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On the Convergence Analysis of Bernstein Operators

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Abstract: In recent years, many studies have been conducted to emphasize the convergence criteria of the q-Bernstein polynomial and their importance. A point often overlooked in the literature is that the kernels of these polynomials depend on the probabilities of the variable parameter binomial distribution. Taking advantage of this property, we develop a simplified form of this polynomial for the binomial variable with variable parameters. This approach not only simplifies the computational processes but also sheds light on the convergence properties of this polynomial. By examining this reduced form, an upper bound for convergence is determined. The findings highlight the versatility and power of the q-Bernstein polynomial in approximation theory and provide a deeper understanding of the mathematical foundations and potential applications of the polynomial.

Key words: Approximation theory, Bernstein polynomial, Bernoulli process, convergence, expected value.

Bernstein Operatörlerinin Yakınsama Analizi Üzerine

Öz: Son yıllarda, q-Bernstein polinomları yaklaşım teorisinde önemli bir konu olarak ortaya çıkmıştır. Çok sayıda çalışma bu polinomun yakınsama kriterlerini incelemiş, bunların önemini ve kullanışlılığını vurgulamıştır. Literatürde sıklıkla göz ardı edilen bir nokta, bu polinomların çekirdeklerinin değişken parametreli binom dağılımının olasılıklarına bağlı olmasıdır. Bu özellikten yararlanarak, binom bağımlılığına ilişkin bu polinomun basitleştirilmiş bir formunu geliştirdik; bu, değişken parametreli bir binom değişkeni için momentlerinin hesaplanmasını kolaylaştırmıştır. Bu yaklaşım yalnızca hesaplama süreçlerini basitleştirmekle kalmaz, aynı zamanda bu polinomun yakınsama özelliklerine de ışık tutar. Bu indirgenmiş formu inceleyerek yakınsama için bir üst sınır belirlenir. Bulgular, q-Bernstein polinomunun yaklaşım teorisindeki çok yönlülüğünü ve gücünü vurgular ve polinomun matematiksel temelleri ve potansiyel uygulamaları hakkında daha derin bir anlayış sağlar.

Anahtar kelimeler: Yaklaşım teorisi, Bernstein polinom, bernoulli süreci, yakınsaklık, beklenen değer.

1. Introduction

Polynomials have a wide range of applications in today's technological age. For instance, they are extensively used in fields such as physics, engineering, economics, and computer programming. In physics, polynomials can describe various physical phenomena, such as motion, force, and energy relations. In engineering, they are crucial in designing and analyzing systems and structures, from bridges to electronic circuits. Economists use polynomials to model economic growth, trends, and other financial data. In computer programming, polynomials are used in algorithms, graphics, and data interpolation [1-5].

In essence, polynomials are fundamental tools for modeling and solving numerous problems across different disciplines. One of the key advantages of polynomials is their ease of differentiation and integration, which simplifies the processing of mathematical functions. This makes them particularly useful in various studies and practical applications.

This study focuses on Bernstein polynomials, a special class of polynomials known for their significant applications. Bernstein polynomials are widely used due to their simple structure and valuable properties. They are particularly important in approximation theory, which deals with approximating complex functions using simpler and more manageable functions. This theory is crucial because working directly with functions with unknown or complicated properties can be challenging. By approximating these functions with well-known, simpler functions, researchers can derive useful results more easily.

Approximation theory addresses whether it is possible to convert complex functions into polynomial functions and how closely such approximations can represent the original functions. This process of approximation is vital in various fields, providing a way to work with functions that are otherwise difficult to handle. By using polynomial approximations, researchers and practitioners can simplify their analyses and obtain more tractable solutions to complex problems. It can be said that approximation by polynomials is perhaps the most important

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branch of approximation theory. Today, Bernstein polynomials are mostly applied in the field of approximation theory [6,7].

