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Research Article

Hausdorff objects

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Abstract

In previous papers, several extensions of the T_2 separation property in topology to a topological category were compared. The aim of this paper is to develop further results relating to these extensions as well as to solve several open problems. Moreover, we show one of these T_2, namely KT_2 limit spaces and reciprocal limit spaces are equivalent and every KT_2 limit space induces the associated complete uniform limit space. Finally, we compare our results and give some applications.

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1. Introduction

Several authors studied Hausdorffness in categorical setting. For example, the notions of Hausdorffness with respect to closure operators was done in [15] for abstract categories, with respect to a factorization structure were defined in [16, 22] for a general category, and with respect to initial lifts, final lifts, and discreteness was defined in [2, 25] for a topological category.
The extension of T_2- axiom has several equivalent descriptions for topological spaces and when these notions are extended to in other topological categories it may rise to distinct concepts. One form of these extensions may be more useful than another in certain applications but looking for the right extension may be meaningless. The notion of T_2 separation to the arbitrary topological categories was formulated in terms of final lifts, initial lifts, and (in)discreteness in [2,10,25] with no reference to points and neighbourhoods since the point (resp. neighbourhood) notion may not be available for non set-based topological categories (resp. for topological categories not related to topological spaces). The relationships among these extensions of T_2, denoted by LT_2, $\overline{T}_2$, T_2', and KT_2 (see Definition 2.1, below), were investigated and it was shown that $LT_2 \Rightarrow \overline{T}_2 \Rightarrow KT_2$ [10] and $T_2' \Rightarrow \overline{T}_2$ [25].
Also, T_2-axiom can be generalized to a topological category by using that the diagonal map embeds as a closed subspace of its product with itself [2, 15].
The main object of this paper consists of three parts:
(1) To find conditions on an object in arbitrary topological category such that when each

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of the reverse implications above holds and when one has $LT_2 = T_2' = \overline{T}_2 = KT_2$ as well as to solve several open problems.
(2) To show that a limit space is KT_2 iff it is reciprocal [21] and as an consequence, the categories **CULim** (completely uniform limit spaces and uniformly continuous functions) and **$\mathbf{KT_2}$Lim** (KT_2 limit spaces and continuous functions) are isomorphic and deduce that every KT_2 limit space induces the associated complete uniform limit space.
(3) To examine the results relating to these extensions in several topological categories and give some applications.

2. Premilinaries

Let $\mathcal{B}$ be a category with finite products and cokernel pairs (i.e., given any morphism $f : A \to B$ in $\mathcal{B}$, the pushout of f with itself) and $B \in Ob(\mathcal{B})$. We denote by B^n, the product of B with itself n times and by $\pi_j : B^n \to B$ the j th projection morphism, $j = 1, 2, ..., n$. If $A \in Ob(\mathcal{B})$ and $f_i : A \to B$ are morphisms in $\mathcal{B}$, then there exists a unique morphism $f = (f_1, f_2, ..., f_n) : A \to B^n$ such that $\pi_i f = f_i$ for $i = 1, 2, ..., n$.

The diagonal $\Delta : B \to B^2$ is given by $\Delta = (1_B, 1_B)$, where $1_B : B \to B$ is the identity morphism. Let $\pi_j, \pi_k : B^2 \to B$ be projections. Define $\pi_{jk} : B^2 \vee_\triangle B^2 \to B$ to be $\pi_j + \pi_k$ for $j, k = 1, 2$, where $B^2 \vee_\triangle B^2$ denotes the cokernel of Δ along itself. More precisely, if i_1 and $i_2 : B^2 \to B^2 \vee_\triangle B^2$ denote the inclusions of B^2 as the first and second factor, respectively, then $i_1\Delta = i_2\Delta$ is a pushout diagram. Note that for morphisms $(\pi_1, \pi_1, \pi_2) : B^2 \to B^3$ and $(\pi_1, \pi_2, \pi_1) : B^2 \to B^3$, $(\pi_1, \pi_1, \pi_2)\Delta = (1_B, 1_B, 1_B) = (\pi_1, \pi_2, \pi_1)\Delta$, and thus, $A = (\pi_{11}, \pi_{12}, \pi_{21}) : B^2 \vee_\triangle B^2 \to B^3$ is the unique morphism called the principal axis morphism in [2] for which $Ai_1 = (\pi_1, \pi_2, \pi_1)$ and $Ai_2 = (\pi_1, \pi_1, \pi_2)$.
Similarly, $(\pi_1, \pi_1, \pi_2)\Delta = (1_B, 1_B, 1_B) = (\pi_1, \pi_2, \pi_2)\Delta$ (resp. $1_{B^2}\Delta = \Delta$) and so, there exists a unique morphism $S = (\pi_{11}, \pi_{12}, \pi_{22}) : B^2 \vee_\triangle B^2 \to B^3$ (resp. $\nabla = (\pi_{11}, \pi_{22}) : B^2 \vee_\triangle B^2 \to B^2$), called the skewed axis (resp. the fold) morphism in [2].
Note that $Si_1 = (\pi_1, \pi_2, \pi_2)$, $Si_2 = (\pi_1, \pi_1, \pi_2) = Ai_2$, $Ai_1 = (\pi_1, \pi_2, \pi_1)$ and $\nabla i_k = 1_{B^2}$ for $k = 1, 2$.

