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Research Article

Unbalanced fully fuzzy solid transportation problem: Solution strategy and some novel prospects

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Abstract

This study investigates the unbalanced solid transportation problem in a fuzzy environment by looking at the importance of solid transportation problem over classical transportation where the supply of sources and the capacity of vehicles are less than the demand for destinations. The solution of such problems obtained by the existing methods involves a dummy source/dummy vehicle or both, but in reality the dummy source or dummy vehicle has no physical significance and the quantity transported either by the dummy source or by the dummy vehicle is not actually transported. In these situations, the demand for some of the destinations remains unfulfilled and the problem is still unsolved in terms of real-life applications. So, the main question is to find the availability of which of the existing sources and the capacity of which vehicle should be increased to fulfill the total destination requirements with the minimum transportation cost possible. To our knowledge, no existing method in the literature could provide us this information. Therefore, a new method has been proposed to fill this gap. By analyzing the optimal solution obtained through the proposed method, we can identify the availability of which sources and the capacity of which vehicles should be increased to fully satisfy demand. Due to the uncertainty occurring in evaluating the parameters of the real-life problem, the data have been considered as triangular fuzzy numbers, and a fuzzy optimal solution is obtained for the same. Finally, a real-life unbalanced solid transport problem is solved to demonstrate the applicability of the suggested methodology.

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1. Introduction

Transportation plays a crucial role in the process of globalization and is particularly significant in the context of the expansion of global trade. The transportation problem is a special kind of linear programming problem which was initially introduced by [20] in 1941. The transportation problem is mainly concerned with efficiently delivering goods from a source to a destination while minimizing the total cost, while satisfying supply and demand constraints. For many companies, transportation costs are a significant component of their expenses, with millions of dollars spent on transporting goods from suppliers to customers. Consequently, proper planning for the transportation of products from sources to destinations is crucial. Therefore, various approaches [3, 4, 10, 21, 23, 39, 41] have been proposed in the literature to find the optimal planning of the transport of resources.

The traditional formulation of the transportation problem is considered with only two constraints: supply constraints and demand constraints, assuming that a single mode of transportation is sufficient to supply the product from sources to destinations. However, in real life, there are multiple transport facilities available such that per unit transportation cost, transporting time, fuel consumption rate, etc. varies with these facilities. Therefore, in a classical transportation framework, if more than one transport facility is considered for transportation, then the extended problem is referred to as a solid transportation problem (STP) [19]. In STP, in addition to supply and demand constraints, other constraints known as vehicle loading capacity constraints are also included.

The parameters such as transportation cost, supply, demand, vehicle loading capacity, etc. may not be precisely defined due to lack of evidence, incomplete information, or instability of the financial market to solve the real-life transportation problem. To deal with this imprecision, Zadeh [44] developed fuzzy set theory in which the uncertainty of the parameters is determined by their membership value $\in [0, 1]$. Numerous researchers [1,6,8,14,22,34,35,42] have used fuzzy set theory to cope with uncertainty or imprecision in real-life transportation problems. For the fuzzy transportation problem, Ebrahimnejad [14] suggested a two-step technique in which all parameters are considered triangular fuzzy numbers. Bagheri et al. [6] developed the DEA approach for the fuzzy transportation problem with parameters in the form of triangular fuzzy numbers. Using parabolic fuzzy numbers, Adhami and Ahmad [1] captured the uncertainty of the transportation problem with multiple objectives. Singh et al. [40] formulated a bilevel transportation problem with neutrosophic numbers. Garg and Rizk-Allah [16] proposed a novel approach to obtain the solution to the rough multi-objective transportation problem. To present the demand and supply aspects in multi-objective multimodal transportation planning problem, Zhang et al. [45] incorporated the fuzzy set theory. Roy et al. [33] the model of the fixed-charge solid-transport problem where the coefficients of the objective functions and constraints are represented by fuzzy rough variables.

The uncertainty in the parameters of the STP is also handled by fuzzy set theory in various studies, such as Kocken and Sivri [25] proposed an approach to generate all optimal solutions of the fuzzy STP parametrically. Das et al. [12] developed the fixed-charge STP model considering the safety arrangements for transporting items. Samanta et al. [38] investigated profit maximization STP in fuzzy environment and solved it by the Genetic Algorithm. To solve transportation problems, various researchers [11,28,29,29,32,37] have employed intuitionistic fuzzy set theory/neutrosophic set theory, both of which extend fuzzy set theory. Roy and Midya [32] introduced the concept of product blending in fixed-charge STP under an intuitionistic fuzzy environment. Recently, in a green supply chain network, Midya et al. [28] studied multistage STP with intuitionistic fuzzy parameters. To solve the multi-objective STP, Chhibber et al. [11] developed an intuitionistic fuzzy approach that incorporates non-linear membership and non-membership functions. Gupta et al. [18] utilized neutrosophic set theory to introduce neutrosophic goal programming as a solution for the multi-objective model of transportation problems.

In the aforementioned articles, researchers primarily focused on obtaining crisp solutions to problems with fuzzy parameters. The fuzzy solution holds more information than the crisp solution and hence provides better insight of the solution so obtained. Due to this characteristic of fuzzy solution, Numerous researchers [5, 12, 13, 17, 43] paid more attention to obtaining a fuzzy solution for the fuzzy problem rather than crisp solutions, and thus the problems were named a full fuzzy transportation problem (FFTP). Dhanasekar et al. [13] developed the fuzzy Hungarian MODI method to solve FFTP with triangular and trapezoidal fuzzy numbers. Ghosh et al. [17] obtained the fully intuitionistic fuzzy solution to the multi-objective fixed-charge solid transportation problem. To solve FFTP, Bagheri et al. [5] proposed a fuzzy DEA-based methodology. These approaches are helpful in finding the fully fuzzy solution to balanced transportation problems. However, very few strategies have been proposed to obtain a fully fuzzy solution to unbalanced transportation problems. for instance, Chakraborty and Jana [9] proposed extensions of the existing approaches such as Vogel's approximation method, least cost method, and modified distribution method to solve the unbalanced FFTP. Muthuperumal [30] proposed a novel method for an unbalanced transportation problem with fuzzy triangular parameters. Rani and Gulati [31] applied the fuzzy programming technique to solve an unbalanced multiproduct transportation problem with trapezoidal fuzzy numbers. For an unbalanced fully fuzzy solid transportation problem (FFSTP), Kaur et al. [24] developed a new methodology with dummy facilities.

In this paper, an unbalanced transportation problem in which the total supply of sources and the total loading capacity of the vehicles is less than the total demand of destinations has been studied. In all existing methods to solve such an unbalanced FFSTP, generally a dummy source and dummy vehicles are introduced to balance the problem. But these fictitious sources and vehicles have no existence in reality. So, any supply from a fictitious source is not transported in actuality, and hence the demand for some of the destinations remains unfulfilled. This drawback of the existing approaches has inspired us to propose such a method for an unbalanced FFSTP that can give a solution which does not involve the dummy source/destination/vehicle in the final solution. Furthermore, the proposed method gives us insight into which source(s) and vehicle(s) should increase capacity to meet the destination's demand at the minimum possible transportation cost.

The key motivations for this proposed work are as follows: Akbari et al. [2] addressed the STP by considering all parameters as crisp numbers. However, in real-life applications, data related to the problem may not be available in crisp numbers due to various economic and environmental factors. To address this uncertainty, our study uses fuzzy numbers. The methods proposed by Srinivasan et al. [43], Ghosh et al. [17] aim to find optimal solutions for FFSTP or fully intuitionistic fuzzy STP, but these methods are not applicable to unbalanced FFSTP. In contrast, our proposed method efficiently handles unbalanced FFSTP. The existing solutions for the unbalanced FFSTP such as those proposed by Kaur et al. [24] and Rani and Gulati [31] involve the use of dummy sources, destinations, or vehicles. However, in real-life applications, these dummy facilities are of no significance and the quantities supposed to be supplied by dummy sources or vehicles are not actually provided to the destinations. Hence, in this study, we propose a novel method to solve the unbalanced FFSTP, ensuring that the destination requirements are fulfilled without the involvement of dummy facilities. Chakraborty and Jana [9] and Mahmoodirad et al. [27] solved the FFTP considering a single mode of transport. However, practical problems often involve multiple transport facilities with varying transportation costs and times. Our proposed method simultaneously handles multiple transport facilities. Samanta et al. [36] proposed the Vogel approximation method to obtain the crisp optimal solution of the STP with fuzzy parameters. However, the solution represented by fuzzy numbers provides more informative and realistic results compared to the crisp solution. In this study, a new method is proposed to calculate the optimal fuzzy solution of the fuzzy problem, which enhances the realism and accuracy of the solution obtained.

This study proposes the following main contributions to effectively address the issues highlighted in the motivation section. The model of an unbalanced FFSTP is formulated, where all the parameters and decision variables are considered as triangular fuzzy numbers. A new methodology is proposed to find the optimal solution of an unbalanced FFSTP in terms of original sources, destinations, and vehicles only and does not involve any dummies in the final solution. The validation of the proposed methodology is done by solving a case study of a rice transport company, and the results are compared with existing / traditional techniques.

The remaining sections of the paper are summarized as follows: Section 2 provides fundamental definitions and concepts related to fuzzy set theory. In Section 3, the mathematical model of an unbalanced FFSTP is formulated. The proposed solution procedure and the related theorems are proved in Section 4. In Section 5, a real-life application of the transportation problem is solved to illustrate the effectiveness and practicality of the proposed approach. The validation and discussions of the results obtained are given in Section 6. Finally, Section 7 presents the conclusions drawn from the study and discusses the potential avenues for future research and development in this area.

2. Preliminaries

In this section, the fundamental definitions related to fuzzy set theory are given.

