Fundamental Journal of Mathematics and Applications, 7(4) (2024), 212-217
Research Paper / Open Access

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1443574

A Note on Statistical Continuity of Functions

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Article Information

Keywords: Statistical continuity; Statistical convergent double sequence; Statistical convergent function

AMS 2020 Classification: 54A20; 40A35

Abstract

In the present paper, we first recall the notion of statistical convergence of double sequences defined on topological spaces and reduced equivalent to the definition, along with some of its basic properties. Later, we define the concept of statistically continuous as a general case of the continuous function using the statistical convergence of double sequences. We define strong and weak statistically continuous functions as final definitions that arise as a direct consequence of statistically continuous functions. In the rest of the paper, we analyze the implications between the given definitions and investigate additional conditions for equality.

1. Introduction

The concepts of convergence and continuity, one of the joint research topics of mathematical analysis and topology, are fundamental issues. These concepts are used to characterize many properties in topological and metric spaces. The usual convergence studied in many topological spaces has been studied in detail to obtain new results and solve many topological problems in terms of convergence. From a more general point of view, different types of convergence can be defined to get new results. Based on this idea, new types of convergence, such as statistical convergence, have emerged.

The notion of statistical convergence, an extension of the usual convergence, was formerly named ”almost convergence” by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [1]. The concept was formally introduced by Fast [2] and Steinhaus [3] and later was reintroduced by Schoenberg [4] and also independently by Buck [5]. Maio introduced and studied statistical convergence in topological spaces [6]. The concept of statistical convergence of double sequences was introduced by Muresaleen and Edely [7] using the double natural density. Later, Renukadevi and Vijayashanthi [8] applied this notion to topological spaces and showed many topological results. In recent years, many papers have been written using the idea of statistical and ideal convergence (see [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]). Specific to this presented work, we note that for the special case G=st-lim in [22], G-sequentially continuous function coincides with continuous in the ordinary sense since statistical limit is a subsequential method. Although statistical convergence was introduced over nearly the last ninety years, it has become an active area of research for forty years with the contributions by several authors, Salat [23], Fridy [24], Di Maio and Kočinac [6], Çakallı and Khan [25].

First, in the article, we present the notion of statistical convergence of double sequences defined on topological spaces and their equivalent case. Then, using the statistical convergence of double sequences, we define the notions of statistically continuous function, weakly statistically continuous function, and strongly statistically continuous function. The rest of the paper analyses the results between the given definitions.

2. Preliminaries

Basic definitions and theorems regarding statistical convergence of the double sequences in this section are given by Renukadevi et al. in [8].

Received: 27-02-2024 **Revised:** 24-08-2024 **Accepted:** 30-09-2024 **Online:** 30-12-2024 **Published:** 31-12-2024

Throughout this paper, unless otherwise stated clearly, all spaces are assumed to be Hausdorff. For the real line with the natural topology, we use $\mathbb{R}$. We often say just space instead of topological space. A double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in a topological space X is said to converge to a point $x\in X$ in Pringsheims sense [26] if for every open set U containing x, there exists $l\in\mathbb{N}$ such that $x_{nm}\in U$ for all $m>l$ and $n>l$.

If $A\subset\mathbb{N}\times\mathbb{N}$, then A_{nm} denotes the set

$$A_{nm}=\{(k,l)\in A: k\leq n, l\leq m\}.$$

The double natural (or double asymptotic) density of A is given by

$$d(A)=\lim_{n,m\to\infty}\frac{|A_{nm}|}{nm},$$

if it exists. A subset A of $\mathbb{N}\times\mathbb{N}$ is statistically dense if $d(A)=1$. We also recall that

$$d((\mathbb{N}\times\mathbb{N})\backslash A)=1-d(A)$$

for $A\subset\mathbb{N}\times\mathbb{N}$.

A double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in a space X is said to statistically convergent to $x\in X$, if for every neighborhood U of x,

$$d(\{(n,m)\in\mathbb{N}\times\mathbb{N}: x_{nm}\notin U\})=0.$$

We denote it by

$$x_{nm}\xrightarrow{st}x \text{ or } st-\lim_{n,m\to\infty}x_{nm}=x,$$

and we call x the statistical limit of (x_{nm}).

Theorem 2.1 ([8])**.** *If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ is convergent, then it is statistically convergent.*

A double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ is said to be st*-convergence [8] to $x\in X$ if there is $A\subset\mathbb{N}\times\mathbb{N}$ with $d(A)=1$ such that

$$\lim_{n,m\to\infty,(n,m)\in A}x_{nm}=x.$$

We denote it by

$$x_{nm}\xrightarrow{st^*}x \text{ or } st^*-\lim_{n,m\to\infty}x_{nm}=x.$$

Theorem 2.2 ([8])**.** *If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ st^*-convergent to $x\in X$, then $(x_{nm})_{n,m\in\mathbb{N}}$ st-convergent to x.*

The converse holds if the space X is first countable.

