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ON DYNAMICS OF QUADRATIC STOCHASTIC OPERATORS GENERATED BY 3-PARTITION ON COUNTABLE STATE SPACE

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Abstract. Quadratic stochastic operator (QSO) theory has advanced significantly since the early 1920s and is still growing due to its numerous applications in a variety of fields, particularly mathematics, where QSOs have inspired mathematicians to use and integrate various mathematical knowledge and concepts to better understand their properties and behaviors. Motivated by the relationship between the number of partitions on an infinite state space and the development of the system of equations corresponding to QSOs, this work sought to investigate the dynamics of QSOs formed by three partitions. First, we define and construct the 3-partition QSOs, which result in a system of equations with three variables. We then provide the formulation of the fixed point form and discuss its behavior using Jacobian matrix analysis. Some scenarios of three-partition QSOs with three different parameters are considered to readily investigate the type of fixed point in such systems. It is demonstrated that the operators can have either an attracting or a saddle fixed point but can never be repelling. We show how the saddle fixed point behaves, by identifying a set of points known as the fixed point's stable manifold.

1. Introduction

Quadratic stochastic operator (QSO) theory has been an appealing topic among researchers in diverse knowledge areas since its establishment in the early 1920s by

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Bernstein [2] through his innovative idea on the synthesis study between Mendel's crossing law and Galton's regression law. The QSO is the simplest nonlinear operator, which refers to a complex system model and such a model is widely applied to describe a dynamical system. Proficiency of the QSO in providing a distribution of the next generation given the distribution of the current generation has led to the acknowledgment of the model as a significant analysis source of dynamical properties and modeling study in various domains running from biology to economy. Due to its immense contributions across fields, the study of QSO has been promptly developing through numerous publications, where the existing studies can be classified into two sets, namely finite and infinite state space. The most prominent QSO study on a finite state space is the study of Volterra QSO [21] due to its accessibility in applying renowned mathematical techniques such as dynamical systems theory, linear algebra, convex analysis, etc. The compelling form of the systems generated by the Volterra QSO has preceded the extension of the investigation to infinite cases [18, 19]. The noteworthy findings of the QSO study on an infinite-dimensional setting allow mathematicians to discover the properties of the operator by introducing different QSO classes on infinite state space [5–11].

Recently, researchers have conducted studies on the classes of QSO on an infinite state space. These works have incorporated the concept of measurable partitions on the state space [13–16]. The research of the dynamics of classes of quadratic stochastic operators, specifically Geometric QSO and Poisson QSO, formed by two measurable partitions on a countable state space, has been thoroughly conducted and extensively described in [13, 14, 16]. Meanwhile, in [15], the concept of measurable partitions is applied to Lebesgue QSO with nonnegative integer parameters that are specified on a continuous state space.

Currently, most studies of the classes of QSO on the countable state space focused on two measurable partitions (see [13, 14, 16]), which limits the analysis to characteristics of two distinct groups. Previous works on Geometric QSO and Poisson QSO [13,14,16] mainly discussed the regular property of such operators through the existence of fixed points, either they are attracting or repelling, since the 2-partition can be represented into a one-dimensional map. Considering the representation of 3-partition by a two-dimensional map may result to the study of an extra behavior of fixed point, namely saddle, we are motivated to extend the study to three measurable partitions to uncover additional properties of these operators. This include a whole process of constructing the QSO generated by 3-partition, followed by the representation of the operators into a system of equations. From here, we will work on the finding of the unique fixed point of the system of equations based on existing theorems and propositions. Some prominent techniques and methods will be used to examine the behavior of the fixed point.

Accordingly, this research paper will establish some forms of QSO classes created by a 3-measurable partition. These classes will be categorized and their dynamics will be further analyzed. Some examples of Geometric QSO and Poisson QSO generated by 3-partition will be demonstrated as a part of the results. Also, we aim to provide evidences of the fixed points to be saddle through an analysis on the presence of a set of points known as a stable manifold of such a saddle fixed point.

The paper is structured in the following manner. Section 2 of the paper introduces the preliminary concepts, including the definitions of QSO and measurable partitions. In Section 3, we outline the process of constructing the QSO created by the 3-partition, provide a detailed study of the dynamics of the operators, present some examples of the trajectory behavior of Geometric QSO and Poisson QSO, and lastly, discuss the behavior of saddle fixed points of such operators through the existence of the stable manifold of the fixed points.

2. Preliminaries

In this section, we provide necessary details to address the key notion of QSO and measurable partitions.

2.1. **Quadratic stochastic operators.**

The quadratic stochastic operator (QSO) has gained significant recognition as a valuable analytical tool for studying dynamical properties and modelling across various fields of study. In a thorough and methodical explanation of the dynamics of quadratic stochastic operators, Ganikhodjaev, Mukhamedov, and Rozikov [12] address the key issues in the QSO theory, including constructions, dynamics, regularity, and more.

Assume X is a state space and $\mathcal{F}$ is a σ-algebra of subsets of X. We denote $(X, \mathcal{F})$ and $S(X, \mathcal{F})$ as a measurable space and a set of all probability measures on such a measurable space, respectively. We then define a family of functions $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ on $X \times X \times \mathcal{F}$ with the following conditions:

(i) for any $x, y \in X$, $P(x, y, \cdot)$ is a probability measure, where $P(x, y, \cdot) : \mathcal{F} \to [0, 1]$,
(ii) $P(x, y, A)$ is a jointly measurable function with a fixed $A \in \mathcal{F}$, and
(iii) $P(x, y, A) = P(y, x, A)$.

A QSO $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$ is defined as follows:

$$(V\mu)(A) = \int_X \int_X P(x, y, A) d\mu(x) d\mu(y) \tag{1}$$

for every $\mu \in S(X, \mathcal{F})$ and $A \in \mathcal{F}$. Note that, this operator is called a quadratic stochastic operator (see [2, 4]).

Given a finite state space $X = \{1, 2, \dots\}$ and a corresponding σ-algebra $\mathcal{F}$ is a power set, $P(X)$. Then, $S(X, \mathcal{F})$ is known as an $(m-1)$-dimensional simplex with the following form:

$$S(X, \mathcal{F}) \equiv S^{m-1} = \{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R} : x_i \geq 0, i = 1, \dots, m, \sum_{i=1}^{m} x_i = 1\}.$$

Provided that the probability measure $P(i, j, \cdot)$ is a discrete measure, where $P(i, j, \{k\})$ can be written as $P_{ij,k}$ and $\sum_{k=1}^{m} P_{ij,k} = 1$, a corresponding QSO V is defined as follows:

Definition 1. *A quadratic stochastic operator V is a mapping of $V : S^{m-1} \to S^{m-1}$ for any $\mathbf{x} = (x_1, \ldots, x_m) \in S^{m-1}$ and $V\mathbf{x}$ is defined as*

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \tag{2}$$

where the coefficients $P_{ij,k}$ conform to the conditions:

$$P_{ij,k} \geq 0, P_{ij,k} = P_{ji,k}, \text{ and } \sum_{k=1}^{m} P_{ij,k} = 1 \text{ for } i, j, k = 1, \ldots, m.$$

In this work, we consider examples of QSO defined on the countable state space X. Thus, we shall provide the definition of Geometric QSO and Poisson QSO as follows:

Definition 2. *A QSO V in (2) is called a Geometric QSO if for any $i, j \in X$, where $X = \{0, 1, \ldots\}$, the probability measure $P(i, j, \cdot)$ is the Geometric distribution $G_{r_{ij}}(k) = (1 - r_{ij})\, r_{ij}^{k}$ with a real parameter $r_{ij} = r_{ji}$, $0 < r_{ij} < 1$.*

Definition 3. *A QSO V in (2) is called a Poisson QSO if for any $i, j \in X$, where $X = \{0, 1, \ldots\}$, the probability measure $P(i, j, \cdot)$ is the Poisson distribution $P_{\Lambda_{ij}}(k) = \exp^{-\Lambda_{ij}} \frac{\Lambda_{ij}^{k}}{k!}$ with a positive real parameter Λ_{ij} such that $\Lambda_{ij} = \Lambda_{ji}$.*

Throughout this article, the specified definitions will be used to construct the QSO. The concept of QSO generated by measurable partitions is presented in the following subsection.

