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RESEARCH ARTICLE

Diffeomorphisms of Foliated Manifolds I

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ABSTRACT

The set $Diff(M)$ of all diffeomorphisms of manifold M onto itself is the group related to composition and inverse mapping. The group of diffeomorphisms of smooth manifolds is of great importance in differential geometry and analysis. It is known that the group $Diff(M)$ is topological group in compact open topology.In this paper we investigate the group $Diff_F(M)$ of diffeomorphisms foliated manifold (M, F) with foliated compact open topology.
In this paper we prove that if all leaves of the the foliation F are closed subsets of M then the foliated compact open topology of the group $Diff_F(M)$ coincides with compact open topology. In addition it is studied the question on the dimension of the group of isometries of foliated manifold is studied when foliation generated by riemannian submersion.

Keywords: *foliated manifold, submersion, foliated compact open topology, diffeomorphisms of foliated manifold, Lie group.*
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1. Introduction

This paper is a continuation of the paper "Diffeomorphisms of foliated manifolds" [1]. In this paper we discuss some properties of foliated compact open topology on the group $Diff_F(M)$ of diffeomorphisms foliated manifold (M, F), which was introduced in the paper [6] and studied in papers [1],[7]. Authors proved that foliated compact open topology of the group $Diff_F(M)$ has a countable base and the group $Diff_F(M)$ is topological group with foliated compact open topology [1].

Let M be a smooth connected manifold of dimension n. Smoothness in this paper means the smoothness of the class C^∞.

Denote by (M, F) the manifold M with the foliation F of dimension k and call foliated manifold, where $0 < k < n$.

The geometry and topology of foliated manifolds have been studied by many authors (see [5],[3]).

Definition 1.1. A diffeomorphism $\varphi : M \to M$ is called a diffeomorphism of the foliated manifold (M, F), if the image $\varphi(L_\alpha)$ of each leaf L_α is a leaf of the foliation F.

The diffeomorphism $\varphi : M \to M$ of the foliated manifold (M, F), is denoted by $\varphi : (M, F) \to (M, F)$. The set of all diffeomorphisms of a foliated manifold is denoted by $Diff_F(M)$. The set $Diff_F(M)$ is a group with respect to the superposition of mappings and is a subgroup of the group $Diff(M)$ of diffeomorphisms of the manifold M.

The group $Diff_F(M)$ was studied in the papers [6], [7] in particular, in [7] it was proved that this group is a closed subgroup of the group $Diff(M)$ with respect to a compact open topology.

Let $Iso_F(M)$ denote the subset of the set $Diff_F(M)$ consisting of isometries of the Riemannian manifold (M, g). This set is a subgroup of the group $Diff_F(M)$. In [7] it was proved that the group $Iso_F(M)$ is a Lie group with a compactly open topology.

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Example 1.1. Let $M = \mathbb{R}^2(x_1, x_2)$ is a Euclidean plane with the Cartesian coordinates (x_1, x_2), foliation F is given by submersion $f(x_1, x_2) = x_2 - x_1^2$. Diffeomorphism of the plane $\varphi_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ determined by the formula

$$\varphi(x, y) = (x_1, x_2 + \lambda f(x_1, x_2))$$

is diffeomorphism of foliated plane $(\mathbb{R}^2, F)$ for every $\lambda \in \mathbb{R}, \lambda \neq -1$. Diffeomorphisms $(x_1, x_2) \to (x_1, x_2 + h)$ and $(x_1, x_2) \to (-x_1, x_2 + h)$ are elements of the group $Iso_F(M)$ for $h \in \mathbb{R}$.

We recall notion of foliated compact open topology on the group $Diff_F(M)$ [6].

Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf L_λ of foliation F and let $\{U_\beta\}$ is the family of all open sets on M. We consider for each pair K_λ and U_β set of all mappings $f \in Diff_F(M)$ for which $f(K_\lambda) \subset U_\beta$. This set of mappings we denote through $[K_\lambda, U_\beta] = \{f : M \to M : f(K_\lambda) \subset U_\beta\}$.

It isn't difficult to show that every possible finite intersection of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. This topology we call foliated compact open topology or in brief $F-$ compact open topology. The space $Diff_F(M)$ with $F-$ compact open topology is Hausdorff topologycal space [6]. Since K runs only over all compact subsets of leaves, $F-$ compact open topology on $Diff_F(M)$ is weaker than induced from $Diff(M)$ usual compact open topology.

Following lemma and the theorem were proved in [1].