Definition 2.1. Let X be a non-empty set, then the fuzzy set $\widetilde{C}$ in X is defined as: $\widetilde{C} = \{(x, \mu_{\widetilde{C}}(x)) : x \in X\}$, where $\mu_{\widetilde{C}}(x) : X \to [0,1]$ represents the membership function with $0 \leq \mu_{\widetilde{C}}(x) \leq 1, \ \forall \ \ x \in X$.

Definition 2.2. For a fuzzy set $\widetilde{C}$ defined in a nonempty set X, the support is denoted $\text{supp}(\widetilde{C})$ and is defined as the set of elements in X for which the membership value is greater than or equal to 0, i.e.,

$$\text{supp}(\widetilde{C}) = \{x \in X : \mu_{\widetilde{C}}(x) > 0\}$$

Definition 2.3. For a fuzzy set $\widetilde{C}$ defined on a non-empty set X, the α-cut is denoted and is defined as the set of elements in X for which the membership value is greater than or equal to α, i.e.,

$$\widetilde{C}(\alpha) = \{x \in X : \mu_{\widetilde{C}}(x) \geq \alpha\}, \quad \alpha \in [0,1]$$

Definition 2.4. A fuzzy set $\widetilde{C}$ defined on a non-empty set X is said to be a fuzzy number if it satisfies the following properties:

(i) $\widetilde{C}$ is a normal fuzzy set, i.e, there exist at least one element $x \in X$ such that $\mu_{\widetilde{C}}(x) = 1$.

(ii) $\widetilde{C}$ is a convex fuzzy set, i.e., $\forall \ \ x_1, \ \ x_2 \in X$ there exist $\delta \in [0,1]$ such that $\mu_{\widetilde{C}}[\delta x_1 + (1-\delta)x_2] \ \geq \ \min\ [\mu_{\widetilde{C}}(x_1), \mu_{\widetilde{C}}(x_2)]$,

(iii) The membership function of $\widetilde{C}$, i.e, $\mu_{\widetilde{C}}(x)$ is piecewise continuous in $\mathbb{R}$.

Definition 2.5. The triangular fuzzy number is represented as $\widetilde{C} = (l, m, n)$ where $l \leq m \leq n$ are real numbers and its membership function $(\mu_{\widetilde{C}}(x))$ is defined as$\mu_{\widetilde{C}}(x) =$

$$\begin{cases} \frac{x-l}{m-l}, & \text{if } l \leq x < m \\ \frac{n-x}{n-m}, & \text{if } m \leq x \leq n \\ 0, & \text{otherwise} \end{cases}$$

The graphic representation of the triangular fuzzy number is shown in Figure 1.

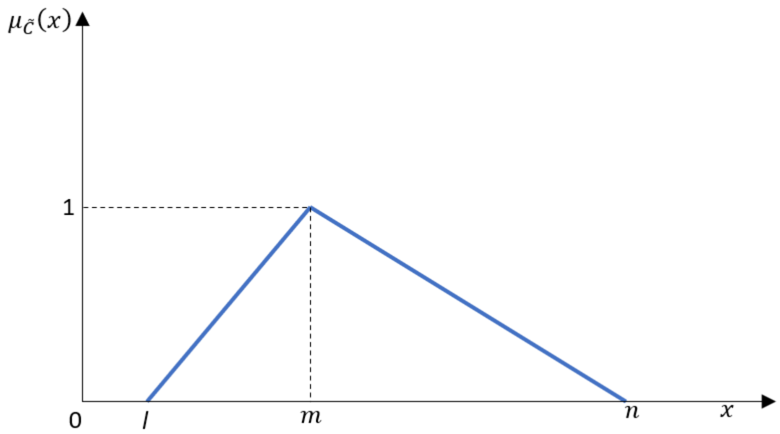

Figure 1. The graphical representation of triangular fuzzy number

Remark 2.6. A triangular fuzzy number $\widetilde{C} = (0,0,0)$ is said to be a zero triangular fuzzy number.

Definition 2.7. Let $\widetilde{C} = (l_1, m_1, n_1)$ and $\widetilde{D} = (l_2, m_2, n_2)$ be two triangular fuzzy numbers. Then

(i) $\widetilde{C} \oplus \widetilde{D} = (l_1 + l_2, m_1 + m_2, n_1 + n_2)$,

(ii) $\widetilde{C} \ominus \widetilde{D} = (l_1 - n_2, m_1 - m_2, n_1 - l_2)$,

(iii) $k\widetilde{C} = \begin{cases} (kl_1, km_1, kn_1), & \text{if } k \geq 0 \\ (kn_1, km_1, kl_1), & \text{if } k < 0, \end{cases}$

(iv) $\widetilde{C} \otimes \widetilde{D} = (\min(l_1l_2, l_1n_2, n_1l_2, n_1n_2), m_1m_2, \max(l_1l_2, l_1n_2, n_1l_2, n_1n_2))$.

Definition 2.8. The left and right integrals for a fuzzy number $\widetilde{C}$ are defined as

$$\text{Left integral} \;\; I_L(\widetilde{C}) = \int_0^1 (\mu^l_{\widetilde{C}})^{-1}(\alpha) d\alpha$$

$$\text{Right integral} \;\; I_R(\widetilde{C}) = \int_0^1 (\mu^r_{\widetilde{C}})^{-1}(\alpha) d\alpha$$

where $(\mu^l_{\widetilde{C}})^{-1}(\alpha)$ and $(\mu^r_{\widetilde{C}})^{-1}(\alpha)$ are the inverse functions of the left and right membership functions $(\mu^l_{\widetilde{C}})(x)$ and $(\mu^r_{\widetilde{C}})(x)$, respectively.

Definition 2.9. The total γ-integral (I^γ_T) of a fuzzy number $\widetilde{C}$ is defined as:

$$I^\gamma_T = \gamma I_R(\widetilde{C}) + (1-\gamma) I_L(\widetilde{C}),$$

where $\gamma \in [0,1]$ indicates the level of optimism of a decision maker.

A larger value of γ represents higher degree of optimism. $\gamma = 0$, I^γ_T represents the pessimistic viewpoint of decision maker, $\gamma = 1$, represents the optimistic viewpoint of decision maker and $\gamma = 0.5$ indicates the moderate viewpoint of decision maker.

For the triangular fuzzy number $\widetilde{C} = (l, m, n)$, we have $(\mu^l_{\widetilde{C}})^{-1}(\alpha) = l + \alpha(m - l)$ and $(\mu^r_{\widetilde{C}})^{-1}(\alpha) = n + \alpha(m - n)$.

Therefore,

$$I_L(\widetilde{C}) = \frac{l+m}{2} \quad \text{and} \quad I_R(\widetilde{C}) = \frac{m+n}{2} \quad .$$

Hence for a moderate decision maker, the total γ-integral of triangular fuzzy number $\widetilde{C}$ also called the ranking function $\Re$, is defined in Definition 9.

Definition 2.10. Let $\widetilde{C} = (l, m, n)$ be a triangular fuzzy number. Then its ranking function $\Re : F(\mathbb{R}) \to \mathbb{R}$, where $F(\mathbb{R})$ is a set of fuzzy numbers defined on a set of real numbers $\mathbb{R}$ is defined as

$$\Re(\widetilde{C}) = \frac{l + 2m + n}{4}$$

Remark 2.11. A triangular fuzzy number $\widetilde{C} = (l, m, n)$ is said to be a non-negative triangular fuzzy number if and only if $\Re(\widetilde{C}) \geq \widetilde{0}$.

Definition 2.12. Let $\widetilde{C} = (l_1, m_1, n_1)$ and $\widetilde{D} = (l_2, m_2, n_2)$ be two triangular fuzzy numbers. Then

(i) $\widetilde{C} \leq \widetilde{D}$ if $\Re(\widetilde{C}) \leq \Re(\widetilde{D})$,
(ii) $\widetilde{C} \geq \widetilde{D}$ if $\Re(\widetilde{C}) \geq \Re(\widetilde{D})$.

3. Mathematical model

To formulate the mathematical model of the proposed unbalanced FFSTP, the following notation and assumptions are used.

Notations	:	Meaning
g	:	index for sources $(g = 1, 2, ..., l)$
h	:	index for destinations $(h = 1, 2, ..., m)$
k	:	index for vehicles $(k = 1, 2, ..., n)$
$(\mathbb{S}_1,\ \mathbb{S}_2,\ ...,\ \mathbb{S}_l)$	:	set of l sources
$(\mathbb{D}_1,\ \mathbb{D}_2,\ ...,\ \mathbb{D}_m)$	:	set of m destinations
$(\mathbb{V}_1,\ \mathbb{V}_2,\ ...,\ \mathbb{V}_k)$	:	set of k vehicles
$\widetilde{\chi}_{ghk}$	:	fuzzy per unit transportation cost of the product from g^{th} source to h^{th} destination through k^{th} vehicle (\$/ton)
$\widetilde{\alpha}_g$	:	fuzzy availability of the product at g^{th} source (in tonnes)
$\widetilde{\beta}_h$	:	fuzzy demand of the product at h^{th} destination (in tonnes)
$\widetilde{\gamma}_k$	:	fuzzy loading capacity of the k^{th} vehicle (in tonnes)
$\widetilde{\xi}_{ghk}$	:	fuzzy units of product transported from g^{th} source to h^{th} destination through k^{th} vehicle (in tonnes)

Assumptions:

(i) All the parameters (transportation cost, availability, demand, vehicle's loading capacity) as well as decision variables (transported amount) are considered as triangular fuzzy numbers.
(ii) $\widetilde{\alpha}_g, \widetilde{\beta}_h, \widetilde{\gamma}_k, \widetilde{\chi}_{ghk}, \widetilde{\xi}_{ghk} \geq 0 \quad \forall \quad g, h, k.$
(iii) $\sum_{g=1}^{l} \widetilde{\alpha}_g < \sum_{h=1}^{m} \widetilde{\beta}_h, \quad \sum_{k=1}^{n} \widetilde{\gamma}_k < \sum_{h=1}^{m} \widetilde{\beta}_h$

In this section, we propose a model for an unbalanced FFSTP, where the total demand at the destinations is not equal to the total supply from the sources and the total loading capacity of the vehicles. The product is transported from l sources $(S_g, \quad 1 \leq g \leq l)$ to m destinations $(D_h, \quad 1 \leq h \leq m)$ through n vehicles $(V_k, \quad 1 \leq k \leq n)$. The transport network with two sources $(g = 2)$, two destinations $(h = 2)$, two vehicles $(k = 2)$ is shown in Figure 2.

