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On Pseudo-cyclic Multipliers in Hilbert Function Spaces

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Abstract. Let $\mathcal{H}$ be a separable complete Pick space of continuous functions on a compact set Ω with multiplier algebra $\mathrm{M}(\mathcal{H})$. The notion of the pseudo-cyclicity is recently defined by Aleman et al. In this short paper, we first extend their definition of the pseudo-cyclic multipliers to all functions f in $\mathcal{H}$. Then we show that whenever one-function corona theorem holds for $\mathrm{M}(\mathcal{H})$ then a function f in $\mathcal{H}$ is in the pseudo-cyclic class $C_n(\mathcal{H})$ if and only if $1/f$ is in the corresponding Pick-Smirnov type class $N_n^+(\mathcal{H})$. Furthermore, we show that non-vanishing functions $f \in \mathcal{H}$ are in the class $C_1(\mathcal{H})$. For functions φ, ψ in $\mathrm{M}(\mathcal{H})$, with at least one being in $C_1(\mathcal{H})$, we also show that the invariant subspace generated by $\varphi\psi$ is equal to the intersection of invariant subspaces generated by φ and ψ.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert function space on a non-empty set Ω. Let $\mathrm{Mult}(\mathcal{H})$ denote the multiplier algebra of $\mathcal{H}$, that is, the set of all complex valued functions φ on Ω such that $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$. A function $f \in \mathcal{H}$ is called cylic if $[f] = \mathcal{H}$, where $[f] = \mathrm{clos}_{\mathcal{H}}\{\varphi f : \varphi \in \mathrm{Mult}(\mathcal{H})\}$. The well-known Hilbert function spaces on the unit disc $\mathbb{D}$ are the Hardy space $H^2(\mathbb{D})$, Bergman space $L_a^2(\mathbb{D})$, and Dirichlet space D. An example of Hilbert function space in several complex variables is the Drury-Arveson space H_d^2 of analytic functions on the unit ball $\mathbb{B}_d$.

One of the problems in the analytic function theory is the investigation of cyclic functions. For the Hardy space $H^2(\mathbb{D})$, as a result of the Beurling theorem [6], it is known that cyclic functions are the outer functions. On the other hand, less is known about cyclic functions in the Bergman space $L_a^2(\mathbb{D})$ and the Dirichlet space D. To see more results about these spaces, see [7, 8, 10]. When working on cyclicity in the Drury-Arveson space and other weighted Besov spaces, Aleman et al. [5] introduce the classes $C_n(\mathcal{H})$ of pseudo-cyclic multipliers. In this paper, we first extend their definition of pseudo-cyclicity to all functions in a separable complete Pick space and introduce Pick-Smirnon type classes $N_n^+(\mathcal{H})$. We then prove the following theorem.

Theorem 1.1. *Suppose $\mathcal{H}$ is a separable complete Pick space of continuous functions on a compact set Ω, $f \in \mathcal{H}$ and $n \geq 1$ is an integer. If the one-function corona theorem holds for* $\mathrm{Mult}(\mathcal{H})$, *then $f \in C_n(\mathcal{H})$ if and only if $1/f \in N_n^+(\mathcal{H})$.*

Further, we show that if f is nonvanishing on Ω, then $f \in C_1(\mathcal{H})$, and hence $f \in C_\infty(\mathcal{H})$. Of course, the converse of this result is true by Lemma 2.2 below. Thus this elementary, yet important observation is our next result.

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Theorem 1.2. *Let $\mathcal{H}$ be a complete Pick space of continuous functions on a compact set Ω and let the one-function corona theorem holds for* Mult($\mathcal{H}$)*. If $f \in \mathcal{H}$ is nonvanishing, then $f \in C_1(\mathcal{H})$. Hence, $f \in C_\infty(\mathcal{H})$.*

Next, we consider the invariant subspaces generated by functions in the multiplier algebra Mult($\mathcal{H}$). If $\mathcal{H} = D$ the Dirichlet space, then the class $C_1(D)$ consists of the outer functions, that is, those functions f in the Hardy space $H^2(\mathbb{D})$ such that

$$\log|f(0)| = \int_0^{2\pi} \log|f(e^{it})|\frac{dt}{2\pi} > -\infty.$$

In this case, it is known that if $\varphi, \psi \in C_1(D)$, then the invariant subspace generated by $\varphi\psi$ is equal to the intersection of the invariant subspaces generated by φ and ψ (see [13], Theorem 4.5). Motivated from this theorem we have the following result.