Theorem 2.3 ([8])**.** *Let X be a first countable space. If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ st-convergent to $x\in X$, then (x_{nm}) st^*-convergence to x .*

Due to this theorem, the definition of statistical convergence of a double sequence is equivalently said that for the first countable space X, there exists a subset A of $\mathbb{N}\times\mathbb{N}$ with $d(A)=1$ such that the double sequence $(x_{nm})_{(n,m)\in A}$ convergent to x, i.e. for every neighborhood V of x there is $n_0\in\mathbb{N}$ such that $n,m\geq n_0$ and $(n,m)\in A$ imply $x_{nm}\in V$.

3. Main results

In this part of this study, we first define the concept of statistical continuity of functions, which is not available in the literature, by using the idea of statistically convergent double sequences.

Theorem 3.1. *If a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ is statistically convergent, then its statistical limit is unique.*

Proof. Suppose that

$$x_{nm}\xrightarrow{st}x_1 \text{ and } x_{nm}\xrightarrow{st}x_2$$

with $x_1\neq x_2$. Let $x\in X$. Let U and V be neighborhood of x_1 and x_2, respectively, such that

$$U\cap V=\emptyset.$$

Since $x_{nm} \xrightarrow{st} x_1$ and $x_{nm} \xrightarrow{st} x_2$, then

$$d(K_1) = d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin U\}) = 0$$

and

$$d(K_2) = d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin V\}) = 0,$$

respectively. Let $K = K_1 \cup K_2$. Thus, $d(K) = 0$ which implies

$$d((\mathbb{N} \times \mathbb{N}) \backslash K) = 1.$$

If $(k,l) \in (\mathbb{N} \times \mathbb{N}) \backslash K$, then

$$x_{kl} \in U \text{ and } x_{kl} \in V.$$

This contradicts the fact that $U \cap V = \emptyset$. Hence $x_1 = x_2$. □

We denote by Y^X the set of all functions from a space X to a space Y.

Definition 3.2. *$f \in Y^X$ is called a statistically continuous function if for every double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X statistically converging to x, $(f(x_{nm}))_{n,m\in\mathbb{N}}$ statistically convergent to $f(x)$.*

Proposition 3.3. *Every continuous function is statistically continuous.*

Proof. Let $f \in Y^X$ be continuous and $x_{nm} \xrightarrow{st} x$. Then, for every neighborhood U of x,

$$d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin U\}) = 0.$$

Since for every open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $U = f^{-1}(V)$ (due to continuity), hence

$$d(\{(n,m) \in \mathbb{N} \times \mathbb{N} : f(x_{nm}) \notin f(U) = V\}) = 0.$$

Therefore, $f(x_{nm}) \xrightarrow{st} f(x)$. □

The converse of Proposition 3.3 does not hold.

Example 3.4. *For the usual topology $\mathcal{U}$ and the countable complement topology τ on $\mathbb{R}$, the identity function $I : (\mathbb{R},\tau) \to (\mathbb{R},\mathcal{U})$ is statistically continuous, but not continuous.*

Now, we define the notion of weak statistical continuity, which is a general case of statistically continuous function for double sequences.

Definition 3.5. *$f \in Y^X$ is called a weakly statistically continuous function if for every double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X converging to x, $(f(x_{nm}))_{n,m\in\mathbb{N}}$ statistically convergent to $f(x)$.*

Proposition 3.6. *Every statistically continuous function is weakly statistically continuous.*

Proof. Let $f \in Y^X$ be statistically continuous and $x_{nm} \longrightarrow x$. Then from Theorem 2.1

$$x_{nm} \xrightarrow{st} x.$$

Since f is statistically continuous, it follows from

$$f(x_{nm}) \xrightarrow{st} f(x).$$

Hence, f is weakly statistically continuous. □

Considering Proposition 3.3, it can give the following result.

