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Some Results on Related Fixed Point Theorems in Two S-Metric Spaces

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Abstract — In this study, by considering the technique in the theorem of Bollenbacher and Hicks, we obtain some related fixed point theorems involving two mappings on two related complete S-metric spaces with certain conditions. As the applications of the main theorem, we then derive some new related fixed point theorems involving a pair of mappings in such spaces. We finally discuss the need for further research.

Keywords *S-metric space, fixed point, related fixed point, related complete S-metric space*

Mathematics Subject Classification (2020) 54H25, 47H10

1. Introduction

Recently, Sedghi et al. [1] have defined a new type of generalized metric space, S-metric space, and presented certain important properties of them. Further, they have proved a version of the famous Banach's theorem on complete S-metric spaces. This novel framework enriches the theoretical landscape and opens avenues for practical applications in diverse fields, such as optimization, computer science, and engineering. Afterward, many authors [2–8] have explored and established various fixed point theorems in S-metric spaces, further solidifying the significance of this emerging area of study.

Moreover, Fisher [9] has provided a theorem involving compositions of two mappings on two complete metric spaces and investigated the relation between the fixed points for these mappings. This theorem is the initial related fixed point result involving two mappings on two complete metric spaces. Several authors [10–17] have established various types of this theorem in many different directions.

In the present study, we introduced the concept related to completeness in two S-metric spaces. In section 3, by considering the technique in the theorem of Bollenbacher and Hicks [18], we establish certain related fixed point theorems involving two mappings in two S-metric spaces under certain conditions. In section 4, as applications of the main theorem, we derive some new related fixed point theorems involving a pair of mappings in such spaces. In addition, we provide an illustrative example of the main result. In the last section, we address whether additional research concerning the aforesaid notions is needed.

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2. Preliminaries

This section presents S-metric spaces and some of their basic properties.

Definition 2.1. [1] An S-metric space (W, S) is a nonempty set W with a non negative real-valued function S on W^3 such that, for all $u, v, y, z \in W$, the following conditions hold,

(SM1) $S(u, v, y) = 0 \Leftrightarrow u = v = y$

(SM2) $S(u, v, y) \leq S(u, u, z) + S(v, v, z) + S(y, y, z)$

Here, the function S is said to be an S-metric on W.

Lemma 2.2. [1] If S is an S-metric on W, then $S(v, v, u) = S(u, u, v)$, for all $u, v \in W$.

Definition 2.3. [1] Let $\{v_k\}$ be a sequence in an S-metric space. Then,

i. $\{v_k\}$ is a convergent to $v \in W$ if, for all $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $S(v_k, v_k, v) < \varepsilon$ whenever $k \geq k_0$

ii. $\{v_k\}$ is a Cauchy sequence if, for all $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $S(v_k, v_k, v_l) < \varepsilon$ whenever $k, l \geq k_0$

Moreover, an S-metric space (W, S) is said to be complete if every Cauchy sequence in W is convergent in W.

Lemma 2.4. [1] Let (W, S) be an S-metric space.

i. If $\{v_k\}$ is a sequence in W such that $\lim\limits_{k\to\infty} v_k = v$, then v is unique

ii. If a sequence $\{v_k\}$ is convergent in W, then $\{v_k\}$ is a Cauchy sequence

iii. If $\{v_k\}$ and $\{w_k\}$ are two sequences in W such that $\lim\limits_{k\to\infty} v_k = v$ and $\lim\limits_{k\to\infty} w_k = w$, then

$$\lim_{k\to\infty} S(v_k, v_k, w_k) = S(v, v, w)$$

Definition 2.5. [17, 19] Let (V, S_1) and (W, S_2) be two S-metric spaces. A real-valued function $\delta : V \times W \to [0, \infty)$ is a weak lower semi-continuous (briefly WLSC) at $(v, w) \in V \times W$ if and only if $\{v_k\}$ and $\{w_k\}$ are sequences in V and W, respectively, and

$$\lim_{k\to\infty} v_k = v \wedge \lim_{k\to\infty} w_k = w \quad \Rightarrow \quad \delta(v, w) \leq \lim_{k\to\infty} \sup \delta(v_k, w_k)$$

Definition 2.6. [17, 19] Let (V, S_1) and (W, S_2) be two S-metric space and $A : V \to W$ and $B : W \to V$ be two mappings. The sequences $\{v_k\}$ in V and $\{w_k\}$ in W defined by

$$v_k = (BA)^k v_0 \quad \text{and} \quad w_k = A(BA)^{k-1} v_0, \quad k \in \mathbb{Z}^+$$

We use these sequences in the main results. Consider the following sets

$$\Re_V(v_0) = \left\{(BA)^k v_0 : k \in \mathbb{Z}^+\right\} \quad \text{and} \quad \Re_W(v_0) = \left\{A(BA)^{k-1} v_0 : k \in \mathbb{Z}^+\right\}$$

Then, the S-metric spaces (V, S_1) and (W, S_2) are said to be related complete if each Cauchy sequence in $\Re_V(v_0)$ and $\Re_W(v_0)$ converges to a point in V and converges to a point in W, respectively.

