-symmetric non-metric connection*, Journal of Engineering Technology and Applied Sciences 6 (1) (2021) 1–8.

[11] E. Karataş, S. Zeren, M. Altin, *Riemannian submersions endowed with a new type of semisymmetric non-metric connection*, Thermal Science 27 (4B) (2023) 3393–3403.

[12] Y. Gündüzalp, *On the geometry of Riemannian submersions*, Master's Thesis İnönü University (2007) Malatya.

[13] B. Şahin, Y. Gündüzalp, *Submersion from semi-Riemannian manifolds onto lightlike manifolds*, Hacettepe Journal of Mathematics and Statistics 39 (1) (2010), 41–53.

[14] B. Şahin, *On a submersion between Reinhart lightlike manifolds and semi-Riemannian manifolds*, Mediterranean Journal of Mathematics 5 (3) (2008) 273–284.

[15] K. L. Duggal, A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer Academic, Dordrecht, 1996.

[16] K. Nomizu, Fundamentals of linear algebra, McGraw-Hill, New York, 1966.

[17] B. Sahin, Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications, Academic Press, 2017.

[18] M. Falcitelli, S. Ianus, A. M. Pastore, Riemannian submersions and related topics, World Scientific, 2004.