Let $\mathcal{B}$ be a category with finite products and cokernel pairs, $U : \mathcal{E} \to \mathcal{B}$ be a topological functor, and $X \in Ob(\mathcal{E})$ with $U(X) = B$.
Let S_B (resp. A_B) be the initial lift of the U-source S (resp. A): $B^2 \bigvee_\Delta B^2 \to U(X^3)$ and $W_{(B^2 \bigvee_\Delta B^2)}$ be the final lift of the U-sink $\{q \circ i_1, q \circ i_2 : U(X^2) \to B^2 \vee_\Delta B^2\}$, where $i_k : B^2 \to B^2 \coprod B^2, k = 1, 2$ are the canonical injection maps and $q : B^2 \coprod B^2 \to B^2 \bigvee_\Delta B^2$ is the quotient map [2].

Definition 2.1. **(1)** If $S_B = A_B$ (resp. $S_B = W_{(B^2 \bigvee_\Delta B^2)}$), then X is called a $Pre\overline{T}_2$ (resp. $PreT_2'$) object.
(2) If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(X^3)$ (resp. $S : B^2 \vee_\Delta B^2 \to U(X^3)$) is discrete, then X is said to be a $\overline{T_0}$ (resp. T_1) object, where D is the discrete functor.
(3) If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(W_{(B^2 \bigvee_\Delta B^2)})$ is discrete, then X is called a T_0' object.
(4) If X is T_0' and $Pre\overline{T}_2$ (resp. $PreT_2'$), then X is called a KT_2 (resp. T_2') object.
(5) If X is $Pre\overline{T}_2$ (resp. $PreT_2'$) and $\overline{T}_0$, then X is called a $\overline{T}_2$ (resp. LT_2) object.

If $\mathcal{B}$ is a category with finite colimits and limits, then (3) of Definition 2.1 is given in [27].

If $\mathcal{B} = \mathbf{Set}$, then Definition 2.1 is given in [2, 10]. A point (x, y) in $B^2 \vee_\triangle B^2$ will be denoted by $(x, y)_1$ (resp. $(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \vee_\triangle B^2$ [2].

Note that $S(x, y)_1 = (x, y, y)$, $S(x, y)_2 = (x, x, y) = A(x, y)_2$, $A(x, y)_1 = (x, y, x)$, and $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [2].

For the category of $\mathcal{E} = \mathbf{Top}$ and $\mathcal{B} = \mathbf{Set}$, T_0' and $\overline{T}_0$ (resp. LT_2, $\overline{T}_2$, KT_2, and T_2') reduce to the usual T_0 (resp. T_2) axiom [2, 10]. A topological space is pre-Hausdorff i.e., $PreT_2' = Pre\overline{T}_2$ iff for any two distinct points, if there is a neighborhood of one missing the other, then the points have disjoint neighborhoods [2]. There is no implication between $Pre\overline{T}_2$ and each of T_0 and T_1. Take the integers set Z with indiscrete and cofinite topologies. In the realm of $PreT_2$ topological spaces, all T_2, T_1, and T_0 spaces are equivalent.

Let $B \neq \emptyset$ and α be a filter on B. The filter α is improper (proper) iff α contains (resp., α does not contain) $\emptyset$, the empty set. We denote by $\alpha \cup \beta$ the smallest filter containing both α and β.
Let K be a function on B that assigns each $x \in B$ a set of filters (the "filters converging to x"). K is said to be a limit structure on B ((B, K) a limit space [26]) iff K satisfies the following three conditions: for each $x \in B$,
(i) $[x] \in K(x)$, where $[U] = \{V \subset B : U \subset V\}$.
(ii) if $\alpha \in K(x)$ and $\beta \supset \alpha$, then $\beta \in K(x)$ for any filter β on B.
(iii) if $\alpha, \beta \in K(x)$, then $\alpha \cap \beta \in K(x)$.
Limit spaces are referred to as convergence spaces in [14, 20]. A limit space (B, K) is called reciprocal if $K(x) \cap K(y) \neq \{[\emptyset]\}$, then $K(x) = K(y)$ for each $x, y \in B$ [21].
A map $f : (B, K) \to (C, L)$ between limit spaces is said to be continuous iff $\alpha \in K(x)$ implies $f(\alpha) \in L(f(x))$ for each $x \in B$, where $f(\alpha) = [\{f(V) : V \in \alpha\}]$.

The category of limit spaces and continuous maps is denoted by **Lim** [26].

Note that every topological space (A, τ) induces a limit space. Indeed, for $a \in A$, let $\eta_a = \{U \subset A : \exists V \in \tau$ such that $a \in V \subset U\}$ be a neighborhood filter at the point a and define $K_\tau(a) = \{\alpha : \alpha \supset \eta_a\}$. Then (A, K_τ) is a limit space. If $f : (A, \tau) \to (B, \sigma)$ is continuous, then a map $f : (A, K_\tau) \to (B, L_\sigma)$ between limit spaces is continuous and **Top** is a bireflective subcategory of **Lim** (see [26], p.145).

Proposition 2.2. *Let $B \neq \emptyset$, $\{(B_i, K_i), i \in I\}$ be a class of limit spaces, and $\{f_i : B \to B_i, i \in I\}$ be a source in the category* **Set**. *A source $\{f_i : (B, K) \to (B_i, K_i), i \in I\}$ in* **Lim** *is an initial lift iff $\alpha \in K(x)$ precisely when $f_i(\alpha) \in K_i(f_i(x))$ for all $i \in I$ [26].*
An epi sink $\{q \circ i_1, q \circ i_2 : (B^2, K) \to (B^2 \bigvee_\Delta B^2, L)\}$ in **Lim** *is a final lift iff for any point $s \in B^2 \bigvee_\Delta B^2$ and any filter α on $B^2 \bigvee_\Delta B^2$, $\alpha \in L(s)$ iff either $\exists k \in \{1, 2\}$, $\exists u \notin \Delta$, $\exists \beta \in K(u)$ such that $\alpha \supset (q \circ i_k)(\beta)$, $(q \circ i_k)(u) = s$ or $\exists u \in \Delta$, $\exists \beta_1, \beta_2 \in K(u)$ such that $\alpha \supset (q \circ i_1)(\beta_1) \cap (q \circ i_2)(\beta_2)$, $(q \circ i_1)(u) = s = (q \circ i_2)(u)$. This is a special case of [26].*