Theorem 1.3. *Let $\mathcal{H}$ be a complete Pick space of continuous functions on a compact set Ω and let the one-function corona theorem holds for* Mult($\mathcal{H}$)*. Let $\varphi, \psi \in$ Mult($\mathcal{H}$) be such that $\psi \in C_1(\mathcal{H})$. Then, $[\varphi\psi] = [\varphi] \cap [\psi]$.*

The plan of the paper is as follows. In section 2, we give definitions and known results that will be important for our results. In Section 3, we first define the classes $N_n^+(\mathcal{H})$, and then prove Theorem 1.1 and Theorem 1.2 as Theorem 3.4 and Theorem 3.5, respectively. Finally, in the Section 4, we prove Theorem 1.3 as Theorem 4.1.

2. Preliminaries

Suppose Ω is a non-empty set. A Hilbert function space $\mathcal{H}$ on Ω is defined to be a Hilbert space of complex valued functions on Ω such that the evaluation functional is continuous on $\mathcal{H}$, i.e., for each $z \in \Omega$ the map $f \mapsto f(z)$ is continuous on $\mathcal{H}$.

For a Hilbert function space $\mathcal{H}$, the multiplier algebra of $\mathcal{H}$, denoted by Mult($\mathcal{H}$), is defined by

$$\text{Mult}(\mathcal{H}) = \{\varphi : \Omega \to \mathbb{C} : \varphi f \in \mathcal{H} \ \text{ for all } \ f \in \mathcal{H}\}.$$

As an application of the closed graph theorem, it is well-known that each multiplier $\varphi \in$ Mult($\mathcal{H}$) defines a bounded multiplication operator M_φ on $\mathcal{H}$, and Mult($\mathcal{H}$) becomes a Banach algebra by setting $\|\varphi\|_{\text{Mult}(\mathcal{H})} = \|M_\varphi\|$.

For a function $f \in \mathcal{H}$, the multiplier invariant subspace generated by f, denoted by $[f]$, is $[f] = \text{clos}_{\mathcal{H}}\{\varphi f : \varphi \in \text{Mult}(\mathcal{H})\}$. A function $f \in \mathcal{H}$ is called cyclic if $[f] = \mathcal{H}$.

When working on cyclicity Aleman et al. [5] introduce the following classes of pseudo-cyclic multipliers. For each integer $n \geq 0$, they define

$$C_n(\mathcal{H}) = \{\varphi \in \text{Mult}(\mathcal{H}) : \varphi \neq 0 \ \text{ and } \ [\varphi^n] = [\varphi^{n+1}]\},$$

and

$$C_\infty(\mathcal{H}) = \{\varphi \in \text{Mult}(\mathcal{H}) : \bigcap_{n=1}^{\infty}[\varphi^n] \neq 0\}.$$

If Mult($\mathcal{H}$) is dense in $\mathcal{H}$, then $C_0(\mathcal{H})$ consists of the cyclic multipliers and one has

$$C_0(\mathcal{H}) \subseteq C_1(\mathcal{H}) \subseteq C_2(\mathcal{H})... \subseteq C_\infty(\mathcal{H}) \tag{2.1}$$

(see [5]). If $\mathcal{H} = H^2(\mathbb{D})$, the Hardy space, then it is known that Mult($H^2(\mathbb{D})$) $= H^\infty$, where H^∞ is the algebra of bounded analytic functions on $\mathbb{D}$. In this case, as is mentioned in [5], we have equality in (2.1), where each set equals the outer functions in H^∞. If $\mathcal{H} = D$, the Dirichlet space, then $C_1(D)$ equals the outer functions in Mult(D), and $C_0(D) \neq C_1(D)$. Moreover, in this case $C_1(D) = C_\infty(D)$, see [5] for details. In [5], Aleman et al. showed that if d is odd, then $\mathbb{C}_{\text{stable}}[z] \subseteq C_{\frac{d-1}{2}}(H^2_d)$, and if d is even then $\mathbb{C}_{\text{stable}}[z] \subseteq C_{\frac{d}{2}-1}(H^2_d)$, where $\mathbb{C}_{\text{stable}}[z]$ denotes the $d-$variable complex polynomials without zeros in $\mathbb{B}_d$.