Recent studies on the approximation properties of blending-type modified Bernstein–Durrmeyer operators have demonstrated that these operators possess strong approximation characteristics [8]. In a recent study, a new type of coupled Bernstein operators for Bezier basis functions was introduced, demonstrating their approximation properties, including the establishment of a local approximation theorem and a convergence theorem for Lipschitz continuous functions [9]. In a recent study, a novel method for approximating the Koopman operator using Bernstein polynomials was proposed. This approach provides a finite-dimensional approximation and characterizes approximation errors with upper bounds expressed in the uniform norm, covering various contexts including univariate and multivariate systems [10]. In a recent study, a new class of Bernstein polynomials based on Bézier basic functions with a shape parameter $\lambda \in [-1,1]$ was examined. The study provides a Korovkin-type approximation theorem and demonstrates improvements in error estimation in some cases by comparing these operators with classical Bernstein operators [11]. In 2024, Aslan investigated some approximation properties of the mixture-type univariate and bivariate Schurer-Kantorovich operators based on shape parameters $\lambda \in [-1, 1]$ [12]. In 2023, Su [13] studied various approximation properties of the Durrmeyer variant of q-Bernstein operators based on the Bézier basis with shape parameters.

Most of the recent work on achievement studies in the Bernoulli trials is in Feller's [14] basic book. Let S_n be the total number of successes in the Bernoulli trials of n independents. When the trials are the same, most of the properties of the S_n distribution and the related theorems are well known and discussed in many books and studies on statistics. Charalambides [15] showed that the probability mass function has a very important place in q-Bernstein polynomial. He calculated the expected value and variance based on this probability function.

In the Bernoulli process, the experiment takes place once and has two possible outcomes. Occurrence of the desired situation is considered a success and the occurrence of the undesired situation is considered a failure. Let *k* be the number of successes in *n* independent Bernoulli experiments.

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \;\; k = 0,1,\dots,n. \tag{1}$$

where x is the probability of success in a single Bernoulli trial as given in Eq. 1. $p_{n,k}(x)$ is the probability of k successes in n trials [16].

The approximation of real-valued continuous functions has attracted attention of researchers for years. The Weierstrass Approximation Theorem is one of the fundamental theorems that is frequently used in functional analysis. According to this approximation theory, polynomials can uniformly approximate any function that is merely continuous over a closed interval. Bernstein proved this approximation theorem in its simplest form in 1912 [17].

Since Bernstein operators are simpler and have very different approach properties, they have become the operators preferred by researchers. Lupaş [18] presented the q-Bernstein theory as a contribution to science in 1987. Acu [19] has studied different generalizations of Bernstein operators. The q-generalization of these operators was examined by Cárdenas-Morales [20] presented the new sequence of linear Bernstein-type operators. It is also useful to say that the Benstein operators have many more generalizations that can contribute to science.

2. Positive Operators Obtained with the Help of the Binomial Distribution

Bernstein polynomials are given in 1912 in the proof of the Weierstrass theorem by Bernstein and were later used in the proof of many theorems. The Korovkin theorem is a typical example. The Bernstein polynomial can be expressed as in Eq. 2.

Let $f\colon [0,1] \to R$. The Bernstein polynomial of f is

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x). \tag{2}$$

Here $B_n(f;x)$ is called Bernstein operators of order n for f [21].

Some Korovkin type approximation theorems were proved in [22] by using statistical convergence, lacunary statistical convergence and statistical summability $(C,1)$, respectively. Now, let's introduce the Korovkin approximation theorem [23].

Theorem 2.1. Let (T_n) be a sequence of positive linear operators from $C[a,b]$, into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$ where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$ [24].

3. Convergence Criteria

Definition 3.1. Let the function f be continuous in the interval $[a,b]$. The $\omega(\delta)$ function defined by the equation

$$\omega(\delta) = sup_{|x_1 - x_2| \le \delta} |f(x_1) - f(x_2)|$$

with $x_1, x_2 \in [a,b]$ for the real number $\delta > 0$, is the modulus of continuity of the function f [25].

This function will take values based on f, the interval $[a,b]$ and the chosen $\delta > 0$. Let's continue the features of $\omega(\delta)$ modulus of continuity.

3.1. For $0 < \delta_1 < \delta_2$, $\omega(\delta_1) \le \omega(\delta_2)$.

3.2. $\lim_{\delta \to 0} \omega(\delta) = 0$, when function f is continuous in the interval $[a,b]$.

3.3. $\omega(\lambda\delta) \le (1+\lambda)\,\omega(\delta)$, for the real number $\lambda > 0$.

Let us express the theorem that enables to evaluate the difference $|B_n(f;x) - f(x)|$ with the help of the modulus of continuity.