Figure 2. Network representation of STP

In the proposed model to address uncertainty, all parameters as well as decision variables are considered as triangular fuzzy numbers. The transportation cost per unit of product on each route from the g^{th} source to the h^{th} destination by k^{th} conveyance is denoted by $\widetilde{\chi}_{ghk}$. Each g^{th} source has a supply of $\widetilde{\alpha}_g$, each h^{th} destination has a demand of $\widetilde{\beta}_h$, and each k^{th} conveyance has a capacity of $\widetilde{\gamma}_k$. The main objective of the proposed unbalanced FFSTP model is to determine the amount of product to be transported from the g^{th} source to the h^{th} destination by k^{th} conveyance in a way that minimizes the total transportation cost while fully satisfying the demand at each destination, without introducing any dummy facilities. The objective function is defined as follows:

Minimize the total transportation cost:

$$\text{Minimize} \quad \widetilde{Z} = \sum_{g=1}^{l}\sum_{h=1}^{m}\sum_{k=1}^{n} \widetilde{\chi}_{ghk} \otimes \widetilde{\xi}_{ghk} \tag{3.1}$$

Demand constraints:

Since $\widetilde{\beta}_h$ denotes the demand for the destination h^{th} $(h = 1, , ..., m)$ and the feasibility condition of the transportation problem is that the demand for each destination should be fully satisfied. So, the constraint (3.2) represents that the amount of product transported to h^{th} destination must be equal to the demand of the product at the same destination.

$$\sum_{g=1}^{l}\sum_{k=1}^{n} \widetilde{\xi}_{ghk} = \widetilde{\beta}_h, \quad h = 1, 2, ..., m, \tag{3.2}$$

Supply constraints:

The proposed transportation problem model is unbalanced in that the total availability of the product at the sources is less than the total demand at the destination points, i.e., $\sum_{g=1}^{l} \widetilde{\alpha}_g < \sum_{h=1}^{m} \widetilde{\beta}_h$. From constraint (3.2), $\sum_{g=1}^{l}\sum_{k=1}^{n} \widetilde{\xi}_{ghk} \geq \widetilde{\beta}_h, \quad h = 1, 2, ..., m,$. Therefore, the supply constraint becomes the following.

$$\sum_{h=1}^{m}\sum_{k=1}^{n} \widetilde{\xi}_{ghk} \geq \widetilde{\alpha}_g, \quad g = 1, 2, ..., l, \tag{3.3}$$

The constraint (3.3) shows that each g^{th} source supply atleast $\widetilde{\alpha}_g$ units of product to the destinations and it may also increase to meet the demand of the destinations.

Loading capacity constraints:

The other unbalanced condition of the model is that the total loading capacity of the vehicle is less than the total demand at the destination points, i.e., $\sum_{k=1}^{n}\widetilde{\gamma}_k < \sum_{h=1}^{m}\widetilde{\beta}_h$. From constraint (3.2), $\sum_{g=1}^{l}\sum_{k=1}^{n}\widetilde{\xi}_{ghk} \geq \widetilde{\beta}_h, \quad h = 1,2,...,m,$. Therefore, the loading capacity constraints becomes

$$\sum_{g=1}^{l}\sum_{h=1}^{m}\widetilde{\xi}_{ghk} \geq \widetilde{\gamma}_k, \quad k = 1,2,...,n, \tag{3.4}$$

The constraint (3.4) shows that each k^{th} vehicle loads at least $\widetilde{\alpha}_g$ units of product to the destinations and it can also increase to meet the demand of the destinations.

Non-negative constraints:
The non-negativity constraints in transportation problems ensure that the quantity transported from sources to destinations (decision variables) is a non-negative value, reflecting the practical reality that negative shipments are neither feasible nor meaningful. These constraints are essential in formulating realistic and implementable transportation optimization models. For the proposed model, the non-negative integral constraints are represented as follows:

$$\widetilde{\xi}_{ghk} \geq \widetilde{0}, \quad \forall \quad g,h,k, \tag{3.5}$$

Therefore, considering the above objective function (3.1) and constraints (3.2−3.5), the proposed model of an unbalanced FFSTP is formulated as follows:

$$\text{Minimize} \quad \widetilde{Z} = \sum_{g=1}^{l}\sum_{h=1}^{m}\sum_{k=1}^{n}\widetilde{\chi}_{ghk} \otimes \widetilde{\xi}_{ghk} \tag{3.6}$$

subject to

$$\sum_{h=1}^{m}\sum_{k=1}^{n}\widetilde{\xi}_{ghk} \geq \widetilde{\alpha}_g, \quad g = 1,2,...,l, \tag{3.7}$$

$$\sum_{g=1}^{l}\sum_{k=1}^{n}\widetilde{\xi}_{ghk} = \widetilde{\beta}_h, \quad h = 1,2,...,m, \tag{3.8}$$

$$\sum_{g=1}^{l}\sum_{h=1}^{m}\widetilde{\xi}_{ghk} \geq \widetilde{\gamma}_k, \quad k = 1,2,...,n, \tag{3.9}$$

$$\widetilde{\xi}_{ghk} \geq \widetilde{0}, \quad \forall \quad g,h,k. \tag{3.10}$$

The dual of (P_1) can be written as (DP_1).

$$(DP_1) \qquad \text{Maximize} \quad \sum_{g=1}^{l}\widetilde{\alpha}_g\widetilde{u}_g + \sum_{h=1}^{m}\widetilde{\beta}_h\widetilde{v}_h + \sum_{k=1}^{n}\widetilde{\gamma}_k\widetilde{w}_k \tag{3.11}$$

subject to

$$\widetilde{u}_g + \widetilde{v}_h + \widetilde{w}_k \leq \widetilde{\chi}_{ghk} \quad \text{for} \quad 1 \leq g \leq l, \quad 1 \leq h \leq m, \quad 1 \leq k \leq n \tag{3.12}$$

$$\widetilde{v}_h \quad \text{is unrestricted in sign} \quad \text{for} \quad 1 \leq h \leq m$$

$$\text{and} \quad \widetilde{u}_g, \widetilde{w}_k \geq 0 \quad \text{for} \quad 1 \leq g \leq l, 1 \leq k \leq n. \tag{3.13}$$

Here, $\widetilde{u}_g$, $\widetilde{v}_h$, $\widetilde{w}_k$ are the dual variables corresponding to the availability, demand and vehicle's loading capacity constraints, respectively.

4. The proposed method

In this section, a new method is proposed to solve the unbalanced FFSTP (P_1). To facilitate the solution process, we prove the following theorem, for which we first formulate the balanced model of (P_1) and the corresponding dual problem as follows:

Let (P_2) represent the balanced FFSTP derived from (P_1) by incorporating a dummy source S_{l+1} and a dummy vehicle V_{n+1}, with a unit fuzzy transportation cost $\widetilde{\chi}_{(l+1)hk} = \min\limits_{1\leq g\leq l} \widetilde{\chi}_{ghk}$; $1 \leq h \leq m, 1 \leq k \leq n$ and $\widetilde{\chi}_{gh(n+1)} = \min\limits_{1\leq k\leq n} \widetilde{\chi}_{ghk}$; $1 \leq g \leq l+1, 1 \leq h \leq m$ respectively.

$$\text{Minimize} \quad \widetilde{Z} = \sum_{g=1}^{l+1}\sum_{h=1}^{m}\sum_{k=1}^{n+1} \widetilde{\chi}_{ghk} \otimes \widetilde{\xi}_{ghk} \tag{4.1}$$

subject to

$$\sum_{h=1}^{m}\sum_{k=1}^{n+1} \widetilde{\xi}_{ghk} = \widetilde{\alpha}_g, \quad g = 1, 2, \ldots, l+1, \tag{4.2}$$

$$\sum_{g=1}^{l+1}\sum_{k=1}^{n+1} \widetilde{\xi}_{ghk} = \widetilde{\beta}_h, \quad h = 1, 2, \ldots, m, \tag{4.3}$$

$$\sum_{g=1}^{l+1}\sum_{h=1}^{m} \widetilde{\xi}_{ghk} = \widetilde{\gamma}_k, \quad k = 1, 2, \ldots, n+1, \tag{4.4}$$

$$\widetilde{\xi}_{ghk} \geq \widetilde{0} \quad \forall\ g, h, k \tag{4.5}$$

where

$$\sum_{g=1}^{l} \widetilde{\alpha}_g + \widetilde{\alpha}_{l+1} = \sum_{h=1}^{m} \widetilde{\beta}_h \quad \text{and} \quad \sum_{k=1}^{n} \widetilde{\gamma}_k + \widetilde{\gamma}_{n+1} = \sum_{h=1}^{m} \widetilde{\beta}_h.$$

The dual of (P_2) with dual variables $\widetilde{u}'_g$, $\widetilde{v}'_h$ and $\widetilde{w}'_k$ can be written as (DP_2).

$$(DP_2) \quad \text{Maximize} \quad \sum_{g=1}^{l+1} \widetilde{\alpha}_g \widetilde{u}'_g + \sum_{h=1}^{m} \widetilde{\beta}_h \widetilde{v}'_h + \sum_{k=1}^{n+1} \widetilde{\gamma}_k \widetilde{w}'_k \tag{4.6}$$

subject to

$$\widetilde{u}'_g + \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{ghk}, \quad 1 \leq g \leq l+1,\ 1 \leq h \leq m,\ 1 \leq k \leq n+1 \tag{4.7}$$

$\widetilde{u}'_g$, $\widetilde{v}'_h$, $\widetilde{w}'_k$ are unrestricted in sign,

for $1 \leq g \leq l+1,\ 1 \leq h \leq m,\ 1 \leq k \leq n+1$.