Lemma 1.1. *The space $Diff_F(M)$ with $F-$ compact open topology is a topological space with a countable base.*

Theorem 1.1. *Let (M, F) be a smooth foliated manifold.Then the group $Diff_F(M)$ is a topological group with $F-$ compact open topology.*

2. Diffeomorphisms of Foliated Manifolds

In this section we will study conditions to manifold M and to foliation F under which foliated compact open topology and usual compact open topology on the space $Diff_F(M)$ coincides.

Theorem 2.1. *Let M be a smooth compact manifold.Then $F-$ compact open topology coincides with compact open topology on $Diff_F(M)$.*

Proof. Let g be some Riemannian metric on M, $d(x, y)$ is the distance between points x and y defined by riemannian metric g.

We define the distance function on $Diff_F(M)$ as

$$\rho(f, g) = max\{d(f(x), g(x)) : x \in M\}.‘$$

We will show that the compact-open topology and the foliated compact-open topology on the diffeomorphism group $Diff_F(M)$ of a foliated manifold (M, F) coincide with the topology of the metric space $(Diff_F(M), \rho)$.

Let us consider a set $A \subset Diff_F(M)$, which has the form

$$A = [K, U] = \{f \in Diff_F(M) : f(K) \subset U\},$$

where K is a compact set, U is a open set in M. If K is a compact subset of a leaf,then A is a element of the base of foliated compact-open topology, in any case it is a element of the base of compact open topology.

For the element $f \in A$ let $K' = f(K)$ and $\delta = d(K', M \setminus U$.) Then the open ball $B_\delta(f) = \{g \in Diff_F(M) : \rho(f, g) < \delta\}$ contained in A and therefore $f \in A$ is a interior point of the set A in the topology of metric space $(Diff_F(M), \rho)$ and hence A is a open set in in the topology of metric space $(Diff_F(M), \rho)$.

Now we prove that open ball $B_\varepsilon(f)$ of metric space $(Diff_F(M), \rho)$ is a open set in foliated compact open topology and in compact open topology.

Let $g \in B_\varepsilon(f)$ and $\rho(f, g) = \varepsilon - a$, where a is a positive number. We put $K' = g(K)$ for any compact set K and put $U_a = \{x : max\{d(x, y), y \in K'\} < a\}$. It follows $g \in [K, U_a]$ and $[K, U_a] \subset B_\varepsilon(f)$. It follows from if K is a subset of some leaf then $g \in B_\varepsilon(f)$ is a interior point of $B_\varepsilon(f)$ in foliated compact open topology and hence $B_\varepsilon(f)$ is an open set foliated compact open topology of $Diff_F(M)$. If K is a any compact subset of it follows from here $B_\varepsilon(f)$ is an open set compact open topology of $Diff_F(M)$. .

Thus we have proved that the compact-open topology and foliated compact open topology on the diffeomorphism group $Diff_F(M)$ coincides with the topology of the metric space $(Diff_F(M), \rho)$. ☐

Theorem 2.2. *Let (M, F) be a smooth foliated manifold and all leaves are closed subsets of M. Then $F-$ compact open topology coincides with compact open topology on $Diff_F(M)$.*

Proof. As $F-$ compact open topology on $Diff_F(M)$ is weaker than induced from $Diff(M)$ usual compact open topology it is sufficient to prove that every open set in compact open topology is a open set in $F-$ compact open topology on $Diff_F(M)$.

Let $A \subset Diff_F(M)$ be a open set in compact open topology. Without loss of generality, we will assume that the set A has the form

$$A = [K, U] = \{f \in Diff_F(M) : f(K) \subset U\},$$

where K is a compact subset of M, U is a open subset of M.

We will show that the set A is a open set in $F-$ compact open topology.

Let us put $K^\alpha = K \bigcap L_\alpha$ for every leaf L_α of the foliation F such that $K \bigcap L_\alpha \neq \emptyset$. As every leaf L_α is a closed subset of M, it follows from [2] canonical injection $i : L_\alpha \to M$ is an embedding, i.e., the topology of the leaf coincides with the topology induced from M. Therefore, the set $K^\alpha = K \bigcap L_\alpha$ has finite number of components. Since each leaf L_α is a closed set, the set K^α is also a compact subset of the leaf L_α.

We have the set $[K^\alpha, U] = \{f : M \to M | f(K^\alpha) \subset U\}$ is a open set in $F-$ compact open topology for each α. Since $f(K^\alpha) \subset U$ for every $f \in A$ for all α it holds relation $A \subset \bigcap_\alpha [K^\alpha, U]$.