Corollary 3.7. *Every continuous function is weakly statistically continuous.*

Theorem 3.8. *For the first countable spaces X and Y, a double sequence $(f_{nm})_{n,m\in\mathbb{N}}$ and a function $f \in Y^X$ be given. Then the following are equivalent:*

1. *f is continuous.*
2. *f is statistically continuous.*
3. *f is weakly statistically continuous.*

Proof. The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are given in Proposition 3.3 and Proposition 3.6, respectively.
$(3) \Rightarrow (1)$. Let K be a closed set in the space Y. It is enough to show that the set $f^{-1}(K)$ is closed in the space X. Take any $x \in \overline{f^{-1}(K)}$. Since X first countable space, there exists a double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X such that

$$x_{nm} \longrightarrow x.$$

By using the weak statistical continuity of f, we get

$$f(x_{nm}) \xrightarrow{st} f(x).$$

From Theorem 2.3, there exists a subset A of $\mathbb{N}\times\mathbb{N}$ with $d(A) = 1$ such that

$$f(x_{n_k m_l}) \longrightarrow f(x).$$

Due to $f(x_{n_k m_l}) \in K$ and $f(x) \in \overline{K} = K$, we have $x \in f^{-1}(K)$ and as a result, $f^{-1}(K)$ is closed in the space X. As can be seen from here, f is continuous. □

Now, we define the notion of strong statistical continuity, which is a special case of statistically continuous functions for double sequences.

Definition 3.9. *$f \in Y^X$ is called a strongly statistically continuous function if for every double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ in X statistically converging to x, $(f(x_{nm}))_{n,m\in\mathbb{N}}$ convergent to $f(x)$.*

Proposition 3.10. *Every strongly statistically continuous function is statistically continuous.*

Proof. Let $f \in Y^X$ be strongly statistically continuous and $x_{nm} \xrightarrow{st} x$. As f is strongly statistically continuous,

$$f(x_{nm}) \longrightarrow f(x).$$

Then from Theorem 2.1

$$f(x_{nm}) \xrightarrow{st} f(x).$$

This shows that f is statistically continuous. □

The converse of Proposition 3.10 does not hold.

Example 3.11. *Consider the statistically continuous function I in Example 3.4. it is not strongly statistically continuous because, when $x_{nm} \xrightarrow{st} x$, the convergence $x_{nm} \longrightarrow x$ does not satisfy.*

Theorem 3.12. *For the first countable spaces X and Y, every strongly statistically continuous function $f \in Y^X$ is continuous.*

Proof. Let f be astrongly statistically continuous function. Since $x_{nm} \longrightarrow x$, then we have $x_{nm} \xrightarrow{st} x$ from Theorem 2.1 and so, from the definition of strong statistical continuity,

$$f(x_{nm}) \longrightarrow f(x).$$

This means that f is sequentially continuous. We know that continuity and sequential continuity are equivalent in the first countable space. Therefore, f is continuous. □

Theorem 3.13. *Let $f \in Y^X$ and $g \in Z^Y$ be statistical continuous functions. Then $g \circ f$ is a statistical continuous function.*

Proof. Let $x_{nm} \xrightarrow{st} x$ in X. Since $f \in Y^X$ is statistical continuous function, then

$$f(x_{nm}) \xrightarrow{st} f(x)$$

in Y. Since $f(x_{nm}) \xrightarrow{st} f(x)$ in Y and $g \in Z^Y$ is a statistically continuous function, then we have

$$g(f(x_{nm})) \xrightarrow{st} g(f(x))$$

in Z. As a result, the $g \circ f$ is shown to be statistically continuous. □

Theorem 3.14. *Let $f, g \in \mathbb{R}^X$ be real-valued statistical continuous functions. Then the following is provided.*

1. *$g + f$ is a statistical continuous function.*
2. *$g.f$ is a statistical continuous function.*

Proof. (1). Let $x_{nm} \xrightarrow{st} x$ in X. Since f and g are real-valued statistical continuous functions,

$$f(x_{nm}) \xrightarrow{st} f(x) \text{ and } g(x_{nm}) \xrightarrow{st} g(x)$$

in $\mathbb{R}$, respectively. Therefore,

$$f(x_{nm}) + g(x_{nm}) \xrightarrow{st} f(x) + g(x)$$

and so

$$(f+g)(x_{nm}) \xrightarrow{st} (f+g)(x).$$

Hence, $g+f$ is a statistical continuous function.

(2). The proof is similar to (1) □

4. Conclusion

Continuous functions have an essential place in the study of topological spaces. Convergence can also be used to investigate the continuity of a function. Therefore, studies on these two concepts further research on topological spaces.

In this study, we investigated the continuity of topological valued functions using the statistical convergence of double sequences. A more general view of the continuity of real-valued functions is introduced by taking topological valued functions.

Since continuous functions are essential in mathematical studies, further research is required. Studies can be extended by using different convergence types and topological open-closed sets. As a continuation of this study, in future studies, uniform continuity with change in the value set can be defined and compared with this work.