Theorem 2.3 ([3])**.** ***(i)*** *Let σ be a filter on $B^2 \vee_\triangle B^2$. If $\sigma_0 = \bigcup \pi_{ij}^{-1} \pi_{ij} \sigma$, $j, i = 1, 2$, then $\sigma_0 \subset \sigma$ and $\pi_{ij}\sigma = \pi_{ij}\sigma_0$ for all $j, i = 1, 2$.*
Let α_{ij}, $i, j = 1, 2$ be proper filters on B.
(ii) *$\sigma = \bigcup_{j,i=1}^{2} \pi_{ij}^{-1} \alpha_{ij}$ is proper if and only if either (a) $(\alpha_{11} \cup \alpha_{12})$ and $(\alpha_{21} \cup \alpha_{22})$ are proper or (b) $(\alpha_{21} \cup \alpha_{11})$ and $(\alpha_{22} \cup \alpha_{12})$ are proper.*
(iii) *There exists a proper filter σ on $B^2 \vee_\triangle B^2$ with $\pi_{ij}\sigma = \alpha_{ij}$ for all $j, i = 1, 2$ if and only if*
(1) *If (a) does not hold, then $\alpha_{22} = \alpha_{12}$ and $\alpha_{21} = \alpha_{11}$.*

(2) *If (b) does not hold, then* $\alpha_{11} = \alpha_{12}$ *and* $\alpha_{21} = \alpha_{22}$.
(3) *If both (a) and (b) hold, then* $\alpha_{22} \cap \alpha_{11} = \alpha_{21} \cap \alpha_{12}$.

The category **RRel** of reflexive relation spaces where objects are sets with a reflexive relation and where morphisms $f : (A_1, R) \to (B_1, S)$ are functions with $f(a)Sf(b)$ if aRb for all $a, b \in A_1$. The category **Prord** of reflexive and transitive relation spaces is the full subcategory of **RRel** [1, 26].

3. Applications

The full subcategories of **Lim** (resp. **ULim**) whose objects are KT_2, $Pre\overline{T}_2$, and reciprocal limit spaces (resp. completely uniform limit spaces) are denoted by $\mathbf{KT_2Lim}$, $\mathbf{Pre\overline{T}_2Lim}$, and **RLim** (resp. **CULim**). We show that the full subcategories $\mathbf{KT_2Lim}$ and **RLim** of **Lim** are isomorphic categories and deduce that every KT_2 limit space induces the associated complete uniform limit space.

Theorem 3.1. *The following are equivalent for a limit space* (B, K).
(a) (B, K) *is pre*$\overline{T}_2$.
(b) (B, K) *is reciprocal.*
(c) (B, K) *is* KT_2.

Proof. $(a) \Rightarrow (b)$. Suppose (B, K) is *pre*$\overline{T}_2$ and $\alpha \in K(x) \cap K(y)$ for $x, y \in B$. Let $\delta \in K(x)$. If δ is improper, then $\delta \in K(y)$. Assume δ is proper.
If $\alpha \cup \delta$ is improper, then let $\alpha_{11} = \delta = \alpha_{12}$ and $\alpha_{21} = \alpha = \alpha_{22}$ in the Theorem 2.3 (iii). Since $\alpha_{22} \cup \alpha_{21} = \alpha$ and $\alpha_{11} \cup \alpha_{12} = \delta$ are proper, $\alpha_{11} \cup \alpha_{21} = \alpha \cup \delta = \alpha_{12} \cup \alpha_{22}$ is improper, $\alpha_{11} = \alpha_{12}$, and $\alpha_{21} = \alpha_{22}$, by Theorem 2.3 (iii)(1), there exists a proper filter β on $B^2 \bigvee_\Delta B^2$ with

$$\pi_1 A\beta = \alpha_{11} = \pi_1 S\beta, \pi_2 A\beta = \alpha_{21} = \pi_2 S\beta,$$

$\pi_3 A\beta = \alpha_{12}$, and $\pi_3 S\beta = \alpha_{22}$. Since

$$\pi_1 A\beta = \pi_1 S\beta = \delta \in K(x), \pi_2 A\beta = \pi_2 S\beta = \alpha \in K(x), \pi_3 S\beta = \alpha \in K(y)$$

and (B, K) is *pre*$\overline{T}_2$, Proposition 2.2 and Definition 2.1, $\pi_3 A\beta = \delta \in K(y)$.
If $\alpha \cup \delta$ is proper, then let $\alpha_{21} = \alpha = \alpha_{22}$ and $\alpha_{11} = \delta = \alpha_{12}$ in the Theorem 2.3 (iii). Note that

$$\alpha_{11} \cup \alpha_{12} = \delta, \alpha_{22} \cup \alpha_{21} = \alpha, \alpha_{22} \cup \alpha_{12} = \alpha \cup \delta = \alpha_{11} \cup \alpha_{21}$$

are proper and $\alpha_{12} \cap \alpha_{21} = \alpha \cap \delta = \alpha_{11} \cap \alpha_{22}$. Hence, by Theorem 2.3 (iii) (3), there exists a proper filter β on $B^2 \bigvee_\Delta B^2$ with

$$\pi_1 A\beta = \alpha_{11} = \pi_1 S\beta, \pi_2 A\beta = \alpha_{21} = \pi_2 S\beta, \pi_3 A\beta = \alpha_{12},$$

and $\pi_3 S\beta = \alpha_{22}$. Since (B, K) is *pre*$\overline{T}_2$, by Proposition 2.2,

$$\pi_1 A\beta = \pi_1 S\beta = \delta \in K(x), \pi_2 A\beta = \pi_2 S\beta = \alpha \in K(x), \pi_3 S\beta = \alpha \in K(y),$$

we get $\pi_3 A\beta = \delta \in K(y)$.
Hence, $K(x) \subset K(y)$ and changing the role of x and y, we get $K(y) \subset K(x)$. Thus, $K(x) = K(y)$ and (B, K) is reciprocal.