Theorem 4.1. *If* $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ *and* $(\widetilde{u}', \widetilde{v}', \widetilde{w}')$ *are the optimal solutions of* (DP_1) *and* (DP_2)*, respectively, where* $(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (\widetilde{u}_1, \widetilde{u}_2, ..., \widetilde{u}_l, \widetilde{v}_1, \widetilde{v}_2, ..., \widetilde{v}_m, \widetilde{w}_1, \widetilde{w}_2, ..., \widetilde{w}_n)$ *and* $(\widetilde{u}', \widetilde{v}', \widetilde{w}') = (\widetilde{u}'_1, \widetilde{u}'_2, ..., \widetilde{u}'_{l+1}, \widetilde{v}'_1, \widetilde{v}'_2, ..., \widetilde{v}'_m, \widetilde{w}'_1, \widetilde{w}'_2, ..., \widetilde{w}'_{n+1})$*, then* $\widetilde{u}_g = \widetilde{u}'_g$, $1 \leq g \leq l$, $\widetilde{v}_h = \widetilde{v}'_h$, $1 \leq h \leq m$ *and* $\widetilde{w}_k = \widetilde{w}'_k$, $1 \leq k \leq n$ *provided* $\widetilde{u}'_{l+1} = \widetilde{0}$, $\widetilde{w}'_{n+1} = \widetilde{0}$.

Proof. To show that $\widetilde{u}_g = \widetilde{u}'_g$, $1 \leq g \leq l$, $\widetilde{v}_h = \widetilde{v}'_h$, $1 \leq h \leq m$ and $\widetilde{w}_k = \widetilde{w}'_k$, $1 \leq k \leq n$ provided $\widetilde{u}'_{l+1} = \widetilde{0}$, $\widetilde{w}'_{n+1} = \widetilde{0}$, we have to prove that $(\widetilde{u}', \widetilde{v}', \widetilde{w}')$ is also an optimal solution of (DP_1). For this, it is sufficient to prove that:

(i) $\widetilde{u}'_g + \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{ghk}, \forall\ 1 \leq g \leq l, 1 \leq h \leq m, 1 \leq k \leq n$ and

(ii) $\widetilde{u}'_g, \widetilde{w}'_k \geq \widetilde{0}, \quad \forall\ 1 \leq g \leq l, 1 \leq k \leq n$

(i) Since $(\widetilde{u'}, \widetilde{v}', \widetilde{w}')$ is an optimal solution of (DP_2). So,

$$\widetilde{u}'_g + \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{ghk}, \quad \forall \ 1 \leq g \leq l+1, 1 \leq h \leq m, 1 \leq k \leq n+1. \tag{4.8}$$

Therefore, $\quad \widetilde{u}'_g + \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{ghk}, \quad \forall \ 1 \leq g \leq l, 1 \leq h \leq m, 1 \leq k \leq n.$

(ii) Let if possible $\widetilde{u}'_g < 0 \ \ (1 \leq g \leq l)$

Now, for $g = l+1$, $1 \leq h \leq m$, $1 \leq k \leq n$, the inequality (4.8) can be written as

$$\widetilde{u}'_{l+1} + \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{(l+1)hk}, \quad \forall \ 1 \leq h \leq m, 1 \leq k \leq n.$$

Since $\widetilde{u}'_{l+1} = \widetilde{0}$

$$\Rightarrow \quad \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{(l+1)hk} \ \ \forall \ 1 \leq h \leq m, 1 \leq k \leq n$$

$$\Rightarrow \quad \widetilde{v}'_h + \widetilde{w}'_k \leq \widetilde{\chi}_{ghk} \quad (\because \widetilde{\chi}_{(l+1)hk} = \min_{1 \leq g \leq l} \ \widetilde{\chi}_{ghk}; \quad 1 \leq h \leq m, 1 \leq k \leq n) \tag{4.9}$$

Since for each g $(1 \leq g \leq l)$ there must exist some q $(1 \leq q \leq m)$ and r $(1 \leq r \leq n)$ such that the cell (g, q, r) is a basic cell.
Therefore, the relation $\widetilde{u}'_g + \widetilde{v}'_q + \widetilde{w}'_r = \widetilde{\chi}_{gqr}$ must hold for the basic cell (g, q, r). Also $\widetilde{\chi}_{gqr} \geq 0$ and $u'_g < 0$ (as per assumption)
Which gives, $\widetilde{v}'_q + \widetilde{w}'_r > \widetilde{\chi}_{gqr}$ which is contradiction to (4.9). Hence, our supposition is wrong. So, $\widetilde{u}'_g \geq 0 \ \ \forall \ \ 1 \leq g \leq l.$

Now let if possible $\widetilde{w}'_k < 0 \ \ \ (1 \leq k \leq n)$

Also for $k = n+1$, $1 \leq g \leq l+1$, $1 \leq h \leq m$, the inequality (4.8) can be written as

$$\widetilde{u}'_g + \widetilde{v}'_h + \widetilde{w}'_{n+1} \leq \widetilde{\chi}_{gh(n+1)}, \quad \forall \ 1 \leq g \leq l+1, 1 \leq h \leq m.$$

Since $\widetilde{w}'_{n+1} = \widetilde{0}$

$$\Rightarrow \quad \widetilde{u}'_g + \widetilde{v}'_h \leq \widetilde{\chi}_{gh(n+1)} \ \ \forall \ 1 \leq g \leq l+1, 1 \leq h \leq m$$

$$\Rightarrow \quad \widetilde{u}'_g + \widetilde{v}'_h \leq \widetilde{\chi}_{ghk} \quad (\because \widetilde{\chi}_{gh(n+1)} = \min_{1 \leq k \leq n} \ \widetilde{\chi}_{ghk}; \quad 1 \leq g \leq l+1, 1 \leq h \leq m) \tag{4.10}$$

Since for each k $(1 \leq k \leq n)$ there must exist some s $(1 \leq s \leq l)$ and t $(1 \leq t \leq m)$ such that the cell (s, t, k) is a basic cell.
Therefore, the relation $\widetilde{u}'_s + \widetilde{v}'_t + \widetilde{w}'_k = \widetilde{\chi}_{stk}$ must hold for the basic cell (s, t, k). Also $\widetilde{\chi}_{stk} \geq 0$ and $w'_k < 0$ (as per assumption)
Which gives, $\widetilde{u}'_s + \widetilde{v}'_t > \widetilde{\chi}_{stk}$ which is contradiction to (4.10). Hence our supposition is wrong. So, $\widetilde{w}'_k \geq 0 \qquad \forall \ \ 1 \leq k \leq n.$

□

Solution methodology:

The proposed method has five main steps, namely

(i) Balancing the given problem
(ii) Finding the basic feasible solution
(iii) Checking the optimality of the obtained basic feasible solution.
(iv) Getting the optimal solution
(v) Finding the optimal solution in terms of the original sources and vehicles

4.1. Balancing the given problem:

Balance the given unbalanced FFSTP problem as P_2 by adding a dummy source S_{l+1} and a dummy vehicle V_{n+1} with the unit fuzzy transportation cost $\widetilde{\chi}_{(l+1)hk} = \min_{1\leq g\leq l} \widetilde{\chi}_{ghk}; \quad 1 \leq h \leq m, 1 \leq k \leq n$ and $\widetilde{\chi}_{gh(n+1)} = \min_{1\leq k\leq n} \widetilde{\chi}_{ghk}; \quad 1 \leq g \leq l+1, 1 \leq h \leq m,$, respectively.

4.2. Finding the basic feasible solution:

Step 4.2.1: Identify the minimum transportation cost in each source, subtract it from the next minimum transportation cost, and write their difference called penalty, along with the corresponding source of the table. In a similar way, calculate the penalties for each destination and vehicle.
Step 4.2.2: Identify the maximum penalty. Find out the corresponding source/destination /vehicle and mark the smallest fuzzy cost in it. Let it be $\widetilde{\chi}_{abc}$. Find the minimum of $\widetilde{\alpha}_a$, $\widetilde{\beta}_b$, $\widetilde{\gamma}_c$. The following cases may arise:
Case 1: If $\min(\widetilde{\alpha}_a, \widetilde{\beta}_b, \widetilde{\gamma}_c) = \widetilde{\alpha}_a$, then allocate $\widetilde{\xi}_{abc} = \widetilde{\alpha}_a$. Discard the a^{th} source to obtain the new FFSTP. Find $\widetilde{\beta}'_b$, $\widetilde{\gamma}'_c$ such that $\widetilde{\beta}'_b + \widetilde{\alpha}_a = \widetilde{\beta}_b$ and $\widetilde{\gamma}'_c + \widetilde{\alpha}_a = \widetilde{\gamma}_c$. Replace $\widetilde{\beta}_b$ and $\widetilde{\gamma}_c$ by $\widetilde{\beta}'_b$ and $\widetilde{\gamma}'_c$,, respectively, in the obtained FFSTP. Go to Step 4.2.3.
Case 2: If $\min(\widetilde{\alpha}_a, \widetilde{\beta}_b, \widetilde{\gamma}_c) = \widetilde{\beta}_b$, then allocate $\widetilde{\xi}_{abc} = \widetilde{\beta}_b$. Discard the b^{th} destination to obtain the new FFSTP. Find $\widetilde{\alpha}'_a$, $\widetilde{\gamma}'_c$ such that $\widetilde{\alpha}'_a + \widetilde{\beta}_b = \widetilde{\alpha}_a$ and $\widetilde{\gamma}'_c + \widetilde{\beta}_b = \widetilde{\gamma}_c$. Replace $\widetilde{\alpha}_a$ and $\widetilde{\gamma}_c$ by $\widetilde{\alpha}'_a$ and $\widetilde{\gamma}'_c$,, respectively, in the obtained FFSTP. Go to Step 4.2.3.
Case 3: If $\min(\widetilde{\alpha}_a, \widetilde{\beta}_b, \widetilde{\gamma}_c) = \widetilde{\gamma}_c$, then allocate $\widetilde{\xi}_{abc} = \widetilde{\gamma}_c$. Discard the c^{th} vehicle to obtain the new FFSTP. Find $\widetilde{\alpha}'_a$, $\widetilde{\beta}'_b$ such that $\widetilde{\alpha}'_a + \widetilde{\gamma}_c = \widetilde{\alpha}_a$ and $\widetilde{\beta}'_b + \widetilde{\gamma}_c = \widetilde{\beta}_b$. Replace $\widetilde{\alpha}_a$ and $\widetilde{\beta}_b$ by $\widetilde{\alpha}'_a$ and $\widetilde{\beta}'_b$,, respectively, in the obtained FFSTP. Go to Step 4.2.3.
Case 4: If $\widetilde{\alpha}_a = \widetilde{\beta}_b = \widetilde{\gamma}_c$, then randomly choose any of the a^{th} source, b^{th} destination, and c^{th} vehicles and follow the corresponding case mentioned above. We call the newly obtained FFSTP reduced FFSTP.
Step 4.2.3: Calculate the new penalties of the reduced FFSTP as per Step 4.2.1. Repeat Step 4.2.2 until all resources are utilized.
Step 4.2.4: Allocate all the obtained values of $\widetilde{\xi}_{ghk}$ in the $(g, h, k)^{th}$ cell of the given FFSTP.