Declarations

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: Conceptualization, İ.O. and E.D.; methodology, İ.O. and E.D.; validation, İ.O. and E.D. investigation, İ.O. and E.D.; resources, İ.O. and E.D.; data curation, İ.O. and E.D.; writing—original draft preparation, E.D.; writing—review and editing,İ.O.; supervision, E.D. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

Use of AI tools: The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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References

[1] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, (2002). [CrossRef]
[2] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241-244. [Web]
[3] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73-74.
[4] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361-375. [CrossRef]
[5] R.C. Buck, *Generalized asymptotic density*, Am. J. Math., **75**(2) (1953), 335-346. [CrossRef]
[6] G. Di Maio and D.R. Kočinac, *Statistical convergence in topology*, Topol. Appl., **156**(1) (2008), 28-45. [CrossRef] [Scopus] [Web of Science]
[7] M. Mursaleen and O.H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288**(1) (2003), 223-231. [CrossRef] [Scopus] [Web of Science]

[8] V. Renukadevi and P. Vijayashanthi, *Statistical convergence of double sequences*, Jordan J. Math. Stat., **14**(4) (2021), 787-808. [CrossRef] [Scopus] [Web of Science]
[9] H. Çakallı, *A new approach to statistically quasi Cauchy sequences*, Maltepe J. Math., **1**(1) (2019), 1-8. [Web]
[10] A. Caserta, G. Di Maio and L.D.R. Kočinac, *Statistical convergence in function spaces*, Abstr. Appl. Anal., **2011**(1)(2011). [CrossRef] [Scopus] [Web of Science]
[11] H.Ş. Kandemir, *On I-deferred statistical convergence in topological groups*, Maltepe J. Math., **1**(2) (2019), 48-55. [Web]
[12] F. Nuray, E. Dündar and U. Ulusu, *Some generalized definitions of uniform continuity for real valued functions*, Creat. Math. Inform., **29**(2) (2020), 165-170. [CrossRef] [Scopus]
[13] F. Nuray, E. Dündar and U. Ulusu, *Wijsman statistical convergence of double sequences of set*, Iran. J. Math. Sci. Inform., **16**(1) (2021), 55-64. [CrossRef] [Scopus] [Web of Science]
[14] R.F. Patterson and H. Çakallı, *Quasi Cauchy double sequences*, Tbilisi Math. J., **8**(2) (2015), 211-219. [CrossRef] [Web of Science]
[15] U. Ulusu, F. Nuray and E. Dündar, *$\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points for functions defined on amenable semigroups*, Fundam. J. Math. Appl., **4**(2) (2021), 45-48. [CrossRef]
[16] E. Gülle and U. Ulusu, *Wijsman deferred invariant statistical and strong p-deferred invariant equivalence of order α*, Fundam. J. Math. Appl., **6**(4) (2023), 211-217. [CrossRef]
[17] Ö. Kişi and E. Güler, *$\mathcal{I}$-Cesaro summability of a sequence of order α of random variables in probability*, Fundam. J. Math. Appl., **1**(2) (2018), 157-161. [CrossRef]
[18] S. Erdem and S. Demiriz, *A study on strongly almost convergent and strongly almost null binomial double sequence spaces*. Fundam. J. Math. Appl., **4**(4) (2021), 271-279. [CrossRef]
[19] M. Candan, *A new aspect for some sequence spaces derived using the domain of the matrix $\hat{\hat{B}}$*, Fundam. J. Math. Appl., **5**(1) (2022), 51-62. [CrossRef]
[20] F. Gökçe, *Compact and matrix operators on the space $\left|\overline{N}_p^{\phi}\right|_k$*, Fundam. J. Math. Appl., **4**(2) (2021), 124-133. [CrossRef]
[21] S. Aydın and H. Polat, *Difference sequence spaces derived by using Pascal transform*, Fundam. J. Math. Appl., **2**(1) (2019), 56-62. [CrossRef]
[22] H. Çakallı , *On G-continuity*, Comput. Math. Appl., **61**(2) (2011), 313-318. [CrossRef] [Scopus] [Web of Science]
[23] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca **30**(2) (1980), 139-150. [Web]
[24] J.A. Fridy, *On statistical convergence*, Anal., **5**(4) (1985), 301-313. [CrossRef]
[25] H. Çakallı and M. K. Khan, *Summability in topological spaces*, Appl. Math. Lett., **24**(3) (2011), 348-352. [CrossRef] [Scopus] [Web of Science]
[26] A. Pringsheim, *Zur Theorie der zweifach unendlichen zahlenfolgen*, Math. Ann., **53**(3) (1900), 289-321. [CrossRef] [Scopus]

Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: İ. Osmanoğlu and E. Dündar, *A note on statistical continuity of functions*, Fundam. J. Math. Appl., **7**(4) (2024), 212-217. DOI 10.33401/fujma.1443574