$(b) \Rightarrow (c)$. Suppose (B, K) is a reciprocal limit space. First, we show (B, K) is T_0'. Let W_K be the limit structure on $B^2 \bigvee_\Delta B^2$ induced by the maps $\bigtriangledown : B^2 \vee_\Delta B^2 \to (B^2, K_d)$ and $id : B^2 \vee_\Delta B^2 \to (B^2 \vee_\Delta B^2, K_q)$, where K_d is the discrete limit structure on B^2 and K_q is the final limit structure on $B^2 \bigvee_\Delta B^2$ induced by the maps $q \circ i_k$, $k = 1, 2$ defined in Section 2.
Suppose $\alpha \in W_K(s)$ for $s \in B^2 \bigvee_\Delta B^2$. By Proposition 2.2 and Definition 2.1, either

$\exists k \in \{1,2\}$, $\exists u \notin \Delta$, $\exists \beta \in K(u)$ such that $(q \circ i_k)(u) = s, \alpha \supset (q \circ i_k)(\beta)$ or $\exists u \in \Delta$, $\exists \beta_1, \beta_2 \in K(u)$ such that

$$\alpha \supset (q \circ i_1)(\beta_1) \cap (q \circ i_2)(\beta_2), (q \circ i_1)(u) = s = (q \circ i_2)(u)$$

and $\nabla(\alpha) = [\emptyset]$ or $[\nabla(\alpha) = \nabla(s)]$.
If $\nabla(\alpha) = [\emptyset]$, then $\alpha = [\emptyset]$.
Assume $\nabla(\alpha) = [\nabla(s)] = [(x,y)]$ for some $(x,y) \in B^2$.
If $x = y$, then $s = (x,x)_k$ and $\alpha = [(x,x)_k]$ for $k = 1,2$.
Suppose $x \neq y$. Then $s = (x,y)_k$, $k = 1,2$ and since α is a filter, $\alpha = [(x,y)_k]$ or $\alpha = [(x,y)_1] \cap [(x,y)_2]$.
Let $s = (x,y)_1$. If $\alpha = [(x,y)_2]$, then $\alpha \supset (q \circ i_1)(\beta)$ for some $\beta \in K^2(x,y)$ with $i_1((x,y)) = s = (x,y)_1$, a contradiction since $x \neq y$ and $\alpha = [(x,y)_2]$.
If $\alpha = [(x,y)_1] \cap [(x,y)_2]$, then $\alpha \supset (q \circ i_1)(\beta)$ for some $\beta \in K^2(x,y)$ with $i_1((x,y)) = s = (x,y)_1$, a contradiction since $x \neq y$.
Hence, $\alpha = [(x,y)_1]$.
Similarly, if $s = (x,y)_2$ with $x \neq y$, then $\alpha = [(x,y)_2]$.
Hence, $\alpha = [s]$ for all $s \in B^2 \bigvee_\Delta B^2$ or $\alpha = [\emptyset]$, i.e., $W_K(s) = \{[s], [\emptyset]\}$ is discrete and by Definition 2.1, (B,K) is T_0'.
Let K_A (resp. K_S) be the initial lift of $\pi_k A$ (resp. $\pi_k S$), $k = 1,2,3$, where $\pi_k : B^3 \longrightarrow B$ are the projections maps. We show (B,K) is $pre\overline{T}_2$, i.e., by Definition 2.1, $K_A = K_S$ and by Proposition 2.2, for any point $s \in B^2 \bigvee_\Delta B^2$ and any filter β on the wedge,

$$\pi_1 A\beta \in K(\pi_1 A(s)), \pi_2 A\beta \in K(\pi_2 A(s)), \pi_3 A\beta \in K(\pi_3 A(s))$$

if and only if

$$\pi_1 S\beta \in K(\pi_1 S(s)), \pi_2 S\beta \in K(\pi_2 S(s)), \pi_3 S\beta \in K(\pi_3 S(s)).$$

Since $\pi_1 A\beta = \pi_1 S\beta$ and $\pi_2 A\beta = \pi_2 S\beta$, we need to show that $\pi_3 A\beta \in K(\pi_3 A(s))$ iff $\pi_3 S\beta \in K(\pi_3 S(s))$.
If $\beta = [\emptyset]$, then nothing to show. If $\beta \neq [\emptyset]$ and $s = (x,y)_1$ for $x, y \in B$ with $x \neq y$, then let

$$\beta_0 = \pi_1^{-1}(\pi_1 A\beta) \cup \pi_1^{-1}(\pi_2 A\beta) \cup \pi_3^{-1}(\pi_3 A\beta) \cup \pi_3^{-1}(\pi_3 S\beta).$$