From other side since $K = \bigcup_\alpha K^\alpha$ it follows $f \in A$ for every $f \in \bigcap_\alpha [K^\alpha, U]$. Therefore we have $A = \bigcap_\alpha [K^\alpha, U]$.

Now we will show that the set $CA = Diff_F(M) \setminus A$ is a closed set in $F-$ compact open topology. Since by the results of [1] the space $Diff_F(M)$ with F- compact open topology is a space with countable base, we can use consequences.

Assume that $f_i \to f$ at $i \to \infty$ in $F-$ compact open topology, where$f_i \in CA$. We have to show that $f \in CA$.

Let g be some Riemannian metric on M, $d(x, y)$ is the distance between points x and y defined by riemannian metric g, $K' = f(K)$, $K'_\varepsilon = \{x : d(x, K') = inf\{d(x, y) : y \in K'\}\} < \varepsilon$.

Let's assume, $f \in A$. Since $f_i \to f$ at $i \to \infty$ in $F-$ compact open topology $f_i \to f$ at $i \to \infty$ at every compact K^α. It follows from here for any $\varepsilon > 0$ for a point $x \in K^\alpha \subset K$ there exists an integer n_x such that $d(f(x), f_i(x)) < \varepsilon$ for $i \geq n_x$. Since $d(f(x), f_i(x))$ is continuos function on x for fixed i there exists neighborhood U_x of x in M such that $d(f(y), f_i(y)) < \varepsilon$ for $y \in U_x$ at $i \geq n_x$.

We can find finite covering $U_{x_1}, U_{x_2}, \cdots, U_{x_m}$ of K, such that $d(f(y), f_i(y)) < \varepsilon$ for $y \in U_{x_i}$ at $i \geq n_{x_i}$, where $i = 1, 2, ..., m$. It follows from here that $d(f(y), f_i(y)) < \varepsilon$ for $y \in K$ at $i \geq p$ where $p = max\{n_{x_1}, n_{x_2}, \cdots, n_{x_m}\}$.

We can choose $\varepsilon > 0$ such that $\varepsilon-$ neighboorhod K'_ε of $K' = f(K)$ is contained in U. Therefore $f_i \in A$ for $i \geq p$. This contradiction shows that $f \in CA$. Therefore the set A is a open set in $F-$ compact open topology. □

Let us consider a submersion

$$\pi : M \to B, \tag{2.1}$$

where M and B are manifolds of dimensions n, m respectively and $n > m$. Connected components of the inverse images of the points of $p \in B$ define a $k = n - m-$ dimensional foliation F on M. To the study of the geometry of the submersions is devoted many investigations [3], [9],[10], in particularly, in [11] obtained fundamental equations of Riemannian submersion.

In this case it is not difficult to show that all leaves of F are closed subsets of M. From the theorem-2 we have following result.

Theorem 2.3. *Let (M, F) be a smooth foliated manifold, where the foliation F generated by submersion.Then $F-$ compact open topology coincides with compact open topology on $Diff_F(M)$.*

Definition 2.1. An isometry $\varphi : M \to M$ is called an isometry of foliated manifold (M, F) if it is diffeomorphism of foliated manifold (M, F).

We will denote by $Iso_F(M)$ the set of all isometries of foliated manifold (M, F). We have that $Iso_F(M) = Diff_F(M) \bigcap Iso(M)$.

(M, g). This set is a subgroup of the group $Diff_F(M)$. In [7] it was proved that the group $Iso_F(M)$ is a Lie group with a compact open topology.

Theorem 2.4. *Let (M, F) be a smooth foliated manifold. Then $F-$ compact open topology on $Iso_F(M)$ coincides with compact open topology on $Iso_F(M)$.*

Proof. As $F-$ compact open topology on $Iso_F(M)$ is weaker than induced from $Diff(M)$ usual compact open topology it is sufficient to prove that every open set in compact open topology is a open set in $F-$ compact open topology on $Iso_F(M)$.

Let $A \subset Iso_F(M)$ be a open set in compact open topology. Without loss of generality, we will assume that the set A has the form

$$A = [K, U] = \{f \in Iso_F(M) : f(K) \subset U\},$$

where K is a compact subset of M, U is a open subset of M.

We will show that the set A is a open set in $F-$ compact open topology. Let $f \in A$, $K' = f(K)$ and $\delta = d(K', M \setminus U.)$

As K is a compact set there exist finite number points $x_1, x_2, ..., x_m$ of the set K such that for every point $y \in K$ and some i it holds $d(x_i, y) < \varepsilon$, where $\varepsilon < \delta$, where $d(x, y)$ is the distance between points x and y defined by some riemannian metric g on M.