2.2. **Quadratic stochastic operators generated by measurable partitions.** This subsection discusses the investigation of QSO generated by measurable partitions. The definition of measurable m-partition is provided below to serve as an overview of the concept of measurable partition that is emphasised in this study.

Definition 4. *A measurable partition of X is a partition such that each of its elements is a measurable set.*

Remark 1. *If $\mathcal{F}$ is a σ-algebra of X and A is a subset of X, then A is called measurable if A is a member of $\mathcal{F}$.*

Let $\xi = \{A_1, \ldots, A_m\}$ be a measurable m-partition of X and $\varsigma = \{B_{ij} : i, j = 1, \ldots, m\}$ be a corresponding partition of $X \times X$, where $B_{ii} = A_i \times A_i$ and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i \neq j$ and $i, j = 1, \ldots, m$. We choose a family of probability measures denoted by $\{\mu_{ij} : i, j = 1, \ldots, m\}$ on a measurable space $(X, \mathcal{F})$ and define a probability measure $P(x, y, A)$ with $(x, y) \in B_{ij}$ as follows:

$$P(x, y, A) = \mu_{ij}(A),$$

for any measurable set $A \in F$. Hence, for an arbitrary $\lambda \in S(X,\mathcal{F})$,

$$\begin{aligned} V\lambda(A) &= \int_X \int_X P(x,y,A)d\lambda(x)d\lambda(y) \\ &= \sum_{i,j=1}^{m} \int_{A_i} \int_{A_j} \mu_{ij}(A)d\lambda(x)d\lambda(y) \\ &= \sum_{i,j=1}^{m} \mu_{ij}(A)\lambda(A_i)\lambda(A_j). \end{aligned}$$

By a mathematical induction, it is evident that

$$\begin{aligned} V^{n+1}\lambda(A) &= \int_X \int_X P(x,y,A)dV^n\lambda(x)dV^n\lambda(y) \\ &= \sum_{i,j=1}^{m} \int_{A_i} \int_{A_j} \mu_{ij}(A)dV^n\lambda(x)dV^n\lambda(y) \\ &= \sum_{i,j=1}^{m} \mu_{ij}(A)V^n\lambda(A_i)V^n\lambda(A_j), \end{aligned}$$

with

$$V^{n+1}\lambda(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k)V^n\lambda(A_i)V^n\lambda(A_j) \tag{3}$$

by assuming that $\{V^n\lambda : n = 0,1,\dots\}$ is the trajectory of the initial point λ, where $V^{n+1}\lambda = V(V^n\lambda)$ with $V^0\lambda = \lambda$.

In measure theory, it is understood that $S(X,\mathcal{F})$ is a weak compact, if X is a compact metric space. For a measurable space $(X,\mathcal{F})$, a sequence μ_n is said to converge strongly to a limit μ if

$$\lim_{n\to\infty} \mu_n(A) = \mu(A),$$

for every set $A \in \mathcal{F}$.

Definition 5. *A quadratic stochastic operator V is called a regular (weak regular), for any initial measure $\lambda \in S(X,\mathcal{F})$, where the strong limit (respectively weak limit),*

$$\lim_{n\to\infty} V^n(\lambda) = \mu,$$

exists.

Consider $x_k^{(n)} = V^n\lambda(A_k)$, where $\left(x_1^{(n)},\dots,x_m^{(n)}\right) \in S^{m-1}$ and $P_{ij,k} = \mu_{ij}(A_k)$. Given a fact that S^{m-1} is the $(m-1)$-dimensional simplex, then the system of equations in (3) can be written as follows:

$$(W\mathbf{x})_k = \sum_{i,j=1}^{k} P_{ij,k} x_i x_j, \tag{4}$$

for all $k = 1, \ldots, m$.

The fundamental system of equations generated for the developed QSOs in this study will be the equation in (4). Upon the construction of the QSOs represented by such as system of equations, we will examine the stability of the system's fixed points and periodic points to analyse the operators' dynamics.

3. Dynamics of Quadratic Stochastic Operators Generated by 3-Partition

In this section, the construction of QSO generated by 3-partition will be demonstrated, followed by the classification of such operators for some cases and their dynamics.

First, let us define a measurable 3-partition $\xi = (A_1, A_2, A_3)$ on the state space X, where its corresponding partition on $X \times X$ is denoted by ς, where $\varsigma = (B_{11}, B_{22}, B_{33}, B_{12}, B_{13}, B_{23})$. We select a family $\{\mu_{ij} : i, j = 1, 2, 3\}$ of Geometric and Poisson distribution with a set of parameters $\{r_{11} = r_1, r_{22} = r_2, r_{33} = r_3, r_{12} = r_4, r_{13} = r_5, r_{23} = r_6\}$ and $\{\Lambda_{11} = \Lambda_1, \Lambda_{22} = \Lambda_2, \Lambda_{33} = \Lambda_3, \Lambda_{12} = \Lambda_4, \Lambda_{13} = \Lambda_5, \Lambda_{23} = \Lambda_6\}$, respectively. Subsequently, we define the probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu_{ij}(A) \ if \ (x, y) \in B_{ij}, i, j = 1, 2, 3, \tag{5}$$

for any $A \in \mathcal{F}$. Then, we describe the following:

$$A(\mu) = \sum_{k \in A_1} \mu(k), B(\mu) = \sum_{k \in A_2} \mu(k), \ and \ C(\mu) = \sum_{k \in A_3} \mu(k).$$