Note that each Hilbert function space $\mathcal{H}$ has a reproducing kernel $k : \Omega\times\Omega \to \mathbb{C}$. Writing $k_w(z) = k(z, w)$, it satisfies $f(w) = \langle f, k_w\rangle$ for all $f \in \mathcal{H}$, $w \in \Omega$ [1, 12]. A reproducing kernel k on Ω is called a normalized complete Pick kernel, if there is $z_0 \in \Omega$ and a function u from Ω into some auxiliary Hilbert space $\mathcal{K}$ such that $u(z_0) = 0$ and

$$k_w(z) = \frac{1}{1 - \langle u(z), u(w)\rangle_{\mathcal{K}}}.$$

If $\mathcal{H}$ is a Hilbert function space of analytic functions, then it is known that $\mathcal{H}$ is separable, and we may assume that $\mathcal{K}$ is separable as well. Following [5], a Hilbert function space $\mathcal{H}$ on Ω is called a complete Pick space, if there is an equivalent norm on $\mathcal{H}$ such that the reproducing kernel of $\mathcal{H}$ is a normalized complete Pick kernel with respect to the new norm. If Ω is the open unit disc $\mathbb{D}$, then important examples of such spaces are the Hardy space $H^2(\mathbb{D})$, where $k_w(z) = \frac{1}{1-\overline{w}z}$, and the Dirichlet space D with the kernel $k_w(z) = \frac{1}{\overline{w}z} \log \frac{1}{1-\overline{w}z}$ ([1], Corollary 7.41). On the other hand, the Bergman space $L^2_a(\mathbb{D})$ has reproducing kernel $k_w(z) = \frac{1}{(1-\overline{w}z)^2}$, which is not a complete Pick kernel (see [9] Example 4.5). Moreover, the Drury-Arveson space H^2_d of analytic functions on the unit ball $\mathbb{B}_d$ is also an example of a space with a normalized complete Pick kernel, where $k_w(z) = \frac{1}{1-\langle z,w\rangle}$ (see [9] Corollary 4.11).

If k is a normalized kernel for $\mathcal{H}$, then the constant function 1 is contained in $\mathcal{H}$, and hence $\mathrm{Mult}(\mathcal{H}) \subset \mathcal{H}$.

A fact known about all complete Pick spaces is that each complete Pick space $\mathcal{H}$ is contained in the corresponding Pick-Smirnov class

$$N^+(\mathcal{H}) = \{\frac{\varphi}{\psi} : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \psi \ \text{ cyclic}\},$$

(see [2], Theorem 1.1).

For the rest of this section, for the sake of completeness, we collect some important results of complete Pick spaces that will be used.

Corollary 2.1. *([5], Corollary 6.2) If $\mathcal{H}$ is a complete Pick space, then* $\mathrm{Mult}(\mathcal{H})$ *is dense in $\mathcal{H}$. In particular, $f \in \mathcal{H}$ is cyclic if and only if $1 \in [f]$.*

Lemma 2.2. *([5], Lemma 2.2)*

Assume that $\mathrm{Mult}(\mathcal{H}) \subseteq \mathcal{H}$.

(a) If $\varphi \in C_\infty(\mathcal{H})$, then $\varphi(z) \neq 0$ for all $z \in \Omega$.

(b) If $n, m \geq 0$ are integers and if $\psi, \varphi \in$ $\mathrm{Mult}(\mathcal{H})$ *such that $[\varphi^n] = [\varphi^{n+1}]$ and $[\psi^m] = [\psi^{m+1}]$, then $[\varphi^n\psi^m] = [\varphi^{n+1}\psi^{m+1}]$.*

The following theorem is a special case of Theorem 1.1 (i) of [3].