Note that two complete S-metric spaces (V, S_1) and (W, S_2) are related complete. However, the reverse is not generally true, as shown in the following example.

Example 2.7. [17] Let $V = (-1, 1]$, $W = [0, 1]$, and $|\,.\,|$ be absolute value function on V and W. Then, $S(u, v, y) = |u - y| + |v - y|$ is an S-metric on V and W. Let the mappings $A : V \to W$ and

$B : W \to V$ be defined by

$$A(v) = \begin{cases} 0, & -1 < v < 0 \\ 1, & 0 \leq v \leq 1 \end{cases} \quad \text{and} \quad B(w) = \begin{cases} -\frac{1}{2}, & 0 \leq w < \frac{1}{2} \\ 1, & \frac{1}{2} \leq w \leq 1 \end{cases}$$

Then, for any $v_0 \in (-1, 0)$, $\Re_V(v_0) = \left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \cdots\right\}$ and $\Re_W(v_0) = \{0, 0, 0, \cdots\}$. Thus, (V, S) and (W, S) are related complete. However, (V, S) is not complete.

3. Main Results

This section proposes the following related fixed point theorem in two related complete S-metric spaces.

Theorem 3.1. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$\max\{S_1(v, v, BAv), S_2(w, w, ABw)\} \leq \alpha(v) - \alpha(BAv) + \beta(w) - \beta(ABw) \tag{3.1}$$

for all $v \in \Re_V(v_0)$ and for all $w \in \Re_W(v_0)$ where $\alpha : V \to [0, \infty)$ and $\beta : W \to [0, \infty)$ are two mappings, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$.

Proof. *i.* From (3.1),

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \max\{S_1(v_n, v_n, v_{n+1}), S_2(w_n, w_n, w_{n+1})\} \\ &= \sum_{n=1}^{k} \max\{S_1(v_n, v_n, BAv_n), S_2(w_n, w_n, ABw_n)\} \\ &\leq \sum_{n=1}^{k} [\alpha(v_n) - \alpha(BAv_n) + \beta(w_n) - \beta(ABw_n)] \\ &= \sum_{n=1}^{k} [\alpha(v_n) - \alpha(v_{n+1}) + \beta(w_n) - \beta(w_{n+1})] \\ &= \alpha(v_1) - \alpha(v_{k+1}) + \beta(w_1) - \beta(w_{k+1}) \leq \alpha(v_1) + \beta(w_1) \end{aligned}$$

Thus, $\{s_k\}$ is bounded above. Moreover, $\{s_k\}$ is non-decreasing. Thus, it is convergent.

Let k and l be any two positive integers with $k < l$. Using (SM2) of S-metric and Lemma 2.2,

$$\begin{aligned} \max\{S_1(v_l, v_l, v_k), S_2(w_l, w_l, w_k)\} &\leq \max\left\{\sum_{n=k}^{l-1} 2S_1(v_n, v_n, v_{n+1}), \sum_{n=k}^{l-1} 2S_2(w_n, w_n, w_{n+1})\right\} \\ &\leq 2\sum_{n=k}^{l-1} \max\{S_1(v_n, v_n, v_{n+1}), S_2(w_n, w_n, w_{n+1})\} \end{aligned} \tag{3.2}$$

Since $\{s_k\}$ convergent, for any $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\sum_{n=k}^{\infty} \max\{S_1(v_n, v_n, v_{n+1}), S_2(w_n, w_n, w_{n+1})\} < \varepsilon/2$$

for all $k \geq n_0$. Hence, from (3.1),

$$\max\{S_1(v_l, v_l, v_k), S_2(w_l, w_l, w_k)\} < \varepsilon$$

for all $k, l \geq n_0$. Thus, $\{v_k\}$ and $\{w_k\}$ are two Cauchy sequence in $\Re_V(v_0)$ and $\Re_W(v_0)$, respectively. Since (V, S_1) and (W, S_2) are related complete, the sequence $\{v_k\}$ converges to a point $z \in V$ and the sequence $\{w_k\}$ converges to a point $u \in W$.