4.3. Checking the optimality of the obtained basic feasible solution:

Step 4.3.1: Using the condition $\widetilde{u}_g + \widetilde{v}_h + \widetilde{w}_k = \widetilde{\chi}_{ghk}$ for all $l + m + n - 2$ basic cells and by taking the value of any of two fuzzy dual variables equal to zero, calculate the rest of the fuzzy dual variables.
Step 4.3.2: Compute $\widetilde{\delta}_{ghk} = \widetilde{\chi}_{ghk} - (\widetilde{u}_g + \widetilde{v}_h + \widetilde{w}_k)$ for all non-basic cells and conclude that
(i) If $\widetilde{\delta}_{ghk} \geq \widetilde{0}, \quad \forall \ g, h, k,$, then the initial feasible basic solution obtained is an optimal solution.
(ii) If at least one $\widetilde{\delta}_{ghk} < \widetilde{0}$, then solution is not optimal.

4.4. Getting the optimal solution:

Step 4.4.1: Choose the cell with the lowest negative $\widetilde{\delta}_{ghk}$ and make an allocation $\widetilde{t}_{ghk}$ to this cell and to all basic cells (g, h, k). For the solution to be feasible, the following condition must be satisfied.

$$\sum_{g=1}^{l+1}\sum_{h=1}^{m}\widetilde{t}_{ghk}=\widetilde{0}, \quad k=1,2,...,n+1,$$
$$\sum_{h=1}^{m}\sum_{k=1}^{n+1}\widetilde{t}_{ghk}=\widetilde{0}, \quad g=1,2,...,l+1,$$
$$\sum_{g=1}^{l+1}\sum_{k=1}^{n+1}\widetilde{t}_{ghk}=\widetilde{0}, \quad h=1,2,...,m,$$

Solve these equations to obtain the new basic feasible solution by allocating as much value as possible to the $\widetilde{t}_{ghk}$ corresponding to the cell with the most negative $\widetilde{\delta}_{ghk}$. Check the optimality of the newly obtained basic feasible solution by following the whole procedure in Step 4.3.
Step 4.4.2: Repeat the above step to get the optimal solution until $\widetilde{\delta}_{ghk} \geq \widetilde{0}, \quad \forall \;\; g,h,k$.

4.5. Finding the optimal solution in terms of the original sources and vehicles

Step 4.5.1: Find the dual variables for the fuzzy optimal solution obtained in Step 4.4 by assuming $\widetilde{u}'_{l+1}, \widetilde{w}'_{n+1}$ (fuzzy dual variable corresponding to the dummy source and dummy vehicle in the FFSTP (P_2)) equal to zero triangular fuzzy number.
Step 4.5.2: By Theorem 4.1, the solution of (DP_1) is same as that of (DP_2). Among all the dual variables obtained in Step 4.5.1, select those $\widetilde{u}_g, \quad 1 \leq g \leq l$ whose rank is zero. By Complementary Slackness Theorem [15], slack/ surplus variable of only these selected sources can have positive values. Therefore, only the availability of these sources can be increased. Similarly, select those $\widetilde{w}_k, \quad 1 \leq k \leq n$ whose rank is zero and increase the capacity of those vehicles accordingly.
Step 4.5.3: Obtain the fuzzy optimal solution in terms of the original sources and vehicles using the fuzzy optimal solution obtained in Step 4.4 as per the following cases:
Case 1: Let $\widetilde{\xi}_{(l+1)qr}$ $(1 \leq q \leq m)$ $(1 \leq r \leq n)$ is a basic variable, consider the following two subcases:
Subcase 1a: If g is such that $\Re(\widetilde{u}_g)=0$ is unique say p, then increase the value of the variable $\widetilde{\xi}_{pqr}$ by $\widetilde{\xi}_{(l+1)qr}$.
Subcase 1b: If g is such that $\Re(\widetilde{u}_g)=0$ is not unique say $g \in G$, then increase the value of the variable $\widetilde{\xi}_{gqr}$ corresponding to the cell (g,q,r) with $\min_{g\in G} \widetilde{\chi}_{gqr}$ by $\widetilde{\xi}_{(l+1)qr}$.
Case 2: Let $\widetilde{\xi}_{\mu\nu(n+1)}$ $(1 \leq \mu \leq l)$ $(1 \leq \nu \leq m)$ is a basic variable, consider the following two subcases:
Subcase 2a: If k is such that $\Re(\widetilde{w}_k)=0$ is unique say ω, then increase the value of the variable $\widetilde{\xi}_{\mu\nu\omega}$ by $\widetilde{\xi}_{\mu\nu(n+1)}$.
Subcase 2b: If k is such that ranking of $\Re(\widetilde{w}_k)=0$ is not unique say $k \in K$, then increase the value of the variable $\widetilde{\xi}_{\mu\nu k}$ corresponding to the cell (μ,ν,k) with $\min_{k\in K} \widetilde{\chi}_{\mu\nu k}$ by $\widetilde{\xi}_{\mu\nu(n+1)}$.
Case 3: Let $\widetilde{\xi}_{(l+1)\lambda(n+1)}$ $(1 \leq \lambda \leq m)$ is a basic variable, consider the following four subcases:
Subcase 3a: If g such that ranking of $\Re(\widetilde{u}_g)=0$ is unique say σ and k such that ranking of $\Re(\widetilde{w}_k)=0$ is unique say η, then increase the value of the variable $\widetilde{\xi}_{\sigma\lambda\eta}$ by $\widetilde{\xi}_{(l+1)\lambda(n+1)}$.
Subcase 3b: If g such that ranking of $\Re(\widetilde{u}_g)=0$ is not unique say $g \in G$ and k such that ranking of $\Re(\widetilde{w}_k)=0$ is unique say τ, then increase the value of the variable $\widetilde{\xi}_{g\lambda\tau}$

corresponding to the of cell (g, λ, τ) with $\min\limits_{g\in G} \widetilde{\chi}_{g\lambda\tau}$ by $\widetilde{\xi}_{(l+1)\lambda(n+1)}$.

Subcase 3c: If g such that ranking of $\Re(\widetilde{u}_g) = 0$ is unique say ρ and k such that ranking of $\Re(\widetilde{w}_k) = 0$ is not unique say $k \in K$, then increase the value of the variable $\widetilde{\xi}_{\rho\lambda k}$ corresponding to the of cell (ρ, λ, k) with $\min\limits_{k\in K} \widetilde{\chi}_{\rho\lambda k}$ by $\widetilde{\xi}_{(l+1)\lambda(n+1)}$.

Subcase 3d: If g such that ranking of $\Re(\widetilde{u}_g) = 0$ is not unique say $g \in G$ and k such that ranking of $\Re(\widetilde{w}_k) = 0$ is also not unique say $k \in K$, then increase the value of the variable $\widetilde{\xi}_{g\lambda k}$ corresponding to the of cell (g, λ, k) with $\min\limits_{g\in G} \min\limits_{k\in K} \widetilde{\chi}_{g\lambda k}$ by $\widetilde{\xi}_{(l+1)\lambda(n+1)}$.

A flowchart representing the steps of the proposed method in brief is given as Figure 3.