Theorem 2.3. *Let $\mathcal{H}$ be a complete Pick space with $k_{w_0} = 1$. For $f : \Omega \to \mathbb{C}$, the following are equivalent:*

(1) *$f \in \mathcal{H}$ and $\|f\| \leq 1$*
(2) *there are multipliers $\varphi, \psi \in$* $\mathrm{Mult}(\mathcal{H})$ *such that*
  (a) $f = \dfrac{\varphi}{1-\psi}$
  (b) $\psi(w_0) = 0$, *and*
  (c) $\|\psi h\|^2 + \|\varphi h\|^2 \leq \|h\|^2$ *for every $h \in \mathcal{H}$.*

Lemma 2.4. *([5], Lemma 6.5) Let $\mathcal{H}$ be a separable Hilbert function space on Ω. If $f = \dfrac{\varphi}{1-\psi} \in \mathcal{H}$, where $\varphi, \psi \in$* $\mathrm{Mult}(\mathcal{H})$ *and $\psi \neq 1$, $\|\psi\|_{\mathrm{Mult}(\mathcal{H})} \leq 1$, then $1 - \psi$ is cyclic in $\mathcal{H}$ and $[f] = [\varphi]$.*

Lemma 2.5. *([5], Lemma 6.6) Let $\mathcal{H}$ be a separable Hilbert function space on Ω. If $f = \dfrac{u}{v} = \dfrac{u_1}{v_1} \in N^+(\mathcal{H})$, where $u, v, u_1, v_1 \in$* $\mathrm{Mult}(\mathcal{H})$*, v, v_1 cyclic, then $[u^n] = [u_1^n]$ for all $n \in \mathbb{N}$.*

3. A Condition for Pseudo-cyclicity

From now on, we will assume that $\mathcal{H}$ is a separable complete Pick space. Then, by Corollary 2.1, $\mathrm{Mult}(\mathcal{H})$ is dense in $\mathcal{H}$. Hence, by the above discussion following the definition of the classes $C_n(\mathcal{H})$, $C_0(\mathcal{H})$ consists of the cyclic multipliers, and we have (2.1). By this observation, we make the following definition.

Definition 3.1. Let $n \geq 0$ be an integer. We define

$$N^+_n(\mathcal{H}) = \{\frac{\varphi}{\psi} : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \psi \in C_n(\mathcal{H})\}.$$

Notice that for each fixed integer $n \geq 0$, a function $\psi \in C_n(\mathcal{H})$ does not vanish anywhere by Lemma 2.2 (a), hence the quotient φ/ψ is defined on all of Ω. Moreover, by the same lemma part (b), since the product of any two functions in $C_n(\mathcal{H})$ is in $C_n(\mathcal{H})$, $N^+_n(\mathcal{H})$ is an algebra. It is clear that

$$N^+_0(\mathcal{H}) \subseteq N^+_1(\mathcal{H}) \subseteq N^+_2(\mathcal{H}) \subseteq ...,$$

and $N^+_0(\mathcal{H}) = N^+(\mathcal{H})$ is the Pick-Smirnov class that corresponds to $\mathcal{H}$.

Moreover, in light of Lemma 2.5 we can extend the membership of classes $C_n(\mathcal{H})$ to all functions in $\mathcal{H}$.

Definition 3.2. We say $f \in \mathcal{H}$ is in $C_n(\mathcal{H})$ if whenever $f = u/v$ with $u, v \in \text{Mult}(\mathcal{H})$, v cyclic, then $u \in C_n(\mathcal{H})$. Further, we say that $f \in C_\infty(\mathcal{H})$ if $u \in C_\infty(\mathcal{H})$.

The definition below is from [2] (see the discussion preceding Remark 5.1).

Definition 3.3. The one-function corona theorem is said to hold for $\text{Mult}(\mathcal{H})$ if whenever f is a multiplier and f is bounded below on Ω, then $1/f$ is a multiplier.

In the rest of the paper, we assume that Ω is compact and the functions in $\mathcal{H}$ are continuous on Ω. For examples of such spaces see Example 5.3 of [2]. In spirit, the following theorem is an extension of Theorem 3.1 of [4] to the classes $C_n(\mathcal{H})$. There authors show that in a complete Pick space $\mathcal{H}$ a function f is cyclic if and only if $1/f$ is in the corresponding Pick-Smirnov class.

We are now ready to prove Theorem 1.1, which for convenience we restate here.