ii. Suppose that $Bu = z$ and $Az = u$ and $\{v_k\}$ and $\{w_k\}$ are sequences in V and W, respectively, such that $v_k \to z$ and $w_k \to u$. Then,

$$P(z,u) = S_1(z,z,Bu) = 0 \leq \limsup S_1(v_k, v_k, Bw_k) = \limsup P(v_k, w_k)$$

and

$$Q(z,u) = S_2(u,u,Az) = 0 \leq \limsup S_2(w_k, w_k, Av_k) = \limsup Q(v_k, w_k)$$

Thus, P and Q are WLSC at (z,u).

Conversely, let $v_k = (BA)^k v_0$, $w_k = A(BA)^{k-1}v_0$, and P and Q are WLSC at (z,u). It follows from *i*,

$$\begin{aligned} 0 \leq S_1(z,z,Bu) = P(z,u) &\leq \limsup P(v_k, w_k) \\ &= \limsup S_1(v_k, v_k, Bw_k) \\ &= \limsup S_1(v_k, v_k, v_k) = 0 \end{aligned}$$

and using Lemma 2.4,

$$\begin{aligned} 0 \leq S_2(u,u,Az) = Q(z,u) &\leq \limsup Q(v_k, w_k) \\ &= \limsup S_2(w_k, w_k, Av_k) \\ &= \limsup S_2(w_k, w_k, w_{k+1}) = 0 \end{aligned}$$

Thus, $Bu = z$ and $Az = u$.

Moreover, assume that *ii* holds. Then, $BAz = z$ and $ABu = u$. □

Remark 3.2. Since the inequality

$$\max\{S_1(v,v,BAv), S_2(w,w,ABw)\} \leq S_1(v,v,BAv) + S_2(w,w,ABw)$$

holds, for all $v \in V$ and for all $w \in W$, then Theorem 3.1 holds for the following inequality

$$S_1(v,v,BAv) + S_2(w,w,ABw) \leq \alpha(v) - \alpha(BAv) + \beta(w) - \beta(ABw) \tag{3.3}$$

If $(V, S_1) = (W, S_2) = (U, S)$, then from Theorem 3.1, we have the following result: The (U, S) is BA-orbitally complete, defined as each Cauchy sequence in the set

$$O_{BA}(v_0, \infty) = \{v_0, Av_0, BAv_0, \cdots, A(BA)^{k-1}v_0, (BA)^k v_0, \cdots\}$$

converges in U, where $v_0 \in U$.

Corollary 3.3. Let (U, S) be an S-metric space and A and B be two mappings of U into itself. If there exists a $v_0 \in U$ such that (U, S) is BA-orbitally complete and

$$\max\{S(v,v,BAv), S(w,w,ABw)\} \leq \alpha(v) - \alpha(BAv) + \beta(w) - \beta(ABw)$$

for all $v, w \in O_{BA}(v_0, \infty)$ where $\alpha, \beta : U \to [0, \infty)$ are two mappings, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : U \times U \to [0, \infty)$, defined by $P(v, w) = S(v, v, Bw)$, and $Q : U \times U \to [0, \infty)$, defined by $Q(v, w) = S(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$ and thus if $u = z$, then $Az = Bz = z$.

Note that a similar result can be proved for (3.3).

If $(V, S_1) = (W, S_2) = (U, S)$ and $A = B$, then the set $\Re_U(v_0) = \{A^k v_0 : k \in \mathbb{Z}^+\}$ is equal to $O_A(v_0, \infty)$, which is called orbit of $v_0 \in U$. Moreover, (U, S) is A-orbitally complete if each Cauchy sequence in $O_A(v_0, \infty)$ converges to U.

If $(V, S_1) = (W, S_2) = (U, S)$, $B = I$, and $\alpha = \beta$ where I is a identity mapping of U, then using (3.3), the following Bollenbacher and Hicks's result [18] in S-metric version is obtained.

Corollary 3.4. Let (U, S) be an S-metric space and $A : U \to U$ be a mapping. If there exists a $v_0 \in U$ such that (U, S) is A-orbitally complete and

$$S(v, v, Av) \leq \alpha(v) - \alpha(Av)$$

for all $v \in O_{BA}(v_0, \infty)$ where $\alpha : U \to [0, \infty)$ is a mapping, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} A^k v_0 = z$

ii. $Az = z$ if and only if $P : U \to [0, \infty)$, defined by $P(v) = S(v, v, Av)$, is WLSC at z.