By Theorem 2.3 (i), $\beta_0 \subset \beta$, $\pi_i A\beta_0 = \pi_i A\beta$, and $\pi_i S\beta_0 = \pi_i S\beta$ for each $i = 1,2,3$.
We apply Theorem 2.3 (iii) with $\alpha_{11} = \pi_1 A\beta = \pi_1 S\beta$, $\alpha_{21} = \pi_2 A\beta = \pi_2 S\beta$, $\alpha_{12} = \pi_3 A\beta$, and $\alpha_{22} = \pi_3 S\beta$. If (1) of Theorem 2.3 (iii) holds, then $\alpha_{22} = \alpha_{12}$, $\alpha_{11} = \alpha_{21}$. Since $\pi_1 A\beta \in K(\pi_1 A(s) = x)$, $\pi_2 A\beta \in K(\pi_2 A(s) = y)$ and $\alpha_{11} = \pi_1 A\beta = \pi_2 A\beta = \alpha_{21}$, we have $\pi_1 A\beta \in K(y) \cap K(x)$. Since (B,K) is a reciprocal limit space, $K(y) = K(x)$. Thus, $\pi_3 S\beta \in K(\pi_3 S(s) = y)$ and $\pi_3 A\beta \in K(\pi_3 A(s) = x)$.
If (2) of Theorem 2.3 (iii) holds, then $\alpha_{22} = \alpha_{21}$, $\alpha_{11} = \alpha_{12}$. Since

$$\pi_1 A\beta \in K(\pi_1 A(s) = x), \pi_2 A\beta \in K(\pi_2 A(s) = y), \alpha_{22} = \pi_3 S\beta = \pi_2 A\beta = \alpha_{21},$$

and $\alpha_{11} = \pi_1 A\beta = \pi_3 A\beta = \alpha_{12}$, we have $\pi_3 A\beta \in K(\pi_3 A(s) = x)$ and $\pi_3 S\beta \in K(\pi_3 S(s) = y)$.
If (3) of Theorem 2.3 (iii) holds, then $\alpha_{11} \cap \alpha_{22} = \alpha_{12} \cap \alpha_{21}$ and $\alpha_{11} \cup \alpha_{21}$ is proper. Since $\alpha_{11} \cup \alpha_{21} \in K(x) \cap K(y)$ and (B,K) is reciprocal, $K(y) = K(x)$. (B,K) is a limit space implies $\alpha_{11} \cap \alpha_{22} = \alpha_{12} \cap \alpha_{21} \in K(y) = K(x)$ and hence, $\pi_3 A\beta \in K(\pi_3 A(s) = x)$ and $\pi_3 S\beta \in K(\pi_3 S(s) = y)$.
Suppose β is proper and $s = (x,y)_2$ for $x, y \in B$, $x \neq y$.
If (1) of Theorem 2.3 (iii) holds, then $\alpha_{22} = \alpha_{12}$, $\alpha_{11} = \alpha_{21}$ and hence,

$$\pi_k A\beta = \pi_k S\beta \in K(\pi_k A(s) = \pi_k S(s) = x), k = 1,2,$$

$$\pi_3 A\beta \in K(\pi_3 A(s) = y), \pi_3 S\beta \in K(\pi_3 S(s) = y).$$

If (2) of Theorem 2.3 (iii) holds, then $\alpha_{22} = \alpha_{21}$, $\alpha_{11} = \alpha_{12}$. Since $\pi_1 A\beta \in K(\pi_1 A(s) = x)$ and $\pi_3 A\beta \in K(\pi_3 A(s) = y)$, we have $\pi_1 A\beta \in K(y) \cap K(x)$ and by the assumption,

$K(y) = K(x)$. Thus, $\pi_3 S\beta \in K(\pi_3 S(s) = y)$.
If (3) of Theorem 2.3 (iii) holds, then $\alpha_{21} \cap \alpha_{12} = \alpha_{22} \cap \alpha_{11}$ and $\alpha_{11} \cup \alpha_{12}$ is proper. Since $\alpha_{11} \cup \alpha_{12} \in K(y) \cap K(x)$ and (B, K) is a reciprocal limit space, $K(y) = K(x)$ and $\alpha_{11} \cap \alpha_{22} = \alpha_{12} \cap \alpha_{21} \in K(y) = K(x)$. Consequently, $\pi_3 A\beta \in K(\pi_3 A(s) = x)$ and $\pi_3 S\beta \in K(\pi_3 S(s) = y)$.
Suppose β is proper and $s = (x, x)_2 = (x, x)_1$ for some $x \in B$. If (1) and (2) of Theorem 2.3 (iii) hold, then $\pi_3 A\beta \in K(\pi_3 A(s) = x)$ and $\pi_3 S\beta \in K(\pi_3 S(s) = x)$. If (3) of Theorem 2.3 (iii) holds, then $\alpha_{11} \cap \alpha_{22} = \alpha_{12} \cap \alpha_{21} \in K(x)$ since (B, K) is a limit space. Hence, $\alpha_{12} = \pi_3 A\beta \in K(\pi_3 A(s) = x)$ and $\alpha_{22} = \pi_3 S\beta \in K(\pi_3 S(s) = x)$.
(B, K) is $pre\overline{T}_2$ and thus, (B, K) is KT_2.
By Definition 2.1, KT_2 implies $pre\overline{T}_2$ which shows that $(c) \Rightarrow (a)$. □

Theorem 3.2. *The categories* $\mathbf{KT_2Lim}$, $\mathbf{RLim}$, $\mathbf{Pre\overline{T}_2Lim}$, *and* $\mathbf{CULim}$ *are pairwise isomorphic. Moreover,* $\mathbf{KT_2Lim}$ *is a topological category.*

Proof. By Proposition 2.1.8, Corollary 2.1.9, and Example 2.3.4 of [14] or Propositions 1.3.5 and 1.3.7 of [20], the categories **RLim** and **CULim** and by Theorem 3.1, the categories $\mathbf{KT_2Lim}$, **RLim**, and $\mathbf{Pre\overline{T}_2Lim}$ are pairwise isomorphic. The second Part follows from the first Part and Theorem 3.4 of [5]. □

Theorem 3.3. *A reflexive space* (A, R) *is* $PreT_2'$ *iff* (A, R) *is discrete.*