Thus, by the family of measures (5), one can define the following operator V:

$$\begin{aligned}
V\mu\,(k) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij,k} \mu(i) \mu(j) \\
&= \sum_{i \in A_1} \sum_{j \in A_1} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_2} \sum_{j \in A_2} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_3} \sum_{j \in A_3} P_{ij,k} \mu(i)\mu(j) \\
&\quad \sum_{i \in A_1} \sum_{j \in A_2} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_2} \sum_{j \in A_1} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_1} \sum_{j \in A_3} P_{ij,k} \mu(i)\mu(j) \\
&\quad \sum_{i \in A_3} \sum_{j \in A_1} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_2} \sum_{j \in A_3} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_3} \sum_{j \in A_2} P_{ij,k} \mu(i)\mu(j) \\
&= \mu_1(k) A^2(\mu) + \mu_2(k) B^2(\mu) + \mu_3(k) C^2(\mu) \\
&\quad + 2\mu_4(k) A(\mu) B(\mu) + 2\mu_5(k) A(\mu) C(\mu) + 2\mu_6(k) B(\mu) C(\mu),
\end{aligned}$$

where by a mathematical induction, it gives us

$$\begin{aligned} V^{n+1}\mu\,(k) &= \mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &\quad + 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &\quad + 2\mu_6(k)B(V^n\mu)C(V^n\mu) \end{aligned} \tag{6}$$

with

$$\begin{aligned} A\left(V^{n+1}\mu(k)\right) = \sum_{k\in A_1} &\{\mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &+ 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &+ 2\mu_6(k)B(V^n\mu)C(V^n\mu)\}, \end{aligned} \tag{7}$$

$$\begin{aligned} B\left(V^{n+1}\mu(k)\right) = \sum_{k\in A_2} &\{\mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &+ 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &+ 2\mu_6(k)B(V^n\mu)C(V^n\mu)\}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} C\left(V^{n+1}\mu(k)\right) = \sum_{k\in A_3} &\{\mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &+ 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &+ 2\mu_6(k)B(V^n\mu)C(V^n\mu)\}, \end{aligned} \tag{9}$$

where $n = 0, 1, \ldots$.

The recurrent equations in (7), (8), and (9) are the constructed QSOs, which can be rewitten as the following system of equations:

$$\begin{aligned} (W\mathbf{x})_1 &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3, \\ (W\mathbf{x})_2 &= b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3, \\ (W\mathbf{x})_3 &= c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + 2c_{23}x_2x_3, \end{aligned} \tag{10}$$

where

$$\begin{aligned} a_{11} = P_{11,1}, a_{22} = P_{22,1}, a_{33} = P_{33,1}, a_{12} = P_{12,1}, a_{13} = P_{13,1}, a_{23} = P_{23,1}, \\ b_{11} = P_{11,2}, b_{22} = P_{22,2}, b_{33} = P_{33,2}, b_{12} = P_{12,2}, b_{13} = P_{13,2}, b_{23} = P_{23,2}, \\ c_{11} = P_{11,3}, c_{22} = P_{22,3}, c_{33} = P_{33,3}, c_{12} = P_{12,3}, c_{13} = P_{13,3}, c_{23} = P_{23,3}, \end{aligned} \tag{11}$$

are arbitrary coefficients in $(0, 1)$. It is clear that these parameters rely on the 3-partition $\xi = \{A_1, A_2, A_3\}$. Note that $P_{ij,k} = \mu_{ij}(A_k)$, then $a_{ij} + b_{ij} + c_{ij} = 1$ for $i, j = 1, 2, 3$.

Saburov and Yusof [20] defined a QSO $Q : S^2 \to S^2$ called a positive QSO as follows:

$$Q(\mathbf{x}) = \left(\sum_{i,j=1}^{3} p_{ij} x_i x_j, \sum_{i,j=1}^{3} q_{ij} x_i x_j, \sum_{i,j=1}^{3} r_{ij} x_i x_j \right)^T, \tag{12}$$

where $p_{ij}, q_{ij}, r_{ij} > 0$ and $p_{ij} + q_{ij} + r_{ij} = 1$ with $p_{ij} = p_{ji}$, $q_{ij} = q_{ji}$, and $r_{ij} = r_{ji}$ for $1 \le i, j \le 3$.

Remark 2. *Let $p_1 \neq p_2$ and $q_1 \neq q_2$. It is apparent that two quadratic equations $x^2 + p_1 x + q_1 = 0$ and $x^2 + p_2 x + q_2 = 0$ have a unique common root if and only if their resultant is equal to zero, i.e.,*

$$(q_2 - q_1)^2 + p_1(q_2 - q_1)(p_1 - p_2) + q_1(p_1 - p_2)^2 = 0.$$

In this case, the only common root is $x = \dfrac{q_2 - q_1}{p_1 - p_2}$.

Now, let us define the following constants.

$$\alpha_{11} = p_{11} - 2p_{13} + p_{33}, \alpha_{22} = p_{22} - 2p_{23} + p_{33}, \alpha_{12} = p_{12} - p_{13} - p_{23} + p_{33},$$
$$\alpha_1 = p_{13} - p_{33}, \alpha_2 = p_{23} - p_{33}, \alpha_0 = p_{33},$$
$$\beta_{11} = q_{11} - 2q_{13} + q_{33}, \beta_{22} = q_{22} - 2q_{23} + q_{33}, \beta_{12} = q_{12} - q_{13} - q_{23} + q_{33},$$
$$\beta_1 = q_{13} - q_{33}, \beta_2 = q_{23} - q_{33}, \beta_0 = q_{33},$$
$$\gamma_0 = \beta_0\alpha_{11} - \alpha_0\beta_{11}, \gamma_1 = (2\beta_2 - 1)\alpha_{11} - 2\alpha_2\beta_{11}, \gamma_2 = \alpha_{11}\beta_{22} - \alpha_{22}\beta_{11},$$
$$\delta_0 = (2\alpha_1 - 1)\beta_{11} - 2\beta_1\alpha_{11}, \delta_1 = \alpha_{12}\beta_{11} - \beta_{12}\alpha_{11}, \Delta_1 = \gamma_2\delta_0^2 - 2\gamma^1\delta_0\delta_1 + 4\gamma_0\delta_1^2,$$
$$\lambda_0 = \alpha_{11}\gamma_0^2 + (2\alpha_1 - 1)\gamma_0\delta_0 + \alpha_0\delta_0^2, \lambda_4 = \alpha_{11}\gamma_2^2 + 4\alpha_{12}\gamma_2\delta_1 + 4\alpha_{22}\delta_1^2,$$
$$\lambda_3 = 2\alpha_{11}\gamma_2\gamma_1 + 2\alpha_{12}\gamma_2\delta_0 + 4\alpha_{12}\gamma_1\delta_1 + 4\alpha_1\gamma_2\delta_1 - 2\gamma_2\delta_1 + 4\alpha_{22}\delta_1\delta_0 + 8\alpha_2\delta_1^2,$$
$$\begin{aligned} \lambda_2 &= 2\alpha_{11}\gamma_2\gamma_0 + \alpha_{11}\gamma_1^2 + 2\alpha_{12}\gamma_1\delta_0 + 4\alpha_{12}\gamma_0\delta_1 + 2\alpha_1\gamma_2\delta_0 + 4\alpha_1\gamma_1\delta_1 \\ &= \gamma_2\delta_0 - 2\gamma_1\delta_1 + \alpha_{22}\delta_0^2 + 8\alpha_2\delta_1\delta_0 + 4\alpha_0\delta_1^2, \end{aligned}$$
$$\lambda_1 = 2\alpha_{11}\gamma_1\gamma_0 + 2\alpha_{12}\gamma_0\delta_0 + 2\alpha_1\gamma_1\delta_0 + 4\alpha_1\gamma_0\delta_1 - \gamma_1\delta_0 - 2\gamma_0\delta_1 + 2\alpha_2\delta_0^2 + 4\alpha_0\delta_1\delta_0.$$