4. Additional Results

This section drives some related fixed point theorems in two S-metric spaces using the Theorem 3.1.

Theorem 4.1. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$S_1(BAv, BAv, (BA)^2 v) + S_2(ABw, ABw, (AB)^2 w) \leq r[S_1(v, v, BAv) + S_2(w, w, ABw)] \tag{4.1}$$

for all $v \in \Re_V(v_0)$ and for all $w \in \Re_W(v_0)$ where $0 \leq r < 1$, then

i. $\lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} (BA)^k v_0 = z$ and $\lim\limits_{k\to\infty} w_k = \lim\limits_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$.

Proof. Define the mappings $\Upsilon(v) = \frac{1}{1-r} S_1(v, v, BAv)$, for all $v \in V$, and $\Gamma(w) = \frac{1}{1-r} S_2(w, w, ABw)$, for all $w \in W$. Then, from (4.1),

$$\begin{aligned} S_1(v, v, BAv) + S_2(w, w, ABw) - r[S_1(v, v, BAv) + S_2(w, w, ABw)] \leq\ & S_1(v, v, BAv) + S_2(w, w, ABw) \\ & -[S_1(BAv, BAv, (BA)^2 v) + S_2(ABw, ABw, (AB)^2 w)] \end{aligned}$$

and thus

$$\begin{aligned} S_1(v, v, BAv) + S_2(w, w, ABw) \leq & \frac{1}{1-r}[S_1(v, v, BAv) + S_2(w, w, ABw)] \\ & - \frac{1}{1-r}[S_1(BAv, BAv, (BA)^2 v) + S_2(ABw, ABw, (AB)^2 w)] \end{aligned}$$

Thus,

$$S_1(v, v, BAv) + S_2(w, w, ABw) \leq \Upsilon(v) - \Upsilon(BAv) + \Gamma(w) - \Gamma(ABw)$$

The results follow from (3.3) and Theorem 3.1. $\square$

Corollary 4.2. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$\max\{S_1(BAv, BAv, (BA)^2v), S_2(ABw, ABw, (AB)^2w)\} \leq \frac{r}{2}[S_1(v, v, BAv) + S_2(w, w, ABw)] \quad (4.2)$$

for all $v \in \Re_V(v_0)$ and for all $w \in \Re_W(v_0)$ where $0 \leq r < 1$, then

i. $\lim_{k\to\infty} v_k = \lim_{k\to\infty} (BA)^k v_0 = z$ and $\lim_{k\to\infty} w_k = \lim_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then $BAz = z$ and $ABu = u$.

Proof. Using (4.2),

$$\begin{aligned} S_1(BAv, BAv, (BA)^2v) + S_2(ABw, ABw, (AB)^2w) &\leq 2\max\{S_1(BAv, BAv, (BA)^2v), S_2(ABw, ABw, (AB)^2w)\} \\ &\leq 2\frac{r}{2}[S_1(v, v, BAv) + S_2(w, w, ABw)] \\ &= r[S_1(v, v, BAv) + S_2(w, w, ABw)] \end{aligned}$$

Then, the results follow immediately from Theorem 4.1. □

Theorem 4.3. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$S_1(BAv, BAv, BAv') + S_2(ABw, ABw, ABw') \leq r[S_1(v, v, v') + S_2(w, w, w')] \quad (4.3)$$

for all $v, v' \in \Re_V(v_0)$ and for all $w, w' \in \Re_W(v_0)$, where $0 \leq r < 1$, then

i. $\lim_{k\to\infty} v_k = \lim_{k\to\infty} (BA)^k v_0 = z$ and $\lim_{k\to\infty} w_k = \lim_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then z is a unique fixed point of BA and u is a unique fixed point of AB.

Proof. Put $v' = BAv$ and $w' = ABw$ in (4.3). Then, (4.1) holds. The results follow immediately from Theorem 4.1.