Proof. Suppose xRy for $x, y \in A$ and (A, R) is $PreT_2'$. Let $s = (y, x)_1$ and $t = (y, x)_2$. Since $\pi_1 S(s) R \pi_1 S(t) = yRy$, $\pi_2 S(s) R \pi_2 S(t) = xRy$, $\pi_3 S(s) R \pi_3 S(t) = xRx$, and (A, R) is $PreT_2'$, it follows $(y, x) R^2 (y, x)$ (R^2 is the product structure on A^2) and $q \circ i_k(y, x) = t, q \circ i_k(y, x) = s$ with $k = 1, 2$. Thus, we must have $x = y$.
Suppose (A, R) is discrete. We show that (I) and (II) are equivalent: for any pair s and t in the wedge,
(I) there exists a pair $(a_1, a_2), (b_1, b_2)$ in A^2 such that $(a_1, a_2) R^2 (b_1, b_2)$ and $q \circ i_k(a_1, a_2) = s, q \circ i_k(b_1, b_2) = t$ with $k = 1, 2$ and
(II) $\pi_1 S(s) R \pi_1 S(t), \pi_2 S(s) R \pi_2 S(t)$, and $\pi_3 S(s) R \pi_3 S(t)$.
If (A, R) is discrete and (I) holds, then $a_1 R b_1$ and $a_2 R b_2$ since $(a_1, a_2) R^2 (b_1, b_2)$. Consequently, $a_1 = b_1$, $a_2 = b_2$, $s = t$ and (II) holds.
Next, we show that (II) implies (I). Suppose $\pi_1 S(s) R \pi_1 S(t), \pi_2 S(s) R \pi_2 S(t)$, and $\pi_3 S(s) R \pi_3 S(t)$ for $s = (x, x)$ or $(x, y)_k$ and $t = (z, z), (z, w)_k$ with $k = 1, 2$ and $x, y, z, w \in A$. If $s = (x, y)_1$ and $t = (z, w)_1$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz$$

and

$$\pi_2 S(s) R \pi_2 S(t) = yRw = \pi_3 S(s) R \pi_3 S(t).$$

Since (A, R) is discrete, $x = z$, $w = y$, and so, $(x, y) R^2 (x, y)$ and $q \circ i_1(x, y) = s = t$.
If $t = (z, w)_2$ and $s = (x, y)_1$, then

$$\pi_1 S(s) R \pi_1 S(t) = xRz, \pi_2 S(s) R \pi_2 S(t) = yRz,$$

$\pi_3 S(s) R \pi_3 S(t) = yRw$. Since (A, R) is discrete, $x = z = y = w$ and consequently, $(x, x) R^2 (x, x)$ and $q \circ i_1(x, x) = s = t$.
Similarly, for the remaining cases of s and t the result holds. Thus, (II) implies (I) and (A, R) is $PreT_2'$.
□

Remark 3.4. **(1)** Theorem 3.3 was proved in different way in Theorem 3.2 of [7].

(2) $PreT_2'$ objects play a role in the theory of geometric realizations, their associated interval and corresponding homotopy structures [25]. It is proved in Corollary 6.3 of [25] that an exact image of a topos in a topological category by a geometric morphism is a

$PreT_2'$ interval. Each interval in a topological category defines a homotopy structure for the category [24] and consequently, any set-based topological category in which $PreT_2'$ objects are discrete admits only trivial interval based homotopy structures [23].

By Theorem 3.3, we obtain:

Theorem 3.5. *The category* **RRel** *is homotopically trivial.*

Remark 3.6. (1) In **Top**, by Proposition 1.5.4 of [27], if B is a finite topological space, then the partitions of B are in one-to-one correspondence with the distinct pre-Hausdorff topologies on B. By Lemma 4.2.3 of [27], every zero-dimensional topological space is pre-Hausdorff.

(2) In **RRel**, Theorem 3.2 of [7],

$$T_2' = LT_2 = \overline{T}_2 \Rightarrow KT_2 = Pre\overline{T}_2 \Rightarrow T_0'$$

$$LT_2 = T_2' = \overline{T}_2 = T_1 \Rightarrow \overline{T}_0 \Rightarrow T_0'.$$

In the presence of $\overline{T}_0$ reflexive spaces, $T_2' = LT_2 = \overline{T}_2 = T_1 = KT_2$ and $\overline{T}_0 = T_0'$. There is no implication between $\overline{T}_0$ and KT_2 reflexive spaces. $(A = \{p, r, s\}, R = \{(p, p), (r, r), (s, s), (p, r)\})$ is $\overline{T}_0$ (i.e., R is anti-symmetric) but it is not KT_2 (i.e., R is an equivalence relation) and the indiscrete reflexive space $(A = \{p, r, s\}, A^2)$ is KT_2 but it is not $\overline{T}_0$.
By Theorem 3.2 of [7], the equivalence (rep. equals) relations can be characterized in terms of KT_2 (resp. $\overline{T}_2$) reflexive spaces.

(3) In **Prord**, by Theorems 4.5 and 6.5 of [11] and Theorem 3.5 of [8],

$$T_2' = LT_2 = \overline{T}_2 = KT_2 = T_1 \Rightarrow PreT_2' = Pre\overline{T}_2$$

$$T_2' = LT_2 = KT_2 = \overline{T}_2 = T_1 \Rightarrow \overline{T}_0 = T_0'.$$

The equivalence (rep. partial, equals) relations can be characterized in terms of $Pre\overline{T}_2$ (resp. $\overline{T}_0, KT_2$) preordered spaces. By Theorem 6.5 of [11], a preordered space is zero-dimensional iff it is pre-Hausdorff.