Theorem 1. *[20] Let $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. The positive quadratic stochastic operator $Q : S^2 \to S^2$ has a unique fixed point (a stationary distribution) if and only if the quartic equation,*

$$\lambda_4 p^4 + \lambda_3 p^3 + \lambda_2 p^2 + \lambda_1 p + \lambda_0 = 0,$$

has a unique real root $p_0 \in (0,1) \setminus \left\{ -\frac{\delta_0}{2\delta_1} \right\}$ which satisfies $0 < P_0 < 1$ and $0 < Q_0 < 1$, where

$$P_0 = \frac{\gamma_2 p_0^2 + \gamma_1 p_0 + \gamma_0}{2\delta_1 p_0 + p_0},$$

$$Q_0 = \frac{(\gamma_2 + 2\delta_1)p_0^2 + (\gamma_1 + \delta_0)p_0 + \gamma_0}{2\delta_1 p_0 + \delta_0}.$$

Moreover, in this case, the only fixed point (a stationary distribution) is $(P_0, p_0, 1 - Q_0)^T$.

According to Theorem 1, it signifies that the system of equations in (10) has a unique fixed point for any 3-measurable partition on the state space X. This implies that we can formulate the form of the fixed point of such a two-dimensional operator W generated by 3-partition ξ.

Suppose that the operator W in (10) has a fixed point. Then, we will have the following system of equations:

$$\begin{aligned} x_1 &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3, \\ x_2 &= b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3, \\ x_3 &= c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + 2c_{23}x_2x_3. \end{aligned} \tag{13}$$

Since the operator W in (10) is in the same form as the operator Q in (12), we shall apply the defined constants with $a_{ij} = p_{ij}$, $b_{ij} = q_{ij}$, and $c_{ij} = r_{ij}$. Hence, the following statement may be established.

Proposition 1. *Let* $W : S^2 \to S^2$. *For the operator* W *in (10), the following statements hold true.*

(1) $|Fix(W)| = 1$,

(2) the unique fixed point $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) \in S^2$ *has the following form:*

$$\begin{aligned} x_1^* &= \frac{\gamma_2 p_0^2 + \gamma_1 p_0 + \gamma_0}{2\delta_1 p_0 + p_0}, \\ x_2^* &= p_0 \in (0,1), \\ x_3^* &= \frac{(\gamma_2 + 2\delta_1)p_0^2 + (\gamma_1 + \delta_0)p_0 + \gamma_0}{2\delta_1 p_0 + \delta_0}. \end{aligned}$$

In Lyubich's study [17], it was proven that a one-dimensional QSO may have either an attracting fixed point or a repelling fixed point that tends to a cycle of second-order depending on the value of discriminant of the following one-variable function:

$$f(x_1) = (a - 2b + c)\, x_1 + 2\,(b - c)\, x_1 + c, \tag{14}$$

where $0 \leq a, b, c \leq 1$ with the value of discriminant Δ of $f(x_1) = x_1$, where

$$\Delta = 4(1-a)c + (1-2b)^2, \tag{15}$$

for the system of equations as follows:

$$\begin{aligned} W(x_1) &= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2, \\ W(x_2) &= b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2, \end{aligned} \tag{16}$$

for $a_{11} = a$, $a_{12} = b$, $a_{22} = c$, and $a_{ij} + b_{ij} = 1$. As a result, the following assertions are established.

Theorem 2. *[17] A fixed point of the transformation (16) is a unique and belongs to the open interval* $(0,1)$*. The fixed point is attracting if* $0 < \Delta < 4$ *and is repelling if* $4 < \Delta < 5$*.*

Theorem 3. *[17] If* $0 < \Delta < 4$*, then all trajectories converge to a fixed point. If* $4 < \Delta < 5$*, then there exists a cycle of second-order and all trajectories tend to this cycle except the stationary trajectory starting with fixed point.*

Apparently, we may utilize the idea of attracting and repelling fixed points on a one-dimensional map to determine the existence of periodic points of period-2 of the system of equations in (16). Meanwhile, for the system of equations in (10), we may use the notion of non-attracting fixed points instead of repelling fixed points due to the consideration of another type of fixed point, i.e., saddle fixed point on a two-dimensional map. It is known that if a fixed point of such a system of equations is non-attracting, then there exist periodic points of period-2.

The first derivative of the quadratic function (14) with respect to one variable and its discriminant are applied to check the local behavior of the fixed point. However, the same method cannot be implied due to the multivariable functions derived from the system of equations in (10).

Definition 6. *[1] Let* $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ *be a map on* $\mathbb{R}^m$*, and let* $\mathbf{x}^* \in \mathbb{R}^m$*. The Jacobian matrix of* $\mathbf{f}$ *at* $\mathbf{x}^*$*, denoted* $J(\mathbf{x}^*)$*, is the matrix*

$$J(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial f_1}{\partial x_m}(\mathbf{x}^*) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial f_m}{\partial x_m}(\mathbf{x}^*) \end{pmatrix}$$

of partial derivatives evaluated at $\mathbf{p}$*.*

Remark 3. *Given a system*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The key to solving the system is by determining the eigenvalues of $\mathbf{A}$*. To find these eigenvalues, we need to derive the characteristic polynomial of* $\mathbf{A}$*.*

$$det(\mathbf{A} - \lambda \mathbf{I}) = det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - (a-d)\lambda + (ad - bc).$$

Surely, $D = det(\mathbf{A}) = ad - bc$ *is the determinant of* $\mathbf{A}$*. Meanwhile, the quantity* $T = a + d$ *is the sum of the diagonal elements of the matrix* $\mathbf{A}$ *is called as the trace of* $\mathbf{A}$ *and written as* $tr(\mathbf{A})$*. It is given that the eigenvalues of* $\mathbf{A}$ *are represented by*

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Consequently, the Jacobian matrix can be implied to investigate the local behavior of the fixed point on a two-dimensional map.

Assume that $\mathbf{x}^* = (x_1^*, x_2^*) = (P_0, p_0)$ and the multivariable functions derived from the system of equations in (10) are as follows:

$$f_1(x_1, x_2) = \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + 2\alpha_{12}x_1x_2 + 2\alpha_1x_1 + 2\alpha_2x_2 + \alpha_0, \tag{17}$$

$$f_2(x_1, x_2) = \beta_{11}x_1^2 + \beta_{22}x_2^2 + 2\beta_{12}x_1x_2 + 2\beta_1x_1 + 2\beta_2x_2 + \beta_0. \tag{18}$$

The Jacobian matrix $J(\mathbf{x}^*)$ of (17) and (18) has the following representation:

$$J(x_1^*, x_2^*) = \begin{pmatrix} 2\alpha_{11}x_1^* + 2\alpha_{12}x_2^* + 2\alpha_1 & 2\alpha_{12}x_1^* + 2\alpha_{22}x_2^* + 2\alpha_2 \\ 2\beta_{11}x_1^* + 2\beta_{12}x_2^* + 2\beta_1 & 2\beta_{12}x_1^* + 2\beta_{22}x_2^* + 2\beta_2 \end{pmatrix}. \tag{19}$$