The sets $G_i = [x_i, K'_{\frac{\varepsilon}{2}}] = \{h \in Iso_F(M) | h(x_i) \subset K'_{\frac{\varepsilon}{2}}\}$ are open in $F-$ compact open topology, where $i = 1, 2, \cdots, m$, $K'_\varepsilon = \{x : d(x, K') = inf\{d(x, y) : y \in K'\}\} < \varepsilon$. If $h \in \bigcap_i G_i$ it holds $h \in A$. It follows that $\bigcap_i G_i \subset A$ and A is a open set in $F-$ compact open topology. □

3. Dimension of the Group of Isometries of Foliated Manifolds

Let us recall some notions which we need. The set $V(M)$ of all smooth vector fields on a manifold M is a linear space over the field of real numbers and a Lie algebra with respect to the Lie brackets. A vector field X on M is called a Killing vector field if if its flow consists of isometries of Riemannian manifold (M, g), that is $L_X g = 0$, where g is riemannian metric, $L_X g$ denotes Lie derivative of the metric g with respect to X. Note that the Lie bracket of two Killing fields and a linear combination of Killing fields over the field of real numbers are Killing fields as well. Therefore, the set $K(M)$ of all Killing vector fields on the manifold M is a Lie algebra over the field of real numbers. In addition, it is well known that the dimension of the Lie algebra $K(M)$ of Killing vector fields on a connected Riemannian manifold M does not exceed $\frac{n(n+1)}{2}$, where $n = dimM$. If $dimK(M) = \frac{n(n+1)}{2}$, then M is a manifold of constant curvature [4].

From the theorem-4 we have that the group $Iso_F(M)$ is a Lie group with foliated compact open topology.

Let us consider a foliated manifold (M, F), where the foliation F generated by submersion (1).

Suppose that L is a leaf of the foliation F, $x \in L$, $T_x L$ is the tangent space of L at the point x, and $H(x)$ is the orthogonal complement of $T_x L$. There arise two subbundles $TF : x \to T_x L$ and $H : x \to H(x)$ of the tangent bundle TM of the manifold M. Each vector field $X \in V(M)$ can be represented in the form $X = X_v + X_h$, where X_v and X_h are the orthogonal projections of X onto TF and H, respectively. If $X_h = 0$, then X is called a vertical field (tangent to F), if $X_v = 0$, then X is called a horizontal field..

Definition 3.1. A vector field X is said to be foliate if for every vertical vector field Y the Lie bracket $[X, Y]$ also is a vertical vector field.

It is known that a vector field X is a foliated field if and only if flow of vector field X translates leaves the foliation F to leaves of this foliation [5]. Every vertical vector field is s foliated field.

Definition 3.2. A submersion of $\pi : M \to B$ is called Riemannian if its differential $d\pi$ preserves the length of horizontal vectors.

To the study of the geometry of Riemannian submersions is devoted many investigations [3], [10],[11], in particularly in [11] obtained fundamental equations of Riemannian submersion. It is known that Riemannian submersion generates riemannian foliation. Foliation F is called a riemannian foliation if each geodesic is orthogonal at some point to the leaf of the foliation F, remains orthogonal to all leaves F in all their points [5, p. 189]. Riemannian foliation without singularities were first introduced and studied by Reinhart in [10].

Theorem 3.1. *Let (M, g) and (B, g^B) are complete riemannian manifolds dimensions n, m respectively and $n > m$, (M, F) be a foliated manifold, where foliation F is generated by Riemannian submersion*

$$\pi : M \to B. \tag{3.1}$$

Then for the the group $Iso_F(M)$ it holds

$$dimIso_F(M) \le \frac{(n-m)^2 + m^2 + n}{2}.$$

Proof. Let us denote by $V_F(M)$ the set of foliated vector fields for the foliation F generated by submersion (3.1). If X_1, X_2 foliated vector fields then Lie bracket $[X_1, X_2]$ also a foliated vector field which follows from Jacobi identity:

$$[[X_1, X_2], Y] + [[X_2, Y], X_1] + [[Y, X_1], X_2] = 0.$$

Thus the set $V_F(M)$ is Lie algebra.

If $X_1, X_2, ..., X_s$ are linear independent infinitesimal generators of the Lie group $Iso_F(M)$, then vector fields $X_1, X_2, ..., X_s$ are foliated Killing vector fields for the foliation F generated by submersion (1) and generate $s-$ dimensional Lie subalgebra of the $V_F(M)$.