3.1. Approximation Theorem

Let $\{X_n : n = 1,2,\dots\}$be a sequence of independent random variables where X_n has a distribution with parameters (n,x). where, n represents the number of trials, and x represents the probability of success. Let f be the real-valued function defined on the real interval [a, b] such that $f^{(m)} \in C[a,b]$ and $L_n(f,x) = Ef(X_n) < \infty$. Then for any $x \in [a,b]$ and any $\delta > 0$

$$\left| L_n(f,x) - \sum_{k=0}^{m} \frac{f^{(k)}(x)}{k!} b_k \right| \le \begin{cases} \frac{1}{m!}(\sqrt{b_{2m}} + \delta b_{m+1})\omega_m(\frac{1}{\delta}) & ,if\ m\ is\ odd \\ \frac{1}{m!}(b_m + \delta\sqrt{b_2 b_{2m}})\omega_m(\frac{1}{\delta}) & ,if\ m\ is\ even \end{cases} \quad (3)$$

where $b_k = b_k(n,x) = E(X_n - x)^k$ for $k = 0,1,\dots,m$ is k th moment of random variable X_n around x, $\omega_m(\frac{1}{\delta})$ is the modulus of continuity of $f^{(m)}$ [26].

4. Reduced Form of *q*- Bernstein Polynomials and Probabilistic Interpretation

Philips [27] gave a generalization of the Bernstein polynomial q in 1996, which changes depending on the integer values of q. After this stage, this issue has been handled by many authors from different angles. Since q-Bernstein polynomials are positive linear operators on *C [0,1]*, the case of $0 < q < 1$ is generally investigated. For each positive integer n, $B_n(f,q;x)$ q- Bernstein polynomials as For each positive integer n, $B_n(f,q;x)$ q-Bernstein polynomials are as in Eq. 4.

$$B_n(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x). \quad (4)$$

When $q = 1$, $B_n(f,q;x)$ is the classical Bernstein operator. The q- Bernstein polynomial shares the shape-preserving properties of the classical Bernstein polynomial.

Let $q > 0$. As in eq. 5, for each nonnegative integer l, the q- integer $[l]$, q- factorial $[l]!$ and q- binomial $\begin{bmatrix} n \\ r \end{bmatrix}$, $(n \ge l \ge 0)$ are defined by

$$[l] := [l]_q := \begin{cases} (1 - q^l)/(1 - q) & q \ne 1 \\ l & q = 1 \end{cases}$$

$$[l]! := \begin{cases} [l][l-1]\dots[1] & q \ne 1 \\ l! & q = 1 \end{cases} \qquad \begin{bmatrix} n \\ l \end{bmatrix} := [n]!/([n-l]!\,[l]!) \quad (5)$$

respectively [28]. Additionally, $[0]! := 1$.

Theorem 4.1. *Let a sequence* $\{q_n\}$ *satisfy* $0 < q_n < 1$ *and* $q_n \to 1$ *as* $n \to \infty$. *If* $f \in C[0,1]$ *then* $B_n(f, q_n; x) \rightrightarrows f(x)$ *for* $x \in [0,1]$ *as* $n \to \infty$ [29].

Direct calculations show that for $0 < q < 1$

$$B_n(t^2, q; x) \rightrightarrows x^2 + (1-q)x(1-x) \neq x^2, \ x \in [0,1] \text{ as } n \to \infty.$$

Therefore in general, the sequence $\{B_n(f, q; x)\}$ is not an approximating one for the function f [16].

4.1. Probabilistic properties of *q*- Bernstein polynomials

Considering probability problems, it is always possible to talk about the probability of a desired situation or event. If two results such as successful or unsuccessful occur for a trial and this trial can be repeated under the same conditions, this experiment is called Bernoulli test. The Bernoulli trial is the basis for discrete distributions. Here, let *k* be the number of successes in *n* independent Bernoulli experiments.