For simplicity, we will use $\alpha = 2\alpha_{11}x_1^* + 2\alpha_{12}x_2^* + 2\alpha_1$, $\beta = 2\alpha_{12}x_1^* + 2\alpha_{22}x_2^* + 2\alpha_2$, $\chi = 2\beta_{11}x_1^* + 2\beta_{12}x_2^* + 2\beta_1$, and $\delta = 2\beta_{12}x_1^* + 2\beta_{22}x_2^* + 2\beta_2$. According to Remark 3, we compute the eigenvalues of the Jacobian $J(\mathbf{x}^*)$, λ_1 and λ_2 in (19), where

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left(\alpha + \delta + \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \beta\chi)}\right), \\ \lambda_2 &= \frac{1}{2}\left(\alpha + \delta - \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \beta\chi)}\right). \end{aligned} \tag{20}$$

Definition 7. *[1] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a second-order autonomous system that has a fixed point at $\mathbf{x}^* \in \mathbb{R}^2$. Suppose that λ_1 and λ_2 be the eigenvalues of $J(\mathbf{x}^*)$. Assuming that neither λ_1 nor λ_2 lies on the boundary of the unit disk, there are three distinct characteristics of the trajectories in the neighborhood of the fixed point $\mathbf{x}^*$.*

- *(i) If $|\lambda_i| < 1$ for $i = 1, 2$, then all trajectories converge to $\mathbf{x}^*$, i.e.,$\mathbf{x}^*$ is an attracting fixed point.*
- *(ii) If $|\lambda_1| < 1$, $|\lambda_2| > 1$ or $|\lambda_1| > 1$, $|\lambda_2| < 1$, then the fixed point $\mathbf{x}^*$ is a saddle fixed point. From the stable direction that corresponds to the eigen-direction for the stable eigenvalue $|\lambda_i|$, where $|\lambda_i| < 1$ for $i = 1, 2$, as $n \to \infty$. From the unstable direction, corresponding to the eigen-direction for the unstable eigenvalue $|\lambda_i|$, where $|\lambda_i| > 1$ for $i = 1, 2$, the trajectories $\mathbf{x}^{(n)}$ move away from $\mathbf{x}^*$ as $n \to \infty$. All other trajectories follow hyperbola-like paths, i.e., at first moving closer to $\mathbf{x}^*$, and then moving away from $\mathbf{x}^*$.*
- *(iii) If $|\lambda_i| > 1$ for $i = 1, 2$, then all trajectories move away from the fixed point $\mathbf{x}^*$, so $\mathbf{x}^*$ is a repelling fixed point.*

From the Jacobian matrix in (19), one may find that $-2 < \alpha, \beta, \chi, \delta < 2$, given the fact that such coefficients are defined from the system of equations in (13). We shall let $\gamma = \alpha\delta - \beta\chi$ and $D = (\alpha+\delta)^2 - 4T$. Based on the form of eigenvalues of $J(\mathbf{x}^*)$ in (19) and Definition 6, we shall classify the eigenvalues as follows:

- (i) if $T > 0$, $(\alpha+\delta)^2 < 4T$, and $(\alpha+\delta) = \pm 2$, then the fixed point is nonhyperbolic;
- (ii) if $T > 0$, $(\alpha+\delta)^2 < 4T$, and $|\alpha+\delta| < 2$, then the fixed point is attracting;
- (iii) if $T > 0$, $(\alpha+\delta)^2 < 4T$, and $|\alpha+\delta| > 2$, then the fixed point is repelling;

(iv) if $T > 0$, $(\alpha+\delta)^2 > 4T$, and $-2 < \alpha+\delta \pm \sqrt{D} < 2$, then the fixed point is attracting;
(v) if $T > 0$, $(\alpha+\delta)^2 > 4T$, and $-2 < \alpha+\delta+\sqrt{D} < 2$, and $\alpha+\delta-\sqrt{D} < -2$, then the fixed point is saddle;
(vi) if $T > 0$, $(\alpha+\delta)^2 > 4T$, and $-2 < \alpha+\delta-\sqrt{D} < 2$, and $\alpha+\delta+\sqrt{D} > 2$, then the fixed point is saddle;
(vii) if $T = 0$ and $|\alpha+\delta| < 1$, then the fixed point is attracting;
(viii) if $T = 0$ and $|\alpha+\delta| > 1$, then the fixed point is saddle;
(ix) if $T < 0$ and $-2 < \alpha+\delta \pm \sqrt{D} < 2$, then the fixed point is attracting;
(x) if $T < 0$, $\alpha+\delta > 0$, $\alpha+\delta+\sqrt{D} > 2$, and $0 < \alpha+\delta-\sqrt{D} < 2$ then the fixed point is saddle;
(xi) if $T < 0$, $\alpha+\delta > 0$, $\alpha+\delta-\sqrt{D} < 2$, and $-2 < \alpha+\delta+\sqrt{D} < 0$ then the fixed point is saddle;
(xii) if $T < 0$, $\alpha+\delta > 0$, $\alpha+\delta+\sqrt{D} > 2$, and $\alpha+\delta-\sqrt{D} < -2$ then the fixed point is repelling;
(xiii) if $T < 0$, $\alpha+\delta < 0$, $\alpha+\delta+\sqrt{D} < -2$, and $\alpha+\delta-\sqrt{D} > 2$ then the fixed point is repelling.

Now, we shall analyze the fixed point of the system of equations in (10) based on the given eigenvalues classification. We shall consider a case of 3-partition ξ to investigate the type of fixed point of such operators by the following conditions of the defined parameters:

(1) $\mu_{11} = \mu_{13} = \mu_{33} \neq \mu_{12} = \mu_{23} \neq \mu_{22}$,
(2) $\mu_{11} = \mu_{12} = \mu_{22} \neq \mu_{13} = \mu_{23} \neq \mu_{33}$,
(3) $\mu_{22} = \mu_{23} = \mu_{33} \neq \mu_{12} = \mu_{13} \neq \mu_{11}$.

Given such conditions, we shall obtain the following systems of equations:

$$\begin{aligned}
(W_1\mathbf{x})_1 &= a_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + a_{22}x_2^2 + 2a_{12}\left(x_1x_2 + x_2x_3\right),\\
(W_1\mathbf{x})_2 &= b_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + b_{22}x_2^2 + 2b_{12}\left(x_1x_2 + x_2x_3\right),\\
(W_1\mathbf{x})_3 &= c_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + c_{22}x_2^2 + 2c_{12}\left(x_1x_2 + x_2x_3\right),
\end{aligned} \tag{21}$$

$$\begin{aligned}
(W_2\mathbf{x})_1 &= a_{22}\left(x_1^2 + 2x_1x_2 + x_2^2\right) + a_{33}x_3^2 + 2a_{23}\left(x_1x_3 + x_2x_3\right),\\
(W_2\mathbf{x})_2 &= b_{22}\left(x_1^2 + 2x_1x_2 + x_2^2\right) + b_{33}x_3^2 + 2b_{23}\left(x_1x_3 + x_2x_3\right),\\
(W_2\mathbf{x})_3 &= c_{22}\left(x_1^2 + 2x_1x_2 + x_2^2\right) + c_{33}x_3^2 + 2c_{23}\left(x_1x_3 + x_2x_3\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
(W_3\mathbf{x})_1 &= a_{33}\left(x_2^2 + 2x_2x_3 + x_3^2\right) + a_{11}x_1^2 + 2a_{13}\left(x_1x_2 + x_1x_3\right),\\
(W_3\mathbf{x})_2 &= b_{33}\left(x_2^2 + 2x_2x_3 + x_3^2\right) + b_{11}x_1^2 + 2b_{13}\left(x_1x_2 + x_1x_3\right),\\
(W_3\mathbf{x})_3 &= c_{33}\left(x_2^2 + 2x_2x_3 + x_3^2\right) + c_{11}x_1^2 + 2c_{13}\left(x_1x_2 + x_1x_3\right).
\end{aligned} \tag{23}$$

We shall denote the operators in (21), (22), and (23) as operators from class $C_1 = \{W_1, W_2, W_3\}$, identified as reducible two-dimensional QSOs due to their ability to be reduced to a one-dimensional setting.

