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RESEARCH ARTICLE

FIBONACCI AND LUCAS NUMBERS AS PRODUCTS OF THEIR ARBITRARY TERMS

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Abstract

This study presents all solutions to the Diophantine equations $F_k = L_m L_n$ and $L_k = F_m F_n$. To be clear, the Fibonacci numbers that are the product of two arbitrary Lucas numbers and the Lucas numbers that are the product of two arbitrary Fibonacci numbers are determined herein. The results under consideration are proven by using the Dujella-Pethő lemma in coordination with Matveev's theorem. All common terms of the Fibonacci and Lucas numbers are determined. Further, the Lucas-square Fibonacci and Fibonacci-square Lucas numbers are given.

Keywords

Fibonacci and Lucas numbers,
Logarithmic height in logarithms,
Matveev theorem,
Dujella - Pethő lemma

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1. INTRODUCTION

Let $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ be the n^{th} terms of the Fibonacci and Lucas numbers, which can be produced by utilizing the recurrence relation $F_{n+1} = F_n + F_{n-1}$ and $L_{n+1} = L_n + L_{n-1}$ for all integers $n \geq 1$ with the initial conditions $(F_0, F_1) = (0, 1)$ and $(L_0, L_1) = (2, 1)$, respectively. It can be observed that the Fibonacci and Lucas numbers are a second-order integer sequence that satisfies the algebraic equation $x^2 - x - 1 = 0$. By considering this algebraic equation with the mentioned initial conditions, one can develop their Binet's formulae for all $n \in \mathbb{N}$:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n \tag{1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. More detail can be referenced to [1-3].

It can be stated immediately that the above integer sequences are the most beloved subject of mathematics and are paid great attention by almost all branches of modern sciences. Today, the respective results and discussions are expanding to an exciting aspect: finding all possible solutions to Diophantine equations, including special integer sequences, i.e., Fibonacci, Lucas, Pell, or Jacobsthal numbers, etc. In [4], Marques investigated the Fibonacci numbers that can be expressible in terms of the generalized Cullen and Woodall numbers. In [5], Chaves and Marques determined all terms of generalized Fibonacci numbers, which are the sum of the powers of the consecutive generalized

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Fibonacci sequence. In [6], Bravo and Gómez considered k-generalized Fibonacci numbers that are the Mersenne numbers. In [7], Pongsriiam found all the Fibonacci and Lucas numbers, which are one away from the product of an arbitrary number of the Fibonacci or Lucas numbers. In [8], Ddamulira et al. solved the Pillai-type problem with k–generalized Fibonacci numbers and powers of 2 for $k > 3$. In [9], Kafle et al. dealt with finding all solutions to the Pell equations related to the product of two Fibonacci numbers. In [10], Qu and Zeng investigated all Lucas numbers that are concatenations of two repdigits. In [11], Şiar et al. found all Fibonacci or Lucas numbers that are products of two repdigits in base b. In [12], Alan and Alan discovered the Mersenne numbers that can be written in terms of the products of two Pell numbers. In [13], Rihane and Togbé obtained terms of k-Fibonacci numbers in the arrays of the Padovan or Perrin numbers.

In the open literature, there are a few more specific papers that study the Diophantine-type equations concerning the Fibonacci numbers or other integer sequences. However, both the above brief literature survey and other source works show that integer sequences in the right-hand side and the left-hand side of problems under consideration are of different characteristic algebraic equations. For example, Fibonacci or Lucas number vs. Pell number by Alekseyev [14], Fibonacci number vs. Pell number by Ddamulira et al. [15], generalized Fibonacci number vs. generalized Pell number by Bravo et al. [16], Fibonacci number vs. Jacobsthal number by Erduvan and Keskin [17], and Leonardo number vs. Jacobsthal number by Bensella and Behloul [18]. Motivated by the results of the current literature, in this paper, we address finding problem of all possible solutions to the following Diophantine equations for positive integers k, m, and n according to the famous Matveev's theorem and the Dujella-Pethő lemma:

$$F_k = L_m L_n \tag{2}$$

and

$$L_k = F_m F_n \tag{3}$$

Here, due to multiplicative symmetry, it is sufficient that the case where $k \geq 1$ and $1 \leq m \leq n$ is considered. However, our equations consist of the Fibonacci and Lucas numbers that have the same characteristic equation. This makes the application of the Dujella-Pethő lemma impossible because some parameters disappear unlike the solution processes in the current literature. One of the novelties of the paper is to display a new approach to this issue.