Let's express it as the sum of the events of Bernoulli such that $S_n = X_1 + X_2 + \cdots + X_n$. Since the sum of the Bernoulli trials will give the binomial distribution, $S_n \sim p_{n,k}(x)$ can be written. Therefore, the following Eq. 6 can be written for S_n

$$P(S_n = k) = \binom{n}{k} x^k (1-x)^{n-k} \tag{6}$$

Since the kernel of the Bernstein polynomial provides the properties $p_{n,k}(x) \geq 0$ and $\sum_{k=0}^{n} p_{n,k}(x) = 1$ for $0 < x < 1$, it can be regarded as the probability function of a random variable. From the equation $P(S_n = k) = p_{n,k}(x), k = 0,1,\ldots,n$, the Bernstein polynomial can be written in the form of the expected value of the S_n random variable with the help of the expected value operator *E* (Eq. 7)

$$B_n(f; x) = Ef\left(\frac{S_n}{n}\right). \tag{7}$$

4.2. Main results

If the Bernstein polynomial is shown as in (2), the term *q* must be added on success possibilities of $X_1, X_2, \ldots, X_n$ Bernoulli in order to express the *q*- Bernstein polynomials in the same way. Accordingly,

$$P(X_j^* = 1) = q^s x$$

$$P(X_j^* = 0) = 1 - q^s x \tag{8}$$

to define the number of *j* experiments can be written as Eq. 8. Also *s* denotes the number of unsuccessful attempts in trials up to *j-1*.

Now let's show that $P(S_n^* = k) = \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1 - q^s x)$ using the method of mathematical induction. For *n = 1, k* values are *0* and *1*, respectively.

$$P(S_1^* = 0) = (1 - x)$$

$$P(S_1^* = 1) = x$$

Let's assume that $P(S_n^* = k) = \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1 - q^s x)$ (9)

for *n = k.*

As ,n Eq. 10, now let's show that for *n = k+1,*

$$P(S_{n+1}^* = k) = \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k}(1 - q^s x).$$

$$P(S_{n+1}^* = k) = P(S_{n+1}^* = k | S_n^* = k-1).P(S_n^* = k-1) + P(S_{n+1}^* = k | S_n^* = k).P(S_n^* = k)$$

$$= q^{n-k+1}x\left(\begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1}\prod_{s=0}^{n-k}(1-q^s x)\right)n + (1-q^{n-k}x)\left(\begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1-q^s x)\right)$$

$$= \frac{[n]![n+1]}{[k]![n-k+1]!}x^k \prod_{s=0}^{n-k}(1-q^s x) = \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k}(1-q^s x). \quad (10)$$

Thus it is proven that Eq. 9 is true.

To obtain an approximate value, it is necessary to calculate the $E(X_n^* - x)^2$. In this case Eq. 11 is obtained.

$$E(X_n^* - x)^2 = b_2(n,x) = E(\frac{S_n}{n} - x)^2 = \frac{1}{n^2}E(S_n^2 - 2nxS_n + n^2x^2) = b_2(n,x) = \frac{1}{n}xq \quad (11)$$

In approximation theorem (Eq.3), if $m = 0$ is taken specially Eq. 12 is obtained.

$$|L_n(f,x) - f(x)| \le (1 + \delta\sqrt{b_2})\omega(\frac{1}{\delta}) \quad (12)$$

is given. According to this, when $L_n = B_{n,q}$ and $\delta = \sqrt{n}$ are taken, Eq. 13 is obtained.

$$|B_n(f,q;x) - f(x)| \le \left(1 + \delta\sqrt{\frac{1}{n}xq}\right)\omega(\frac{1}{\delta})$$

$$B_n(f,q;x) = f(x) + o\left(\frac{1}{n}\right), n \to \infty \quad (13)$$

5. Conclusion

In this paper, q-Bernstein polynomials based on q integers are introduced. These polynomials are constructed in a different form and an upper bound for the approximation of operators is obtained by the approximation theorem. It follows that all operators with kernel-part probability functions can be used in different variations in approximation theory.

References

[1] Kingora K, Sadat-Hosseini H. A novel interpolation-free sharp-interface immersed boundary method. J Comput Phys 2022; 453, 110933.

[2] Tu S, Yang H, Dong L, Huang Y. A stabilized moving Kriging interpolation method and its application in boundary node method. Eng Anal Boundary Elem 2019; 100: 14-23.

[3] Kochurin E. A Simulation of the Wave Turbulence of a Liquid Surface Using the Dynamic Conformal Transformation Method. JETP Letters 2023; 118(12): 893-898.