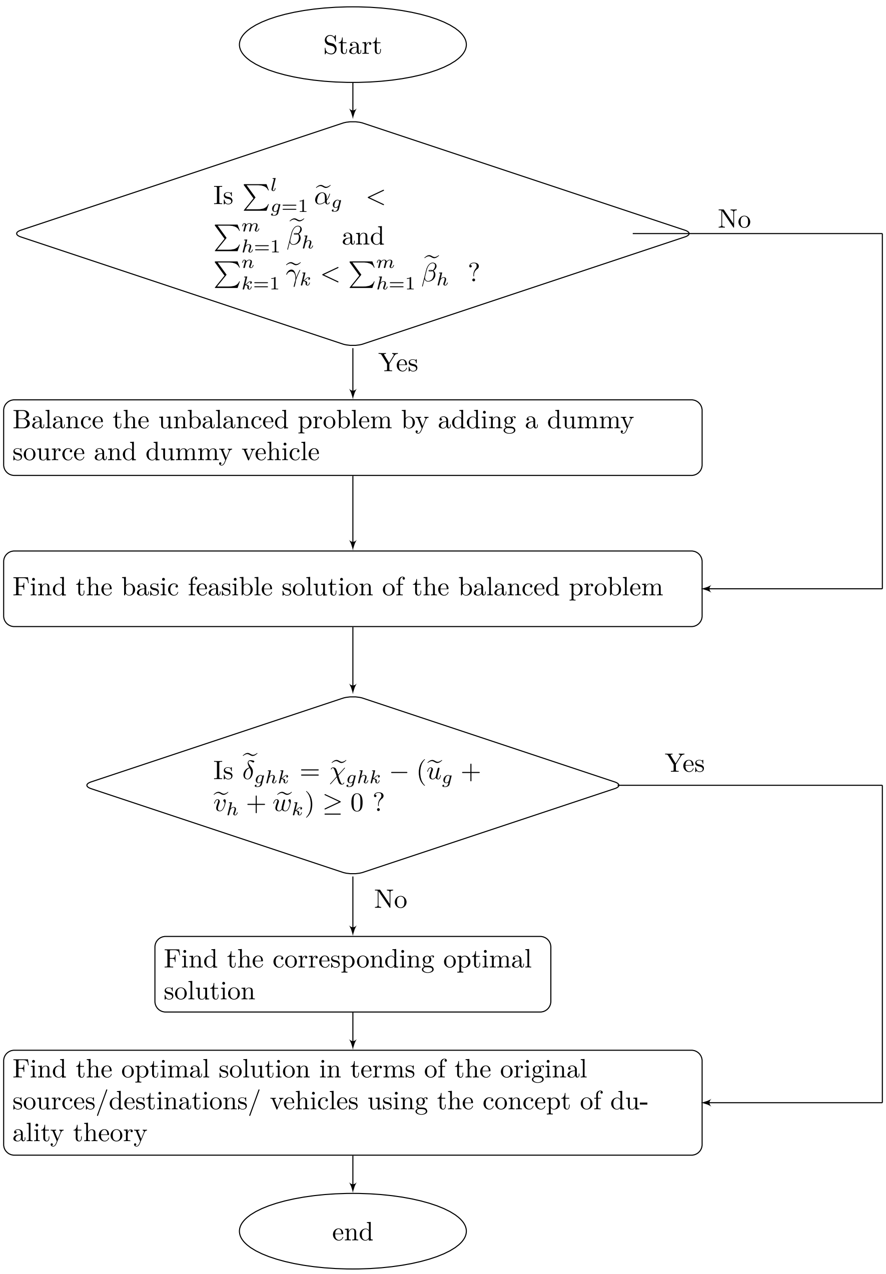

Figure 3. Flow chart of the proposed algorithm

Advantages of the proposed methodology:

(i) Existing methods to solve an unbalanced FFSTP give an optimal solution that involves the dummy source/destination/vehicle and therefore are not practical and useful when applied to a real-life problem. In contrast, the proposed method gives the final solution without any of these dummies. This is the main contribution and benefit of the proposed approach in solving an unbalanced FFSTP.

(ii) The proposed technique is specifically designed to address unbalanced solid transportation problems in a fully fuzzy environment. Using fuzzy set theory, it effectively handles the inherent vagueness and impreciseness commonly encountered in real-life problems. As a result, the modeling of the problem closely resembles practical situations, enhancing its applicability.

(iii) One of the main advantages of the proposed method is the use of fuzzy numbers to represent all parameters and decision variables. This realistic modeling approach provides the decision maker with a broader perspective and better insights into the solution, as it encompasses a wider range of possibilities.

(iv) The proposed method deals with multiple transport facilities simultaneously.

(v) The proposed approach to solve an unbalanced FFSTP is simple to understand and easy to apply.

5. A case study

A rice manufacturing company has three rice mills ($g = 1, 2, 3$) in India. One is located in Karnal (Mill A), the other is in Ludhiana (Mill B), and the third is in Ghaziabad (Mill C) from where the rice is distributed to three ($h = 1, 2, 3$) different wholesale traders markets at Amritsar (Market M_1), Shimla (Market M_2) and Pauri Garhwal (Market M_3). Two types of vehicles ($k = 1, 2$), a 40 ft OPEN-TRAILOR ODC (Vehicle V_1) and a 32 ft OPEN-TRAILOR ODC (Vehicle V_2) are used to transport rice from the three mills to three wholesalers. Due to various uncontrollable factors, such as sudden bad weather in hilly areas and market price fluctuation, the rice company's manager has a tentative idea of the cost of transportation between these three supply sites (Karnal, Ludhiana, Ghaziabad) and three demand points (Amritsar, Shimla, Pauri Garhwal). In addition to this, the availability of rice in mills and the demand of wholesalers also continue to fluctuate for some economic and social reasons. Therefore, a triangular fuzzy number is chosen for the best representation of the uncertainty inherent in various parameters related to this problem. The triangular fuzzy value of these parameters such as transportation cost, supply, demand and loading capacity of the vehicles are given in Table 1. The main objective of the decision maker is to calculate the amount of rice to be transported from each mill to each wholesaler at the minimum possible transportation cost while fully satisfying the demand of all wholesalers.

Table 1. Input data for the case study

	Market M_1		Market M_2		Market M_3		Supply
	Vehicle V_1	Vehicle V_2	Vehicle V_1	Vehicle V_2	Vehicle V_1	Vehicle V_2	
Mill A	$(17, 19, 21)$	$(12, 14, 17)$	$(6, 9, 12)$	$(6, 8, 12)$	$(8, 11, 14)$	$(6, 9, 12)$	$(11, 13, 15)$
Mill B	$(1.5, 3, 4.5)$	$(1, 2, 3)$	$(7, 10, 12)$	$(5, 7, 10)$	$(18, 22, 26)$	$(16, 20, 24)$	$(9, 11, 13)$
Mill C	$(23, 24, 26)$	$(20, 21, 23)$	$(18, 21, 23)$	$(13, 16, 18)$	$(17, 20, 23)$	$(15, 17, 21)$	$(5, 8, 9)$
Loading capacity	$(15, 18, 21)$	$(10, 12, 15)$	$(15, 18, 21)$	$(10, 12, 15)$	$(15, 18, 21)$	$(10, 12, 15)$	
Demand	$(10, 13, 15)$		$(12, 15, 18)$		$(8, 10, 12)$		

The total availability of rice ($\sum_{g=1}^{3} \widetilde{\alpha}_g$) in three mills is (25, 32, 37) tonnes, total loading capacity ($\sum_{k=1}^{2} \widetilde{\gamma}_k$) of vehicles is (25, 30, 36) tonnes and the total demand of rice

($\sum_{h=1}^{3} \widetilde{\beta}_h$) at three wholesaler markets is (30, 38, 45) tonnes. So, the problem is an unbalanced problem, which can not be solved directly. Therefore, to solve this problem, the proposed methodology is applied as follows:

Step 5.1: Add a dummy rice mill (Mill D) with rice availability $(5, 6, 8)$ tons and a dummy vehicle (Vehicle V_3) with loading capacity $(5, 8, 9)$ tonnes to cover the problem in a balanced form (Table2). The per unit transportation cost of rice from the dummy mill (Mill D) to three wholesaler markets (Market M_1, Market M_2, Market M_3) and through the dummy vehicle (Vehicle V_3) is taken as follows:

$$
\begin{aligned}
&\widetilde{\chi}_{411} = \min\{(17, 19, 21), (1.5, 3, 4.5), (23, 24, 26)\} = (1.5, 3, 4.5),\\
&\widetilde{\chi}_{412} = \min\{(12, 14, 17), (1, 2, 3), (20, 21, 23)\} = (1, 2, 3),\\
&\widetilde{\chi}_{421} = \min\{(6, 9, 12), (7, 10, 12), (18, 21, 23)\} = (6, 9, 12),\\
&\widetilde{\chi}_{422} = \min\{(6, 8, 12), (5, 7, 10), (13, 16, 18)\} = (5, 7, 10),\\
&\widetilde{\chi}_{431} = \min\{(8, 11, 14), (18, 22, 26), (17, 20, 23)\} = (8, 11, 14),\\
&\widetilde{\chi}_{432} = \min\{(6, 9, 12), (16, 20, 24), (15, 17, 21)\} = (6, 9, 12),
\end{aligned}
$$

and

$$
\begin{aligned}
&\widetilde{\chi}_{113} = \min\{(17, 19, 21), (12, 14, 17)\} = (12, 14, 17),\\
&\widetilde{\chi}_{213} = \min\{(1.5, 3, 4.5), (1, 2, 3)\} = (1, 2, 3),\\
&\widetilde{\chi}_{313} = \min\{(23, 24, 26), (20, 21, 23)\} = (20, 21, 23),\\
&\widetilde{\chi}_{413} = \min\{(1.5, 3, 4.5), (1, 2, 3)\} = (1, 2, 3),\\
&\widetilde{\chi}_{123} = \min\{(6, 9, 12), (6, 8, 12)\} = (6, 8, 12),\\
&\widetilde{\chi}_{223} = \min\{(7, 10, 12), (5, 7, 10)\} = (5, 7, 10),\\
&\widetilde{\chi}_{323} = \min\{(18, 21, 23), (13, 16, 18)\} = (13, 16, 18),\\
&\widetilde{\chi}_{423} = \min\{(6, 9, 12), (5, 7, 10)\} = (5, 7, 10),\\
&\widetilde{\chi}_{133} = \min\{(8, 11, 14), (6, 9, 12)\} = (6, 9, 12),\\
&\widetilde{\chi}_{233} = \min\{(18, 22, 26), (16, 20, 24)\} = (16, 20, 24),\\
&\widetilde{\chi}_{333} = \min\{(17, 20, 23), (15, 17, 21)\} = (15, 17, 21),\\
&\widetilde{\chi}_{433} = \min\{(8, 11, 14), (6, 9, 12)\} = (6, 9, 12).
\end{aligned}
$$