Proposition 2. *Let $\mathbf{x}^* \in S^2$ be a fixed point of the operator W in (10) and λ_i for $i = 1, 2$ are eigenvalues of Jacobian $J(\mathbf{x}^*)$ in (19). For the operators from class C_1, the fixed point $\mathbf{x}^*$ is either attracting or saddle.*

Proof. Let us consider the first operator from class C_1, i.e., the operator W_1 in (21). Referring to the system of equations in (21) and the Jacobian $J(\mathbf{x}^*)$ in (19), we will obtain the following Jacobian matrix,

$$J(\mathbf{x}^*) = \begin{pmatrix} 0 & \beta \\ 0 & \delta \end{pmatrix}.$$

Hence, we have $T = 0$ and $D > 0$. It follows that $\lambda_1 = 0$ and $\lambda_2 = \delta$ when $\delta < 0$, while $\lambda_1 = \delta$ and $\lambda_2 = 0$ when $\delta > 0$. Apparently, if $|\delta| < 1$, then $\mathbf{x}^*$ is an attracting fixed point. Meanwhile, if $|\delta| > 1$, then $\mathbf{x}^*$ is a saddle fixed point.

Next, we shall consider the operator in (22). Considering the Jacobian $J(\mathbf{x}^*)$ in (19), we will get,

$$J(\mathbf{x}^*) = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix},$$

where $\alpha = \beta \neq \chi = \delta$. Consequently, $T = 0$ and when $\alpha + \delta < 0$, we have $\lambda_1 = 0$ and $\lambda_2 = \alpha + \delta$, while when $\alpha + \delta > 0$, we have $\lambda_1 = \alpha + \delta$ and $\lambda_2 = 0$. Therefore, for the operator W_2, the fixed point $\mathbf{x}^*$ is attracting if $|\alpha + \delta| < 1$, and is saddle if $|\alpha + \delta| > 1$.

Lastly, for the operator W_3 in (23), we may obtain the following Jacobian $J(\mathbf{x}^*)$, where

$$J(\mathbf{x}^*) = \begin{pmatrix} \alpha & 0 \\ \chi & 0 \end{pmatrix}.$$

This follows that $T = 0$ and $D = \alpha^2$. Subsequently, we get $\lambda_1 = 0$ and $\lambda_2 = \alpha$ when $\alpha < 0$, while when $\alpha > 0$, we have $\lambda_1 = \alpha$ and $\lambda_2 = 0$. Then, it is not difficult to verify that $\mathbf{x}^*$ is an attracting fixed point if $|\alpha| < 1$ and $\mathbf{x}^*$ is a saddle fixed point if $|\alpha| > 1$.

Thus, according to Definition 6, evidently if $|\alpha + \delta| < 1$, then $\mathbf{x}^*$ is an attracting fixed point, where $|\lambda_1| < |\lambda_2| < 1$ or $|\lambda_2| < |\lambda_1| < 1$, while if $|\alpha + \delta| > 1$, then $\mathbf{x}^*$ is a saddle fixed point, where $|\lambda_1| < 1 < |\lambda_2|$ or $|\lambda_2| < 1 < |\lambda_1|$. The analysis of the eigenvalues of the Jacobian of the operators from the class C_1 shows that for such operators, the fixed point $\mathbf{x}^*$ is either attracting or saddle as shown in condition (vii) and (viii). The proof is complete. □

In accordance with Proposition 2, one may discover that for the operator W in (10) classified under the class C_1, there exists either an attracting fixed point or a saddle fixed point for some defined partitions and parameters. Also, it is proven that for such operators, the fixed point can never be repelling.

Assume that the behavior of the operators in the class C_1 may represent the behavior of the QSO W in (10). Accordingly, we may establish the following statements.

Corollary 1. *Let* $\mathbf{x}^*$ *be a fixed point of the operator* W *in (10). Then, the fixed point* $\mathbf{x}^*$ *is either attracting or saddle.*

Proposition 3. *Let* $\mathbf{x}^*$ *be a fixed point of the operator* W *in (10). Then, the following statements hold true.*

(i) If the fixed point $\mathbf{x}^*$ *is attracting, then the trajectory converges to that fixed point.*

(ii) If the fixed point $\mathbf{x}^*$ *is saddle, then there exists a second-order cycle.*

We shall provide some examples using Geometric QSO and Poisson QSO to support the above statements.

Example 1. *Let* $A_1 = \{0, 1, 2\}$, $A_2 = \{6, 7, \dots\}$, *and* $A_3 = \{3, 4, 5\}$ *be the measurable 3-partition for Geometric QSO generated by 3-partition with six parameters. We define* $r_1 = 0.975$, $r_2 = 0.5$,$r_3 = 0.95$, $r_4 = 0.25$, $r_5 = 0.9$ *and* $r_6 = 0.2$. *Due to Proposition 1, the fixed point of such an operator* W *in (10) is as follows:*

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (0.5959277932, 0.3461854580, 0.05788674882) \tag{24}$$

We also obtain the following functions, where

$$\begin{aligned} f_1(x_1, x_2) &= -0.326234375x_1^2 - 0.966375x_2^2 - 2(0.136)x_1x_2 \\ &\quad + 2(0.128375)x_1 + 2(0.849375)x_2 + 0.142625, \\ f_2(x_1, x_2) &= 0.5312781916x_1^2 + 0.7505888906x_2^2 + 2(0.2038310312)x_1x_2 \\ &\quad - 2(0.2036508906)x_1 - 2(0.7350278906)x_2 + 0.7350918906. \end{aligned} \tag{25}$$

Then, the Jacobian $J(\mathbf{x}^*)$ *is as follows:*

$$J(x_1^*, x_2^*) = \begin{pmatrix} -0.2262367068 & 0.8675676963 \\ 0.3670317770 & -0.7074327102 \end{pmatrix},$$

where $T = -0.1583776666$, $\alpha + \delta = -0.933669417$, $\alpha + \delta + \sqrt{D} = 0.2932165790$, *and* $\alpha + \delta - \sqrt{D} = -2.160555413$. *These conform to the condition of a saddle fixed point as stated in (xi), in which* $T < 0$, $\alpha + \delta < 0$, $0 < \alpha + \delta + \sqrt{D} < 2$, *and* $\alpha + \delta = \sqrt{D} < -2$. *Following the Jacobian matrix, the eigenvalues are as follows:*

$$\begin{aligned} \lambda_1 &= 0.1466082895, \\ \lambda_2 &= -1.080277706. \end{aligned}$$