[4] Christodoulou KN, Scriven LE. Discretization of free surface flows and other moving boundary problems. J Comput Phys 1992; 99(1): 39-55.

[5] Yang X, Zhang X, Li Z, He GW. A smoothing technique for discrete delta functions with application to immersed boundary method in moving boundary simulations. J Comput Phys 2009; 228(20): 7821-7836.

[6] Cetin E, Applications of Bernstein polynomials. Master Thesis, Uludağ Uni., Graduate School of Natural and Applied Sciences, Bursa, 2011.

[7] Kılın BK, Bernstein-Chlodowsky Type Polynomials and Some Applications. Master Thesis, Gaziantep Uni., Graduate School of Natural and Applied Sciences, Gaziantep, 2019.

[8] Liu YJ, Cheng WT, Zhang WH, Ye PX. Approximation Properties of the Blending-Type Bernstein–Durrmeyer Operators. Axioms 2022; 12(1): 5.

[9] Ayman-Mursaleen M, Nasiruzzaman M, Rao N, Dilshad M, Nisar KS. Approximation by the modified λ-Bernstein-polynomial in terms of basis function. AIMS Math 2024; 9(2): 4409-4426.

[10] Yadav R, Mauroy A. Approximation of the Koopman operator via Bernstein polynomials. 2024, arXiv preprint arXiv:2403.02438.

[11] Aslan R, Mursaleen M. Some approximation results on a class of new type λ-Bernstein polynomials. J Math Inequal 2022; 16(2): 445-462.

[12] Aslan R. Rate of approximation of blending type modified univariate and bivariate λ-Schurer-Kantorovich operators. Kuwait J Sci 2024; 51(1): 100168.

[13] Su LT, Aslan R, Zheng FS, Mursaleen M. On the Durrmeyer variant of q-Bernstein operators based on the shape parameter λ. J Inequal Appl 2023; 2023(1): 56.

[14] Feller W. An Introduction to Probability Theory and Its Applications. New York, Wiley, 1968.
[15] Charalambides CA. The q-Bernstein basis as a q-binomial distribution. J Stat Plann Inference 2010; 140(8): 2184-2190.
[16] II'inskii A, Ostrovska S. Convergence of generalized Bernstein polynomials based on the q-integers. Constructive theory of functions 2003; 309-313.
[17] Bernstein S. Demonstration du theoreme de Weierstrass Fondee Sur le Calcul des Probabilities. Communications of the Kharkov Mathematical Society 1912; 13: 1-2. IP: 88.236.115.136.
[18] Lupaş A. A q-analogue of the Bernstein Operator. Seminar on Numerical and Statistical Calculus 1987; 9: 85-92.
[19] Acu MA. Stancu–Schurer–Kantorovich Operators Based on q-integers. Appl Math Comput 2015; 259: 896–907.
[20] Cárdenas-Morales D, Garrancho P, Rasa I. Bernstein-type Operators which Preserve Polynomials. Comput Math Appl 2011; 62(1): 158–163.
[21] Ostrovska S. On the Lupaş q-analogue of the Bernstein operator. The Rocky Mountain J Math 2006;1615-1629.
[22] Patterson RF, Savas E. Korovkin and Weierstrass approximation via lacunary statistical sequences. J Math Stat 2005; 1(2): 165-167.
[23] Korovkin PP. Linear Operators and Approximation Theory. Delhi, Hindustan Publ. Co., 1960.
[24] Kim T. A note on q-Bernstein polynomials, Russ J Math Phys 2011; 18(1): 73-82.
[25] Rıvlin JP. An Introduction to the Approximation of Functions, New York, Bronx, 1978.
[26] Gurcan M, Colak C. Generalization of Korovkin Type Approximation by Appropriate Random Variables and Moments and an Application in Medicine. Pak J Statist 2011; 27(3): 283-297.
[27] Phillips GM. On generalized Bernstein polynomials. In Numerical Analysis 1996; 263-269.
[28] Dalmanoglu O. Approximation by Kantorovich type q-Bernstein operators, In Proceedings of the 12th WSEAS International Conference on Applied Mathematics 2007; 113-117.
[29] Gupta V, Heping W. The rate of convergence of q-Durrmeyer operators for $0<q<1$. Math Methods Appl Sci 2008; 31(16): 1946-1955.