(4) In **Lim**, let $\mathbf{T_k}$, $k = 0, 1, 2$ denote the usual T_k limit spaces. By Theorem 2.4, Remark 2.8 of [10] and Theorem 3.1, $T_2' = \overline{T}_2 = LT_2 = \mathbf{T_2} \Rightarrow KT_2$ and $T_2' = \overline{T}_2 = LT_2 = \mathbf{T_2}$ $\Rightarrow \mathbf{T_1}= T_1 \Rightarrow \mathbf{T_0}=\overline{T}_0 \Rightarrow T_0'$ but the reverse of each implication is not true. There is no implication between KT_2 and each of $\overline{T}_0$ and T_1 limit spaces. $(R, F(R))$, R the real numbers, is both KT_2 and T_0' but it is neither $\overline{T}_0$ nor T_1 nor LT_2. Let $B = \{x, y\}$, $K(y) = \{[y], [\emptyset]\}$ and $K(x) = F(B)$. (B, K) is $\overline{T}_0$ but it is not KT_2. Let (R, τ) be a cofinite topological space. Then the induced limit space (R, K_τ) is T_1 but it is not KT_2.
In limit spaces the concepts of uniform continuity and completeness are not available but in KT_2 limit spaces, these concepts are available. If (B, K) is a KT_2 limit space, then by Proposition 2.1.8 of [14] and Theorem 3.1, K induces the associated uniform limit structure on B which is complete.

(5) If an extended pseudo-quasi-semi metric space (A, d) is KT_2, then by Theorem 3.13 of [18] and Theorem 3.20 of [6], open and closed subsets of A are the same and A has a partition consisting of closed subsets. The relationship among various form of $preT_2$, T_0, T_1, and T_2 extended pseudo-quasi-semi metric spaces is investigated in [6, 13].

(6) In the category **Chy** of Cauchy spaces, the relationship among various form of the T_0, T_1, and T_2 Cauchy spaces is studied in [17, 19].

4. General results

In this section, we want to find under what conditions each of the reverse implications $LT_2 \Rightarrow \overline{T}_2 \Rightarrow KT_2$, $T_2' \Rightarrow \overline{T}_2$, and $\overline{T}_0 \Rightarrow T_0'$ hold in an arbitrary topological category and to solve several open problems.

Theorem 4.1. *Let $\mathcal{B}$ be a category with finite products and cokernel pairs and $U : \mathcal{E} \to \mathcal{B}$ be a topological functor.*
(A) *(a) $PreT_2'$ (resp. $\overline{T}_0$) implies $Pre\overline{T}_2$ (resp. T_0').*
(b) $T_2' = LT_2$ and there is no implication between KT_2 and each of $\overline{T}_0$ and T_1, in general.
(B) *In the realm of $\overline{T}_0$ objects, we have $KT_2 \Rightarrow \overline{T}_2$.*
(C) *In the realm of $PreT_2'$ objects,*
(a) one has $\overline{T}_2 \Rightarrow T_2'$ and $T_0' \Rightarrow \overline{T}_0$.
(b) the following are equivalent:
(i) T_0'.
(ii) $\overline{T}_0$.
(iii) T_1.
(iv) $\overline{T}_2$.
(v) KT_2.
(vi) LT_2.
(vii) T_2'.
(D) *In the realm of $Pre\overline{T}_2$ objects, the following are equivalent:*
(i) $\overline{T}_0$.
(ii) T_1.
(iii) $\overline{T}_2$.

Proof. Assume $\mathcal{E}$ is a topological category over $\mathcal{B}$ and $Y \in Ob(\mathcal{E})$.
(**A**) (a) Let $T = i_1 \circ (\pi_2, \pi_1) + i_2$, where $\pi_i : B^2 \to B$ are projection morphisms for $i = 1, 2$ and i_1, $i_2 : B^2 \to B^2 \vee_\Delta B^2$ are the inclusions of B^2 as the first and second factor. Note that for the category $\mathcal{B} = \mathbf{Set}$, $T(x, y)_2 = (x, y)_2$ and $T(x, y)_1 = (y, x)_1$. The proof of part (a) is similar to the proof of Theorem 3.1 of [4] and Theorem 3.2 of [9].
(b) If Y is LT_2, then Y is $\overline{T}_0$ and by part (a), Y is T_0'. Hence, Y is T_2'.
If Y is T_2', then Y is T_0' and so, the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(W_{B^2 \bigvee_\Delta B^2})$ is discrete, where $U(Y) = B$. Since Y is T_2', Y is $PreT_2'$ and by part (a), we get Y is $Pre\overline{T}_2$ and $W_{(B^2 \bigvee_\Delta B^2)} = S_B = A_B$. Hence, the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(Y^3)$ is discrete, i.e., Y is $\overline{T}_0$ and hence, Y is LT_2.
By Remark 3.4 (4), there is no implication between KT_2 and each of $\overline{T}_0$ and T_1.
(**B**) If Y is KT_2, then Y is $Pre\overline{T}_2$ and by assumption, Y is $\overline{T}_0$. Thus, Y is $\overline{T}_2$.
(**C**) Let Y be a $PreT_2'$ object.
(a) If Y is $\overline{T}_2$, then Y is $\overline{T}_0$ and by part A (a), Y is T_0'. By assumption, Y is $PreT_2'$ and hence, Y is T_2'.
If Y is T_0', then the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(W_{B^2 \bigvee_\Delta B^2})$ is discrete, where $U(Y) = B$. Since Y is $PreT_2'$ and by part A (a), Y is $Pre\overline{T}_2$ and $W_{(B^2 \bigvee_\Delta B^2)} = S_B = A_B$. Hence, the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(Y^3)$ is discrete, i.e., Y is $\overline{T}_0$.
(b) By Part (a), we get $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (iii)$. Suppose Y is $\overline{T}_0$. Then the initial lift of the U-source $A : B^2 \vee_\Delta B^2 \to U(Y^3)$ and $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ is discrete and since Y is $PreT_2'$, by part A (a), Y is $Pre\overline{T}_2$, i.e., $S_B = A_B$ and hence, the initial lift of the U-source $S : B^2 \vee_\Delta B^2 \to U(Y^3)$ and $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ is discrete. Thus, Y is T_1.
$(iii) \Rightarrow (iv)$. Since Y is T_1 and $Pre\overline{T}_2$, i.e., $S_B = A_B$, we get Y is $\overline{T}_0$ and hence, Y is $\overline{T}_2$.