Table 2. Transformed balanced problem

	Market M_1			Market M_2			Market M_3			Supply
	Vehicle V_1	Vehicle V_2	Vehicle V_3	Vehicle V_1	Vehicle V_2	Vehicle V_3	Vehicle V_1	Vehicle V_2	Vehicle V_3	
Mill A	$(17, 19, 21)$	$(12, 14, 17)$	$(12, 14, 17)$	$(6, 9, 12)$	$(6, 8, 12)$	$(6, 8, 12)$	$(8, 11, 14)$	$(6, 9, 12)$	$(6, 9, 12)$	$(11, 13, 15)$
Mill B	$(1.5, 3, 4.5)$	$(1, 2, 3)$	$(1, 2, 3)$	$(7, 10, 12)$	$(5, 7, 10)$	$(5, 7, 10)$	$(18, 22, 26)$	$(16, 20, 24)$	$(16, 20, 24)$	$(9, 11, 13)$
Mill C	$(23, 24, 26)$	$(20, 21, 23)$	$(20, 21, 23)$	$(18, 21, 23)$	$(13, 16, 18)$	$(13, 16, 18)$	$(17, 20, 23)$	$(15, 17, 21)$	$(15, 17, 21)$	$(5, 8, 9)$
Mill D	$(1.5, 3, 3.5)$	$(1, 2, 3)$	$(1, 2, 3)$	$(6, 9, 12)$	$(5, 7, 10)$	$(5, 7, 10)$	$(8, 11, 14)$	$(6, 9, 12)$	$(6, 9, 12)$	$(5, 6, 8)$
Loading capacity	$(15, 18, 21)$	$(10, 12, 15)$	$(5, 8, 9)$	$(15, 18, 21)$	$(10, 12, 15)$	$(5, 8, 9)$	$(15, 18, 21)$	$(10, 12, 15)$	$(5, 8, 9)$	
Demand	$(10, 13, 15)$			$(12, 15, 18)$			(8,10,12)			

Step 5.2: On solving the transformed balanced problem by applying steps 4.2.1 to 4.2.4, the fuzzy basic feasible solution is obtained as:

$$
\begin{aligned}
&\widetilde{\xi}_{121} = (3, 3, 3), \widetilde{\xi}_{132} = (8, 10, 12), \widetilde{\xi}_{211} = (9, 11, 13), \widetilde{\xi}_{322} = (2, 2, 3),\\
&\widetilde{\xi}_{323} = (3, 6, 6), \widetilde{\xi}_{411} = (1, 2, 2), \widetilde{\xi}_{421} = (2, 2, 3), \widetilde{\xi}_{423} = (2, 2, 3)
\end{aligned}
$$

and the remaining $\widetilde{\xi}_{ghk}$ are zero triangular fuzzy numbers.

Step 5.3 As in Step 4.3, using condition $\widetilde{u}_g + \widetilde{v}_h + \widetilde{w}_k = \widetilde{\chi}_{ghk}$ for all the eight basic cells obtained in Step 5.2 and taking the value of any two fuzzy dual variables equal to zero (say $\widetilde{u}_4 = 0,\ \widetilde{w}_3 = 0$), the value of the rest of the dual variables is calculated as follows:

$$\widetilde{u}_1 = (-11, 0, 11), \widetilde{u}_2 = (-3, 0, 3), \widetilde{u}_3 = (3, 9, 13), \widetilde{v}_1 = (0.5, 1, 2.5), \widetilde{v}_2 = (5, 7, 10),$$
$$\widetilde{v}_3 = (-15, 9, 33), \widetilde{w}_1 = (-4, 2, 7), \widetilde{w}_2 = (-10, 0, 10)$$

and the values of $\widetilde{\delta}_{ghk} = \widetilde{\chi}_{ghk} - (\widetilde{u}_g + \widetilde{v}_h + \widetilde{w}_k)$ are found to be

$\widetilde{\delta}_{111} = (-3.5, 16, 35.5)$, $\widetilde{\delta}_{112} = (-11.5, 13, 37.5)$, $\widetilde{\delta}_{113} = (-0.5, 13, 27.5)$,
$\widetilde{\delta}_{122} = (-25, 1, 28)$, $\widetilde{\delta}_{123} = (-15, 1, 18)$, $\widetilde{\delta}_{131} = (-43, 0, 44)$,
$\widetilde{\delta}_{133} = (-38, 0, 38)$, $\widetilde{\delta}_{212} = (-14.5, 1, 15.5)$, $\widetilde{\delta}_{213} = (-4.5, 1, 5.5)$,
$\widetilde{\delta}_{221} = (-13, 1, 14)$, $\widetilde{\delta}_{222} = (-18, 0, 18)$, $\widetilde{\delta}_{223} = (-8, 0, 8)$,
$\widetilde{\delta}_{231} = (-25, 11, 48)$, $\widetilde{\delta}_{232} = (-30, 11, 52)$, $\widetilde{\delta}_{233} = (-20, 11, 42)$,
$\widetilde{\delta}_{311} = (0.5, 12, 26.5)$, $\widetilde{\delta}_{312} = (-5.5, 11, 29.5)$, $\widetilde{\delta}_{313} = (5.5, 11, 29.5)$,
$\widetilde{\delta}_{321} = (-12, 3, 19)$, $\widetilde{\delta}_{331} = (-36, 0, 39)$, $\widetilde{\delta}_{332} = (-41, -1, 43)$,
$\widetilde{\delta}_{333} = (-31, -1, 33)$, $\widetilde{\delta}_{412} = (-11.5, 1, 12.5)$, $\widetilde{\delta}_{413} = (-1.5, 1, 2.5)$,
$\widetilde{\delta}_{422} = (-15, 0, 15)$, $\widetilde{\delta}_{431} = (-32, 0, 33)$, $\widetilde{\delta}_{432} = (-37, 0, 37)$,
$\widetilde{\delta}_{433} = (-27, 0, 27)$.

Since, from Remark 2.11, $\widetilde{\delta}_{ghk} \geq 0,\ \forall\, g, h, k$. Therefore, the initial feasible feasible solution obtained is an optimal solution.

Step 5.4: The solution obtained in the previous step is an optimal solution. However, this solution involves a dummy mill/ dummy vehicle, which is not acceptable for real-life applications. Therefore, the optimal solution in terms of the existing original mills and vehicles is obtained as follows:

Step 5.4.1: As in Step 4.5.1 by assuming the dual variables $\widetilde{u}'_4 = \widetilde{w}'_3 = (0, 0, 0)$, the values of the remaining fuzzy dual variables are obtained as

$$\widetilde{u}'_1 = (-11, 0, 11), \widetilde{u}'_2 = (-3, 0, 3), \widetilde{u}'_3 = (3, 9, 13), \widetilde{v}'_1 = (0.5, 1, 2.5), \widetilde{v}'_2 = (5, 7, 10),$$
$$\widetilde{v}'_3 = (-15, 9, 33), \widetilde{w}'_1 = (-4, 2, 7), \widetilde{w}'_2 = (-10, 0, 10).$$

Remark 5.1. *The relation $\widetilde{u}'_g + \widetilde{v}'_h + \widetilde{w}'_k = \widetilde{\chi}_{ghk}$ holds for the variables occurring in optimal solution and here the initial basic feasible solution is an optimal solution. Therefore, the value of dual variables obtained in Step 5.4.1 is the same as that obtained in Step 5.3.*

Step 5.4.2: It is found that the fuzzy dual variables $\widetilde{u}'_1$, $\widetilde{u}'_2$ and $\widetilde{w}'_2$ have zero rank, i.e., $\Re(\widetilde{u}'_1) = \Re(\widetilde{u}'_2) = \Re(\widetilde{w}'_2) = 0$. Therefore, the availability of rice in Mill A, in Mill B, and the loading capacity of Vehicle V_2 can be increased to fully meet the demand of the three wholesalers.

Step 5.4.3: Since $\Re(\widetilde{u}'_g) = 0$ is not unique, i.e., $\Re(\widetilde{u}'_1) = \Re(\widetilde{u}'_2) = 0$. Therefore, according to Section 1b of Step 4.5.3, corresponding to the dummy transport network $\widetilde{\xi}_{421}$, increase the value of the fuzzy basic variable $\widetilde{\xi}_{121}$ by $\widetilde{\xi}_{421}$ and corresponding to $\widetilde{\xi}_{411}$, increase the value of fuzzy basic variable $\widetilde{\xi}_{211}$ by $\widetilde{\xi}_{411}$ as $\widetilde{\xi}_{121} = (3, 3, 3) + (2, 2, 3) = (5, 5, 6)$, $\widetilde{\xi}_{211} = (9, 11, 13) + (1, 2, 2) = (10, 13, 15)$, respectively.

In addition, $\Re(\widetilde{w}'_k) = 0$ is unique, that is, $\Re(\widetilde{w}'_2) = 0$. Therefore, according to Section 2a of Step 4.5.3, increase the value of fuzzy basic variable $\widetilde{\xi}_{322}$ by $\widetilde{\xi}_{323}$ as $\widetilde{\xi}_{322} =$

$(2,2,3)+(3,6,6)=(5,8,9)$.

For fuzzy basic variables, $\widetilde{\xi}_{423}$, $\Re(\widetilde{u}'_g)=0$ is not unique but $\Re(\widetilde{w}'_k)=0$ is unique. Therefore, according to Section 3b of Step 4.5.2, increase the value of fuzzy basic variable $\widetilde{\xi}_{222}$ by $\widetilde{\xi}_{423}$ as $\widetilde{\xi}_{222}=(2,2,3)$.

Hence, the required fuzzy optimal solution in terms of original existing mills and vehicles is obtained as

$$\begin{aligned}
\widetilde{\xi}_{132} &= (8,10,12),\\
\widetilde{\xi}_{121} &= (5,5,6),\\
\widetilde{\xi}_{211} &= (10,13,15),\\
\widetilde{\xi}_{222} &= (2,2,3),\\
\widetilde{\xi}_{322} &= (5,8,9)
\end{aligned}$$

and the remaining variables are zero triangular fuzzy numbers.