To prove uniqueness, if z' is a second fixed point of BA, then from (4.3), for $w = w'$,

$$\begin{aligned} S_1(z, z, z') &= S_1(BAz, BAz, BAz') + S_2(ABw, ABw, ABw) \\ &\leq r[S_1(z, z, z') + S_2(w, w, w)] \\ &= rS_1(z, z, z') \end{aligned}$$

proving that $z = z'$, since $r < 1$. Similarly, u is a unique fixed point of AB. □

Corollary 4.4. Let (V, S_1) and (W, S_2) be two S-metric spaces and $A : V \to W$ and $B : W \to V$ be two mappings. If there exists a $v_0 \in V$ such that (V, S_1) and (W, S_2) are related complete and

$$\max\{S_1(BAv, BAv, BAv'), S_2(ABw, ABw, ABw')\} \leq \frac{r}{2}[S_1(v, v, v') + S_2(w, w, w')] \quad (4.4)$$

for all $v, v' \in \Re_V(v_0)$ and for all $w, w' \in \Re_W(v_0)$ where $0 \leq r < 1$, then

i. $\lim_{k\to\infty} v_k = \lim_{k\to\infty} (BA)^k v_0 = z$ and $\lim_{k\to\infty} w_k = \lim_{k\to\infty} A(BA)^{k-1} v_0 = u$

ii. $Bu = z$ and $Az = u$ if and only if $P : V \times W \to [0, \infty)$, defined by $P(v, w) = S_1(v, v, Bw)$, and $Q : V \times W \to [0, \infty)$, defined by $Q(v, w) = S_2(w, w, Av)$, are WLSC at (z, u)

Moreover, if *ii.* is true, then z is a unique fixed point of BA and u is a unique fixed point of AB.

Proof. Using (4.4),

$$\begin{aligned}S_1(BAv,BAv,BAv')+S_2(ABw,ABw,ABw') &\le 2\max\{S_1(BAv,BAv,BAv'),S_2(ABw,ABw,ABw')\}\\ &\le 2\frac{r}{2}[S_1(v,v,v')+S_2(w,w,w')]\\ &= r[S_1(v,v,v')+S_2(w,w,w')]\end{aligned}$$

The results follow immediately from Theorem 4.3. □

We provide an example relevant main result.

Example 4.5. Let $V=(-1,1]$, $W=[0,1]$, and $|\,.\,|$ is absolute value function on V and W. Then, $S(u,v,y)=|u-y|+|v-y|$ is an S-metric space on V and W. Define the mappings $A:V\to W$ and $B:W\to V$ by

$$A(v)=\begin{cases}0, & -1<v<0\\ \frac{1}{2}, & 0\le v\le 1\end{cases} \qquad \text{and} \qquad B(w)=\begin{cases}-\frac{1}{3}, & 0\le w<\frac{1}{2}\\ \frac{1}{2}, & \frac{1}{2}\le w\le 1\end{cases}$$

We have

$$AB(w)=\begin{cases}0, & 0\le w<\frac{1}{2}\\ \frac{1}{2}, & \frac{1}{2}\le w\le 1\end{cases} \qquad \text{and} \qquad BA(v)=\begin{cases}-\frac{1}{3}, & -1<v<0\\ \frac{1}{2}, & 0\le v\le 1\end{cases}$$

If $-1<v_0<0$, then

$$\Re_V(v_0)=\left\{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\cdots\right\} \qquad \text{and} \qquad \Re_W(v_0)=\{0,0,0,\cdots\}$$

Thus, (V,S) and (W,S) are related complete and (3.1) holds for $v\in\Re_V(v_0)$ and $w\in\Re_W(v_0)$ where $\alpha:V\to[0,\infty)$ and $\beta:W\to[0,\infty)$ are two mappings. Further,

$$\lim_{k\to\infty}(BA)^k v_0=-\frac{1}{3} \qquad \text{and} \qquad \lim_{k\to\infty}A(BA)^{k-1}v_0=0$$

and $P(v,w)=2|v-Bw|$ and $Q(v,w)=2|w-Av|$ are WLSC at $\left(-\frac{1}{3},0\right)\in V\times W$. It can be observed that

$$B(0)=-\frac{1}{3}, \quad A\left(-\frac{1}{3}\right)=0, \quad BA\left(-\frac{1}{3}\right)=-\frac{1}{3}, \quad \text{and} \quad AB(0)=0$$

5. Conclusion

In this study, by considering the paper of Bollenbacher and Hicks [18], we proved some new related fixed point theorems for pair of mappings in two related complete and S-metric spaces under suitable conditions. Furthermore, we derived some related fixed-point theorems from the main results herein. Further, we introduced the concept related to completeness in two S-metric spaces. Note that two complete S-metric spaces are related complete. However, the reverse is not generally true. In future studies, considering the concept of related completeness, some related fixed point theorems can be proved in various types of metric spaces under suitable conditions, such as b-metric spaces, fuzzy metric spaces, and G^*-metric spaces.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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