From this, we get $|\lambda_1| < 1 < |\lambda_2|$. *Hence,* $\mathbf{x}^*$ *in (24) is a saddle point. This demonstrates that there exists a cycle of second-order for such an operator.*

Example 2. *Let $A_1 = \{0, 1\}$, $A_2 = \{2, 3\}$, and $A_3 = \{4, 5, \dots\}$ be the measurable 3-partition for Poisson QSO generated by 3-partition with six parameters. Define $\Lambda_1 = 5.25$, $\Lambda_2 = 5.0, \Lambda_3 = 1.75$, $\Lambda_4 = 4.75$, $\Lambda_5 = 0.95$ and $\Lambda_6 = 1.0$. Due to Proposition 1, we shall obtain the fixed point of such an operator W in (10) as follows:*

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (040777949974, 0.2535537737, 0.3386512289) \tag{26}$$

Also, the following functions are obtained:

$$\begin{aligned} f_1(x_1, x_2) &= -0.9976146574x_1^2 - 0.9532117384x_2^2 - 2(0.9622782864)x_1x_2 \\ &\quad + 2(0.2762666512)x_1 + 2(0.2578805378)x_2 + 0.4778783446, \\ f_2(x_1, x_2) &= 0.1606229184x_1^2 + 0.1554036174x_2^2 + 2(0.1984161132)x_1x_2 \\ &\quad - 2(0.1915307380)x_1 - 2(0.1760583449)x_2 + 0.4213113057. \end{aligned} \tag{27}$$

Then, the Jacobian $J(\mathbf{x}^*)$ is as follows:

$$J(x_1^*, x_2^*) = \begin{pmatrix} -0.7490898126 & -0.7524443338 \\ -0.1514407223 & -0.1114841458 \end{pmatrix},$$

where $T = -0.03043907551$, $\alpha + \delta = -0.8605739584$, $\alpha + \delta + \sqrt{D} = 0.0680507451$, and $\alpha + \delta - \sqrt{D} = -1.789198662$. These conform to the condition of a saddle fixed point as stated in (viii), in which $T < 0$ and $-2 < \alpha + \delta \pm \sqrt{D} < 2$. Consequently, the eigenvalues are as follows:

$$\lambda_1 = 0.0340253726,$$
$$\lambda_2 = -0.8945993310.$$

It is notable that $|\lambda_1| < |\lambda_2| < 1$. Hence, $\mathbf{x}^*$ in (26) is an attracting point. This shows that the trajectory of such an operator converges to this fixed point.

From the given examples, it has been demonstrated that such operators may have either an attracting or a saddle fixed point depends on the value of parameters. The discovery of non-attracting fixed point on the two-dimensional setting as a saddle fixed point is considered significant due to an initial assumption that the fixed point should be repelling based on the study of QSOs on one-dimensional simplex. Hence, in the next subsection, we shall discuss the behavior of saddle fixed point to provide a comprehensive finding on the dynamics of such operators generated by 3-partition.

3.1. **Behavior of the saddle fixed point of quadratic stochastic operators generated by 3-partition.**

Remark 4. *[1] A saddle fixed point is unstable. Most initial values near it will move away under iteration of the map. However, unlike the case of a repelling fixed point (source), not all nearby initial values will move away. The set of initial values that converge to the saddle will be called the stable manifold of the saddle.*

Definition 8. *[1] Let f be a smooth map on $\mathbb{R}^2$, and let $\mathbf{p}$ be a saddle fixed point or periodic saddle point for f. The stable manifold of $\mathbf{p}$, denoted $S(\mathbf{p})$ is the set of points $\mathbf{v}$ such that $|f^n(\mathbf{v}) - f^n(\mathbf{p})| \to 0$ as $n \to \infty$. The unstable manifold of $\mathbf{p}$, denoted $U(\mathbf{p})$, is the set of points $\mathbf{v}$ such that $|f^{-n}(\mathbf{v}) - f^{-n}(\mathbf{p})| \to 0$ as $n \to \infty$.*

From Definition 7, Remark 4, and Definition 8, the fact that the operator in (10) with a saddle fixed point is unstable, i.e., from a stable direction corresponds to the stable eigenvalue, the trajectories converge to the fixed point, while from an unstable direction corresponds to the unstable eigenvalue, the trajectories move away from such fixed point. Hence, this conforms the fact that the saddle fixed point indicates the existence of a second-order cycle of the system of equations in (10).

Verification of the saddle fixed point as the unstable fixed point of the operator in (10) and the fixed point of such an operator can never be repelling is rather ambiguous. This comes from the fact that the behavior of a repelling fixed point is quite similar to the behavior of a saddle fixed point, where all trajectories move away from the fixed point except when the initial point is the fixed point itself. Meanwhile, for a saddle fixed point, it behaves as an attractor for some trajectories and a repeller for others. Herewith, we can find a set of points $\mathbf{x} \in S^2$, where $\mathbf{x} \neq \mathbf{x}^*$ in which such points will eventually converge to the saddle fixed point.

Next, we will consider some examples of the saddle fixed point case in Example 1, where the presence of the set of points $\mathbf{x} \in S^2$, denoted by $\rho_{\mathbf{a}}$ for $n \to \infty$, where $\rho_{\mathbf{a}} = \left\{\mathbf{a} \in S^2 : \mathbf{a} \neq \mathbf{x}^*, |W^n(\mathbf{a}) - \mathbf{x}^*| \to 0\right\}$ will be provided.

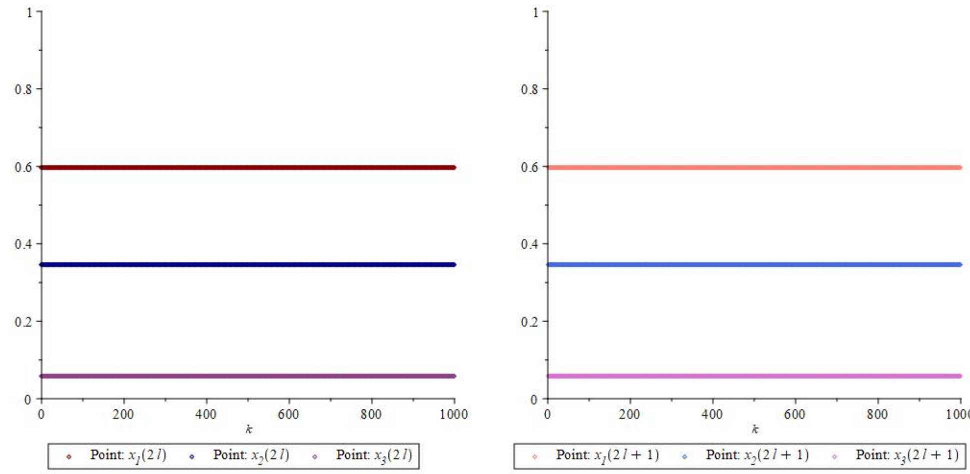
Assume that $\mathbf{a} = (x_1 + \epsilon, x_2 + \epsilon, 1 - x_1 - x_2 - 2\epsilon) = (x_1 + \epsilon, x_2 + \epsilon, x_3 + \epsilon)$, where $\epsilon = m \times 10^{-10}$ with $m = [-100, 100]$. For the operator W in (10) from Example 1, we can find the initial values near the saddle fixed point $\mathbf{x}^*$, where such an operator is regular (see Figure 1), as both even and odd number iterations of x_1 , x_2, and x_3 converge to the same value. Computationally, we obtain that when $-5.5 > m > 4.5$, the trajectories $\mathbf{x}^{(\mathbf{n})}$ approach $\mathbf{x}^*$ as $n \to \infty$.