$(iv) \Rightarrow (v)$. If Y is $\overline{T_2}$, then by part A (a), Y is T_0' and by Definition 2.1, Y is KT_2.
$(v) \Rightarrow (vi)$. If Y is KT_2, then Y is T_0' and by Part (a), Y is $\overline{T}_0$. Thus, Y is LT_2.
By Part (A) and Definition 2.1, we have $(vi) \Rightarrow (vii) \Rightarrow (i)$.
(**D**) Let Y be a $Pre\overline{T}_2$ object. By Definition 2.1, we have $S_B = A_B$ and as in the proof of $(ii) \Rightarrow (iii) \Rightarrow (iv)$, we get Y is T_1 iff Y is $\overline{T}_0$ iff Y is $\overline{T}_2$.
Recall from [12] that Y is called $\overline{T_0}$ (resp. T_0') sober if Y is $\overline{T_0}$ (resp. T_0') and quasi-sober (every nonempty irreducible [6] closed [2] subset of Y is the closure of a point) for $Y \in Ob(\mathcal{E})$ with $\mathcal{E}$ is a set-based topological category.
In **Top**, $\overline{T_0}$ (resp. T_0') sober spaces reduce to the usual sober spaces [12].

By Theorem 4.1, we obtain the following results that were given in [8, 10] as unsolved problems. □

Corollary 4.2. *In any topological category, in the realm of $PreT_2'$ objects, we have:*
(i) $LT_2 = T_2' = \overline{T}_2 = KT_2$ and $\overline{T}_0 = T_0'$.
(ii) $\overline{T}_0 = T_1 = KT_2$.
(iii) T_0' sober and $\overline{T}_0$ sober objects are equivalent.

5. Comments

In **Top**, one has

$$T_2' = LT_2 = \overline{T}_2 = KT_2 \Rightarrow T_1 \Rightarrow T_0' = \overline{T}_0,$$
$$T_2' = LT_2 = \overline{T}_2 = KT_2 \Rightarrow soberity \Rightarrow \overline{T}_0 = T_0',$$
$$T_2' = LT_2 = \overline{T}_2 = KT_2 \Rightarrow PreT_2' = Pre\overline{T}_2,$$

and there is no implication between $Pre\overline{T}_2$ (resp. soberity) and each of $\overline{T}_0, T_1$, and soberity (resp. T_1). Take the Sierpinski space and an infinite set X with indiscrete and cofinite topologies.

For $\mathcal{B} = \mathbf{Set}$ in Definition 2.1, Theorem 4.1 (**A**) (a) and (**D**) generalize the results in [5, 9, 10] and Corollary 4.2 solves several open problems in [8, 10].

If $\mathcal{D}$ is a full subcategory of $\mathcal{E}$ and $U : \mathcal{E} \to \mathbf{Set}$ is a topological functor such that the restriction $U_1 = U|_{\mathcal{D}} : \mathcal{D} \to \mathbf{Set}$ is still topological, then for an object $X \in \mathcal{D}$ we have two notions of T_0, T_1, T_2 etc. one with respect to U and one with respect to U_1. One may expect that both notions may differ. Take $\mathcal{E} = \mathbf{Lim}$ and $\mathcal{D} = \mathbf{Top}$ or take $\mathcal{E} = \mathbf{RRel}$ and $\mathcal{D} = \mathbf{Prord}$. Then by Remark 3.5 and Theorems 3.1 and 3.3,

$$\mathbf{T_0'Lim} = \mathbf{Lim}, \mathbf{KT_2Lim} = \mathbf{RLim}, \mathbf{LT_2Lim} = \mathbf{T_2Lim},$$
$$\mathbf{T_0'Top} = \mathbf{T_0Top}, \mathbf{KT_2Top} = \mathbf{LT_2Top} = \mathbf{T_2Top},$$
$$\mathbf{T_0'RRel} = \mathbf{RRel}, \mathbf{KT_2RRel} = \mathbf{EqRRel},$$
$$\mathbf{T_0'Prord} = \mathbf{T_0Prord}, \mathbf{KT_2Prord} = \mathbf{LT_2Prord} = \mathbf{LT_2RRel} = \mathbf{T_2Prord},$$

where **EqRRel** is the category of equivalence relation spaces.

In the realm of $PreT_2$ topological spaces, by Theorem 4.3 of [8], all of T_0, T_1, T_2, and sober topological spaces are equivalent.

Is there any relation between soberity and separation properties in Definition 2.1 for a topological category? Under what conditions could these notions be the same?

Pre-Hausdorff objects are used to define each of Hausdorff, regular, and normal objects in topological categories [2] and Definition 2.1 opens the way to the investigation of separation properties in a much larger non set-based topological categories.

Theorem 4.1 and Corollary 4.2 give the relationship among various forms of T_0, T_1, and T_2 objects in a topological category.

How are various forms of T_3 and T_4 objects related to each other in a topological category?

What is the relationship between each of T_k, $k = 3, 4$, and each of T_i objects, $i = 0, 1, 2$ in a topological category?

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