It should be noted that in [19], Carlitz considered the same problems the first time by employing divisibility properties and some elementary identities. However, the author's results are either incorrect or incomplete. More precisely, the author asserted that while the unique solution of equation (2) is $(k, m, n) = (8,4,2)$ for $1 < n \leq m$, Equation (3) has no solution for $2 < n \leq m$. Further, in [20], based on the elementary properties and inequalities, Wang et al. stated that while Equation (3) has no solution, the triple $(k, m, n) = (4,2,1)$ is one solution to Equation (2). As can be seen, the results of both studies are also contradictory to each other. The results of our paper will both eliminate this deficiency and will end this debate.

2. BASIC TOOLS

This section introduces essential tools and definitions, lemmas, and theorems required in the rest of the paper. Our proof process is based on the Matveev's theorem, which uses the linear forms in logarithms to limit the variables of the problem, and the Dujella-Pethő lemma, which allows us to reduce the bounds.

Let η be an algebraic number of degree d with the minimal polynomial

$$f(x) := \sum_{j=0}^{d} a_j x^{d-j} = a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x]$$

where $a_0 > 0$ is the leading coefficient, a_j's are integers, and $\eta^{(i)}$ is the i^{th} conjugate of η. The logarithmic height, denoted by $h(\eta)$ of η is defined by

$$h(\eta) = \frac{1}{d}\left(log\ a_0 + \sum_{i=1}^{d} log\left(max\left\{\left|\eta^{(i)}\right|, 1\right\}\right)\right).$$

Let $\eta_1, \eta_2, \cdots, \eta_s$ be positive algebraic numbers in the real number field $\mathcal{F}$ of degree D and let b_1, b_2, $\cdots$, b_s be nonzero rational numbers. Introduce the notations

$$\Lambda \coloneqq {\eta_1}^{b_1}{\eta_2}^{b_2} \cdots {\eta_s}^{b_s} - 1 \text{ and } \mathrm{B} \coloneqq max\{|b_1|, |b_2|, \cdots |b_s|\}.$$

Let $A_1, A_2, \cdots, A_s$ be the positive real numbers such as

$$A_j \geq max\left\{Dh(\eta_j), \left|log\eta_j\right|, 0.16\right\} \text{ for all } j = 1,2, \cdots s.$$

In this case, we can give the famous Matveev's theorem [21] and the Dujella-Pethő lemma [22].

Theorem 1 (Matveev [21]) *The following inequality holds for non-zero Λ over real field $\mathcal{F}$:*

$$log\Lambda > -1.4 \times 30^{s+3} \times D^2 \times (1 + logD) \times (1 + log\ B) \times A_1 \times A_2 \times \cdots \times A_s.$$

Lemma 2 (Dujella and Pethő [22]) *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon \coloneqq \|\mu q\| - M\|\tau q\|$, where $\|.\|$ is the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (x, y, z) of inequality*

$$0 < x\tau - y + \mu < AB^{-z}$$

with

$$x \leq M \text{ and } z \geq \frac{log(Aq/\varepsilon)}{logB}.$$

The following lemmas will be used later.

Lemma 3 Let n be a positive integer. Then,

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}, \tag{4}$$

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n, \tag{5}$$

$$|\beta|^{-(n-2)} \leq F_n \leq |\beta|^{-(n-1)}, \tag{6}$$

and

$$|\beta|^{-(n-1)} \leq L_n \leq |\beta|^{-(n+1)}. \tag{7}$$

Proof. The proof can be made by using the induction method on n.

Lemma 4 (Ddamulira et al. [15]) *For all* $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $|x| < 2|e^x - 1|$ *is satisfied.*

3. MAIN RESULTS

In this section, we will present all solutions to Equations (2) and (3) and will prove our results.

Theorem 5 *Let k, m, and n be a positive integer. Then,*

- Equation (2) is satisfied only for the triples of
$$(k, m, n) \in \{(1,1,1), (2,1,1), (4,1,2), (8,2,4)\}. \tag{8}$$
- Equation (3) holds only for the triples of
$$(k, m, n) \in \{(1,1,1), (1,1,2), (1,2,2), (2,1,4), (2,2,4), (3,3,3)\}. \tag{9}$$

Proof. Here, to reduce the size of the paper, we will only share a detailed proof for Equation (2), neglecting that of Equation (3).