Theorem 3.4. *Suppose $\mathcal{H}$ is a separable complete Pick space of continuous functions on a compact set Ω, $f \in \mathcal{H}$ and $n \geq 1$ is an integer. If the one-function corona theorem holds for* $\text{Mult}(\mathcal{H})$*, then $f \in C_n(\mathcal{H})$ if and only if $1/f \in N_n^+(\mathcal{H})$.*

Proof. Assume $\|f\| = 1$, and let $n \geq 1$ be a fixed integer. Then, by Theorem 2.3 $f = \dfrac{\varphi}{1-\psi}$, where φ, ψ are contractive multipliers. Moreover, it follows from Lemma 2.4 that $1 - \psi$ is cyclic in $\mathcal{H}$.

Now, if $f \in C_n(\mathcal{H})$, that is, $\varphi \in C_n(\mathcal{H})$, then

$$\frac{1}{f} = \frac{1-\psi}{\varphi} \in N_n^+(\mathcal{H}).$$

Conversely, if $\dfrac{1}{f} \in N_n^+(\mathcal{H})$, then $\dfrac{1}{f} = \dfrac{u}{v}$ where u, v are multipliers and $v \in C_n(\mathcal{H})$. Then,

$$f = \frac{\varphi}{1-\psi} = \frac{v}{u} \implies (1-\psi)v = \varphi u.$$

Since $1-\psi$ is cyclic and $v \in C_n(\mathcal{H})$, it follows from Lemma 2.2 that $(1-\psi)v \in C_n(\mathcal{H})$. Thus, $(1-\psi)v$ is non-vanishing on Ω by Lemma 2.2. Then, by continuity of functions in $\mathcal{H}$ on compact set Ω, $(1-\psi)v$ is bounded below and by corona hypothesis $\dfrac{1}{(1-\psi)v} \in \text{Mult}(\mathcal{H})$. Therefore,

$$1 = \frac{1}{(1-\psi)v}\varphi u \in [u],$$

that is, u is cyclic by Corollary 2.1. This says that $f = \dfrac{v}{u}$, where u, v are multipliers with u cyclic and $v \in C_n(\mathcal{H})$. Hence, $f \in C_n(\mathcal{H})$.

□

Next, we prove Theorem 1.2. We restate the theorem here for convenience.

Theorem 3.5. *Let $\mathcal{H}$ be a complete Pick space of continuous functions on a compact set Ω and let the one-function corona theorem holds for* $\text{Mult}(\mathcal{H})$*. If $f \in \mathcal{H}$ is nonvanishing, then $f \in C_1(\mathcal{H})$. Hence, $f \in C_\infty(\mathcal{H})$.*

Proof. Since $\mathcal{H}$ is contained in the Pick-Smirnov class, we can write $f = \dfrac{u}{v}$, where $u, v \in \text{Mult}(\mathcal{H})$ and v is cyclic. Hence, we need to show that $u \in C_1(\mathcal{H})$. Since u is a multiplier, it is clear that $[u^2] \subseteq [u]$. To show the converse, by hypothesis $u(z) \neq 0$ for all $z \in \Omega$, and hence by continuity and compactness u is bounded below on Ω. Therefore, one-function corona hypothesis implies that $\dfrac{1}{u}$ is a multiplier. Thus, $u = \dfrac{1}{u}u^2 \in [u^2]$. This proves the theorem.

□

4. Invariant Subspaces Generated by Pseudo-cyclic Multipliers

In this section we focus on invariant subspaces generated by the functions in the class $C_1(\mathcal{H})$. We show that the invariant subspace generated by the product of two multipliers, one being in the class $C_1(\mathcal{H})$, is equal the intersection of the invariant subspaces generated by those functions. Namely, we have the following result.

Theorem 4.1. *Let $\mathcal{H}$ be a complete Pick space of continuous functions on a compact set Ω and let the one-function corona theorem holds for* Mult($\mathcal{H}$). *Let $\varphi, \psi \in$* Mult($\mathcal{H}$) *be such that $\psi \in C_1(\mathcal{H})$. Then, $[\varphi\psi] = [\varphi] \cap [\psi]$.*

Proof. Since $\varphi\psi \in [\varphi]$ and $\varphi\psi \in [\psi]$ we have $\varphi\psi \in [\varphi] \cap [\psi]$. Therefore, $[\varphi\psi] \subseteq [\varphi] \cap [\psi]$. Conversely, let $f \in [\varphi] \cap [\psi]$. Then there are sequences of multipliers p_n, q_n such that $p_n\varphi \to f$ and $q_n\psi \to f$ in $\mathcal{H}$. Thus, $p_n\varphi\psi \to f\psi$ in $\mathcal{H}$, so $f\psi \in [\varphi\psi]$. Since $\psi \in C_1(\mathcal{H})$, it is nonvanishing on Ω and hence bounded below on Ω. Moreover, since the one-function corona theorem holds for Mult($\mathcal{H}$), we have $\frac{1}{\psi} \in$ Mult($\mathcal{H}$). Thus $f = \frac{1}{\psi} f\psi \in [\varphi\psi]$. That finishes the proof.