For this set of solutions, the total transportation cost is obtained as \$$(168,316,475.5)$ and is presented graphically in Figure 4.

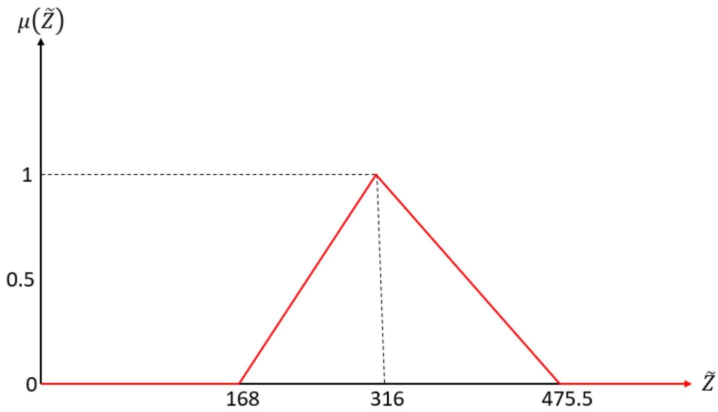

Figure 4. Fuzzy optimal transportation cost

The following information can be interpreted from the obtained membership function:
(i) The transportation cost is \$168 or more but less than \$475.5.
(ii) The probability that the transportation cost will be \$316, is maximum.
(iii) Satisfaction level of the decision maker for the transportation cost, say (χ) is $(\mu(\chi)\times 100)\%$,

where

$$\mu(\chi)=\begin{cases}\frac{\chi-168}{316-168}, & \text{if } 168\le\chi\le 316\\ \frac{475.5-\chi}{475.5-316}, & \text{if } 316\le\chi\le 475.5\\ 0, & \text{otherwise}\end{cases}$$

6. Validation of results

The considered problem of rice transportation considered is an unbalanced problem in which the total availability of rice in the mills and the total loading capacity of the vehicles are less than the total demand for rice of the wholesaler. To solve such types of problem, existing techniques [24, 31] use the concept of dummy facilities (mills/vehicles), which is not acceptable or valid in real-life problems. This is because the decision maker has no idea about the location of the dummy mills or the type of dummy vehicle to be used. Therefore, the concept of dummy facilities does not seem valid for real-life problems. Unlike existing techniques, the proposed method provides an optimal solution in terms of the original mills as well as the facilities of the original vehicles.

For the considered case study, the proposed approach tells us that to meet the total demand at a minimum cost, the availability of rice in mill A should be increased by $(2, 2, 3)$ tonnes, in mill B by $(3, 4, 5)$ tonnes, and the loading capacity of vehicle V_2 should be increased by $(5, 8, 9)$ tonnes. To validate these results, the number of cases is studied by increasing the amount of deficit supply and loading capacity at different mills and vehicles, respectively. A total of seven cases are shown in which the amount of deficit in supply and deficit in loading capacity is increased, respectively, at eachand vehicle one by one and also at all the facilities equally. The problems are solved with Lingo 20.0 and the results obtained are as shown in Table 3. The comparison between the solutions obtained is made based on their ranking and is shown in Figure 5. From Table 3 and Figure 5, it is concluded that the proposed method gives the minimum possible cost among all.

Table 3. Results

Case	Increasing the supply at	Increasing the loading capacity of	Transportation Cost
1	Mill A	Vehicle V_1	$(187.5, 350, 523.5)$
2		Vehicle V_2	$(180.5, 340, 502)$
3	Mill B	Vehicle V_1	$(180, 326, 500.5)$
4		Vehicle V_2	$(171, 319, 475.5)$
5	Mill C	Vehicle V_1	$(232.5, 400, 592.5)$
6		Vehicle V_2	$(222.5, 388, 568.5)$
7	Equally dividing among all the mills	Equally dividing among all the vehicles	$(186.6, 343.7, 517.3)$
8	As suggested by the proposed method	As suggested by the proposed method	$(168, 316, 475.5)$

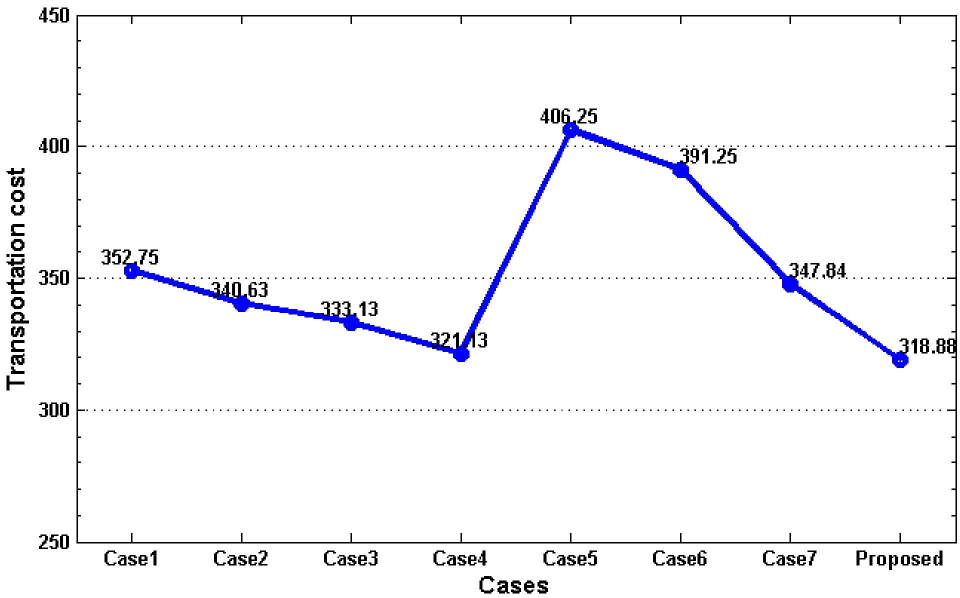

Figure 5. Comparison of the solutions obtained in Table 3

From the optimal solution obtained, it is found that the quantity of rice that is transported from the dummy mill (mill D) is adjusted to the original mills (Mill A, Mill B) and the amount that must be transported by the dummy vehicle V_3 is adjusted to the original vehicle V_2. That is, the supply at Mill A, Mill B, and the loading capacity of vehicle V_2 can be increased to meet the entire demand of wholesalers. In addition, the obtained solution is compared with those obtained by the existing approaches in Table 4.

Table 4. Solution comparison for case study 5

Methods	Fuzzy optimal transportation cost	Transported amount
Kumar and Kaur [26], Chakraborty and Jana [9], Muthuperumal [30]	Not applicable	–
Rani and Gulati [31], Kaur et al. [24]	$(75, 138, 219)$	$(20, 24, 28)$
Proposed method	$(168, 316, 475.5)$	(30, 38, 45)

From Table 4, it can be seen that the fuzzy optimal cost obtained by the proposed method is higher than the solution obtained by [24, 31]. It is only because the existing methods [24, 31] obtained the fuzzy optimal solution by incorporating the dummy mills/vehicles and the cost that has to be calculated by the these existing methods is only to transport the $(20, 24, 28)$ tons of rice. The remaining amount of rice is transported to wholesalers through dummy facilities, which is practically not acceptable. In contrast, the proposed method obtained the fuzzy optimal cost of the unbalanced problem by transporting the entire amount of rice through the original mills / vehicles, and hence the demand for all the wholesale markets of rice is fully satisfied. Therefore, by eliminating the need for dummy facilities, the proposed method identifies which specific sources and vehicle capacities should be enhanced to meet the entire demand. This realistic approach improves long-term efficiency by ensuring that all destinations are fully served, thereby reducing delivery failures. Furthermore, it supports better logistical planning, as it provides actionable insights on where to invest resources either by increasing supply at certain sources or enhancing the loading capacity of specific vehicles. In doing so, the model improves logistical reliability, promotes better resource management, and ultimately contributes to greater customer satisfaction. Although there is an initial increase in cost, this is offset by the long-term benefits of operational feasibility, service completeness, and sustainability making the solution both practical and impactful in the real-world logistics systems.

7. Conclusion

Most of the existing methods for solving unbalanced FFSTP provide the optimal solution involving the dummy source/destination/vehicle. Since these dummies have no physical relevance and these do not exist in reality, the unbalanced problem is not providing a solution that can benefit the decision maker while solving the real-world problems. In order to address this limitation, a novel approach is introduced, whose applicability is demonstrated by solving a real-life unbalanced solid transportation problem in a fuzzy environment. The solution obtained by the proposed approach does not involve the dummy source/dummy vehicle/dummy destination. Although the minimum transportation cost obtained by the proposed method is more than that obtained by the existing methods, our cost is when, in actual, the total demand is met. So, the proposed approach may be more helpful and fruitful in solving real-life transportation problems than the existing ones. Furthermore, the proposed algorithm offers the advantage of providing a fuzzy solution to the problem within a fuzzy environment. This sets it apart from existing methods that provide only crisp solutions to fuzzy problems. Using triangular fuzzy numbers to represent the vagueness of the real world, the proposed algorithm manages indeterminacy and impreciseness in a better way. The triangular fuzzy numbers have been used to represent the real-world vagueness. These numbers are chosen for the easy understanding and readability of the proposed method. One may choose the fuzzy numbers that best suits his/her requirement according to the real-world situation. In the future, the interested researchers may explore the extension of our proposed approach to address an unbalanced multi-objective or multi-item solid transportation problem. However, investigating the applicability of the algorithm in any other uncertain environment is a future challenge. In addition, the present approach can be extended to solve real-life transportation problems with fuzzy parameters with hesitant intuitionistic fuzzy parameters with interval values [7].

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Ali Ebrahimnejad: Formal analysis, Conceptualization, Investigation, review & editing.

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