We assume that $X_1, X_2, ..., X_{s_1}$ are vertical vector fields. If L is a leaf of the foliation F generated by submersion (3.1), the restriction of every vertical vector field X_i to the leaf L is a Killing vector field on L, $i = 1, 2, ..., s_1$.

It is known that each leaf of the foliation F with the induced Riemannian metric is a complete Riemannian manifold of dimension $n - m$ [6], [8]. It follows

$$s_1 \leq \frac{(n-m)(n-m+1)}{2}.$$

For every foliated vector field X from $X_{s_1+1}, X_{s_1+2}, ..., X_s$ consider the vector field $\pi_* X$ on B. where π_* is the differential of the submerion (3.1). It holds $\pi_* X = 0$ if and only if when X is a vertical vector field. As the vector fields $X_{s_1+1}, X_{s_1+2}, ..., X_s$ are linear independent non vertical vector fields and the submersion has maximal rank it follows that

$$\pi_* X_{s_1+1}, \pi_* X_{s_1+2}, ..., \pi_* X_s$$

linear independent vector fields.

Now we show that the vector fields $\pi_* X_{s_1+1}, \pi_* X_{s_1+2}, ..., \pi_* X_s$ are Killing vector fields.

Let Y_1, Y_2 be a vector fields on B. We have to show that

$$\pi_* X g^B(Y_1, Y_2) = g^B([\pi_* X, Y_1], Y_2) + g^B(Y_1, [\pi_* X, Y_2]),$$

where $X \in \{X_{s_1+1}, X_{s_1+2}, ..., X_s\}$.

As π is a riemannian and M is complete riemannian manifold there exist horizontal vector field Z_1, Z_2 (lifts) on M, such that $\pi_* Z_i = Y_i$[3].

As the vector field X from $X_{s_1+1}, X_{s_1+2}, ..., X_s$ is Killing field we have

$$Xg(Z_1, Z_2) = g([X, Z_1], Z_2) + g([X, Z_2], Z_1),$$

where $[X, Y]$ is Lie bracket of vector fields.

By the property of Lie bracket of vector fields we have

$$[\pi_* X, Y_i]) = \pi_*[X, Z_i]$$

Since the mapping π_* is an isometry on the space of horizontal vectors, we have

$$g^B(Y_1, Y_2) = g(Z_1, Z_2),$$

Considering the equality

$$g([X, Z_i], Z) = g([X, Z_i]_h, Z)$$

for every horizontal vector field Z we have

$$g^B([\pi_* X, Y_1], Y_2) = g([X, Z_1], Z_2), g^B(Y_1, [\pi_* X, Y_2]) = g(Z_1, [X, Z_2].$$

Let ∇, ∇^* are Levi-Chivita connections on Riemannian manifolds M, B correspondingly. As ∇^* is Riemannian connection it holds

$$\pi_\star X g^B(Y_1, Y_2) = g^B(\nabla^*_{\pi_* X} Y_1, Y_2) + g^B(Y_1, \nabla^*_{\pi_* X}, Y_2).$$

It was proven in [11] that

$$\pi_*[\nabla_X Z_i]_h = \nabla^*_{\pi_* X} Y_i, i = 1, 2. \tag{3.2}$$

It follows following equality

$$\pi_\star X g^B(Y_1, Y_2) - g^B([\pi_* X, Y_1], Y_2) - g^B(Y_1, [\pi_* X, Y_2]) = 0,$$

i.e. $L_{\pi_* X} g^B(Y_1, Y_2) = 0$.

Thus, vector fields $\pi_* X_{s_1+1}, \pi_* X_{s_1+2}, ..., \pi_* X_s$ are killing fields and their number does not exceed

$$\frac{m(m+1)}{2}.$$

Therefore we have got

$$s \leq \frac{(n-m)^2 + m^2 + n}{2}.$$

□

Example 3.1. Orthogonal projection

$$\pi : R^{n+1} \to \mathbb{R}^1 \tag{3.3}$$

is a Riemannian submersion. In this case leaves of the foliation F, generated by submersion (2) are parallel hyperplanes. We assume that leaves are given by equation $x_{n+1} = const$. Linear independent infinitesimal generators of the Lie group $Iso_F(M)$ are vertical Killing vector fields, which tangent to parallel hyperplanes and horizontal foliated Killing field $X = \frac{\partial}{\partial x_{n+1}}$. It follows that for dimension of the group $Iso_F(\mathbb{R}^{n+1})$ of isometries of foliated manifold $(\mathbb{R}^{n+1}, F)$ we have

$$dim Iso_F(\mathbb{R}^{n+1}) = \frac{n(n+1)}{2} + 1.$$

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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