□

Remark 4.2. In Theorem 4.1, we assume that the one-function corona theorem holds for the multiplier algebra of a complete Pick space $\mathcal{H}$ of continuous functions on a compact set Ω. One may ask can we relax this conditions on $\mathcal{H}$. In [2], the authors used an example of Salas [14] and constructed a complete Pick space of continuous functions, called the Salas space, on the closed unit disc $\overline{\mathbb{D}}$ such that the one-function corona theorem fails (see Theorem 5.5 of [2]). So the one-function corona theorem is crucial.

Remark 4.3. As it was already mentioned in the Introduction for the Dirichlet space D, we have $[\varphi\psi] = [\varphi] \cap [\psi]$ for functions $\varphi, \psi \in C_1(D)$. The corona theorem, hence the one-function corona theorem, holds for the Dirichlet space D (see [11], for a more general version). But it is known that there are functions in the Dirichlet space D that are not continuous on the closed unit disc $\overline{\mathbb{D}}$. This observation raises the following question.

Question: Let $\mathcal{H}$ be a complete Pick space such that one-function corona theorem holds for Mult($\mathcal{H}$) and $\varphi, \psi \in C_1(\mathcal{H})$. Then is $[\varphi\psi] = [\varphi] \cap [\psi]$?

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author have read and agreed to the published version of the manuscript.

References

[1] Agler, J., McCarthy, J.E., Pick Interpolation and Hilbert Function Spaces. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2002.

[2] Aleman, A., Hartz, M., McCarthy, J.E., Richter, S., *The Smirnov class for spaces with the complete Pick property*, J. Lond. Math. Soc. (2), **96**(1)(2017), 228–242.

[3] Aleman, A., Hartz, M., McCarthy, J.E., Richter, S., *Factorizations induced by complete Nevanlinna-Pick factors*, Adv. Math., **335**(2018), 372–404.

[4] Aleman, A., Perfekt, K.-M., Richter, S., Sundberg, C., Sunkes, J., *Cyclicity and iterated logarithms in the drury-arveson space*, Preprint at https://arxiv.org/abs/2301.10091, (2023).

[5] Aleman, A., Perfekt, K.-M., Richter, S., Sundberg, C., Sunkes, J., *Cyclicity in the drury-arveson space and other weighted besov spaces*,Preprint at https://arxiv.org/abs/2301.04994, (2023).

[6] Beurling, A., *On two problems concerning linear transformations in Hilbert space*, Acta Math., **81**(1949), 239–255.

[7] Duren, P., Schuster, A., Bergman Spaces. Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2004.

[8] El-Fallah, O., Kellay, K., Mashreghi, J., Ransford, T., A Primer on the Dirichlet Space. Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2014.

[9] Hartz, M., An Invitation to the Drury–arveson Space. In: Mashreghi, J. (ed.) Lectures on Analytic Function Spaces and Their Applications, Springer, Fields Institute Monographs, 2023.

[10] Hedenmalm, H., Korenblum, B., Zhu, K., Theory of Bergman Spaces, Graduate Texts in Mathematics, Springer, New York, 2000.

[11] Luo, S., *Corona theorem for the Dirichlet-type space*, J. Geom. Anal., **32**(3)(2022), 74–20.

[12] Paulsen, V.I., Raghupathi, M., An Introduction to the Theory of Reproducing Kernel Hilbert Spaces. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2016.

[13] Richter, S., Sundberg, C., *Multipliers and invariant subspaces in the Dirichlet space*, J. Operator Theory, **28**(1) (1992), 167–186.

[14] Salas, H.N., *A note on strictly cyclic weighted shifts*, Proc. Amer. Math. Soc., **83**(3)(1981), 555–556.