Figure 1(a) shows Example 1, which indicates the points, x_1, x_2, and x_3 for even iterations, while Figure 1(b) displays the points of x_1, x_2, and x_3 for odd iterations. This demonstrates that both even and odd iterations of the saddle fixed point case operator will converge to the same value when we choose any initial points that belong to the stable manifold.

Contrarily, when we choose any initial values, which are very close to the saddle fixed point, in which $m \leq -5.5$ or $m \geq 4.5$, one can see the behavior of even and odd number iterations of all coordinates do not converge to the same values (refer Figure 2).

We use Figure 2 to illustrate the behavior of points x_1, x_2, and x_3 of the saddle fixed point case operator in Example 1 with six different colors to represent $x_i(2l)$ and $x_i(2l + 1)$ for $i = 1, 2, 3$ and $l = 0, \ldots, 500$.

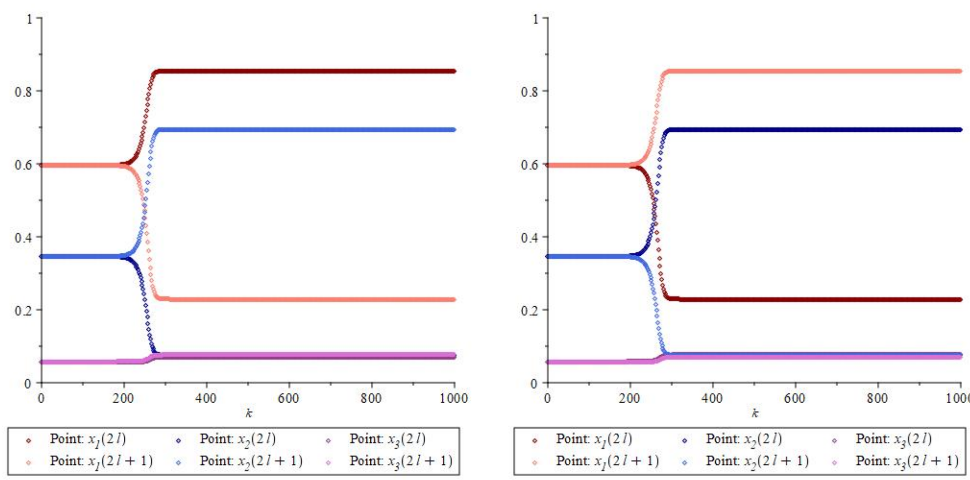
In Figure 1, we show that for some initial values close to the saddle fixed point, the trajectories will eventually converge to the fixed point, indicating the existence



(a) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = -5.4$

(b) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = 4.45$

FIGURE 1. Trajectory behavior of regular transformation of operator W in (10) from Example 1 for $l = 0, \ldots, 500$



(a) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = -5.5$

(b) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = 4.65$

FIGURE 2. Trajectory behavior of nonregular transformation of operator W in (10) from Example 1 for $l = 0, \ldots, 500$

of the set of points $\rho_{\mathbf{a}}$ known as the stable manifold of $\mathbf{x}^*$. Meanwhile, in Figure 2, it is shown that for some relatively close initial values, where $\mathbf{x}^{(0)} \notin \rho_{\mathbf{a}}$, the trajectories will move away from the saddle fixed point $\mathbf{x}^*$ after a number of iterations. Hence, it is evident that the saddle fixed point of the operator W behaves as an attractor for some trajectories and as a repeller for others.

With the given examples as evidence of the existence of the stable manifold of the saddle fixed point of the operator W in (10), we may establish the following statement.

Remark 5. *Let* $\rho_{\mathbf{a}} = \{\mathbf{a} \in S^2 : \mathbf{a} \neq \mathbf{x}^*, |W^n(\mathbf{a}) - \mathbf{x}^*| \to 0, n \to \infty\}$ *be the set of points that belong to the stable manifold of any saddle fixed point* $\mathbf{x}^*$ *of the operator* $W : S^2 \to S^2$ *in (10). Then, the following statements hold true.*

i If $\mathbf{x}^{(0)} \in \rho_{\mathbf{a}}$*, then the operator* W *is regular.*

ii If $\mathbf{x}^{(0)} \notin \rho_{\mathbf{a}}$*, then the operator* W *is nonregular.*

4. Conclusion

The construction of QSOs generated by 3-partition and the formulation of the fixed point form of the system of equations corresponds to such QSOs were presented throughout the paper. By implementing the analysis of the quadratic function (14) on a one-dimensional map, we can determine the existence of periodic points of period-2 through the repelling behavior of the unique fixed point. Unlike the case of one-dimensional map, where we addressed a repelling fixed point to signify the existence of the periodic points of period-2, in the case of a two-dimensional map, we used the notion of non-attracting fixed point to represent both unstable fixed points; i.e., repelling and saddle. Based on the eigenvalues of the Jacobian matrix in (19) of the system of equations (10), we classified the fixed point accordingly.

Further investigation on the dynamics of the QSOs generated by 3-partition was carried out by considering three cases of 3-partition with three parameters, where the corresponding systems of equations denoted as class C_1 can be reduced to a one-dimensional setting. These cases were then implied to explore the behavior of the fixed point through the classification of eigenvalues of the Jacobian matrix in (19), where we established Proposition 2, in which it is proven that such operators may have either an attracting or a saddle fixed point and the fixed point can never be repelling.

We provide some examples using Geometric QSO and Poisson QSO to demonstrate the behavior of the fixed point of the operators through the classification of their eigenvalues. From the obtained results, it is remarked that an attracting fixed point implies the existence of a strong limit, hence the operator is regular. Another example showed that the saddle fixed point indicates the existence of the second-order cycle, where the operator is nonregular.

To illustrate the fact that the fixed point of the operator in (10) can never be repelling, it is necessary to find a set of points denoted by $\rho_{\mathbf{a}}$ that belongs to the stable manifold of the saddle fixed point. We utilized Example 1 with a saddle fixed point and searched for the set of points $\rho_{\mathbf{a}}$. It is shown that for any saddle fixed point of the operator W in (10), there exist some relatively close initial values to the saddle fixed point, which will converge to such a fixed point, while most of the initial values will move away from it. From this, we established the statements in Remark 4.

Author Contribution Statements S. N. Karim and N. Z. A. Hamzah conceived of the presented idea. S. N. Karim developed the theory and performed the computations. N. Z. A. Hamzah verified the analytical methods and encouraged S. N. Karim to investigate the stable manifold notion and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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