From Equation (2) and Lemma 3, we can write

$$\alpha^{k-2} \leq F_k = L_m L_n \leq |\beta|^{-n-m-2}$$

and naturally

$$(k-2)log\alpha \leq -(n+m+2)log|\beta| \Rightarrow 2 - (n+m+2)\frac{log|\beta|}{log\alpha} \Rightarrow k < 4n.$$

Considering Binet's formulas in Equation (1) and the fact that $\alpha = -\beta^{-1}$, we can arrange Equation (2) as follows:

$$\Lambda_1 := \left|\alpha^{-k}|\beta|^{n+m}\sqrt{5} - 1\right| < \frac{8}{\alpha^{2m}}. \tag{10}$$

In this case, we can consider the case $s = 3, \eta_1 = \alpha,\ \eta_2 = |\beta|, \eta_3 = \sqrt{5}, b_1 = -k, b_2 = n + m$ and $b_3 = 1$ in Theorem 1. To be clear, $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{5})$ and $\mathcal{F} = \mathbb{Q}(\sqrt{5})$ of degree $D = 2$. Here, since $\alpha^k\ |\beta|^{-n-m} = \sqrt{5}$ is not satisfied when computing the square of its both sides, $\Lambda_1 \neq 0$. In addition,

$$h(\eta_1) = h(\eta_2) = \frac{1}{2}log\alpha, h(\eta_3) = log\sqrt{5}, A_1 = A_2 = log\alpha, \text{ and } A_3 = 2log\sqrt{5}.$$

Further, for $B\ =\ 4n, B \geq max\{|-k|, n+m, 1\}$. Then, with these values, Theorem 1 implies that

$$log(\Lambda_1) > -3.62 \times 10^{11} \times (1 + log4n). \tag{11}$$

Also, with Equation (10), we obtain

$$log(\Lambda_1) < log8 - 2m\ log\alpha. \tag{12}$$

As a result, we get

$$m < 3.77 \times 10^{11} \times (1 + log4n)\ . \tag{13}$$

Further, coming back to Equation (2), after some mathematical arrangements, we can write

$$\Lambda_2 := \left|\alpha^{-k}|\beta|^n\left(\sqrt{5}L_m\right) - 1\right| < \frac{33}{\alpha^n} \tag{14}$$

which implies that $s = 3, \eta_1 = \alpha,\ \eta_2 = |\beta|, \eta_3 = \sqrt{5}L_m, b_1 = -k, b_2 = n$ and $b_3 = 1$. Here, $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ and $\mathcal{F} = \mathbb{Q}\left(\sqrt{5}\right)$ of degree $D = 2$. On the other hand, one can prove that $\Lambda_2 \neq 0$ by applying the same procedure above. As a result,

$$h(\eta_1) = h(\eta_2) = \frac{1}{2}log\alpha, \text{ and } A_1 = A_2 = log\alpha.$$

Further, since η_3 is the root of the polynomial $x^2 - 5L_m^2, h(\eta_3) = log\left(\sqrt{5}L_m\right)$ and $A_3 = 6mlog\alpha$. In addition, $B \geq max\{|-k|, n, 1\}$ for $B\ =\ 4n$. From Theorem 1, we can write

$$log(\Lambda_2) > -6.49 \times 10^{11} \times m \times (1 + log4n)\ . \tag{15}$$

Solving Equations (13) and (15) together, we get

$$log(\Lambda_2) > -2.45 \times 10^{23} \times m \times (1 + log4n)^2\ . \tag{16}$$

Also, from Equation (14), we obtain

$$log(\Lambda_2) < log33 - n\ log\alpha\ . \tag{17}$$

Considering Equations (13), (16) and (17), we find

$$n < 2.18 \times 10^{27}\ . \tag{18}$$

After applying a similar process into Equation (3), we determine the bounds

$$m < 7.52 \times 10^{11} \times (1 + log4n) \text{ and } n < 2.25 \times 10^{27}. \tag{19}$$

Summing up, it is sufficient that we consider the following lemma in order to complete the proof.

Lemma 6 *Both Equations (2) and (3) are satisfied for all the ordered triples of* (k, m, n) *over the ranges* $k < 4n$, $1 \leq m \leq n$, *and* $n < 2.25 \times 10^{27}$.

According to Lemma 6, there is a finite number of solutions. But, the bounds are huge, and thereby, we must obtain a more favorable condition. To do this, we will use Dujella-Pethő lemma for two different cases.

Case I: Introducing the notation

$$\Gamma_1 \coloneqq -k log\alpha + (n+m) log|\beta| + log\sqrt{5}.$$

we can write

$$\Lambda_1 = |exp(\Gamma_1) - 1| < \frac{8}{\alpha^{2m}}.$$

From Lemma 4, we obtain

$$0 < \left| k\frac{log\alpha}{log|\beta|} - (n+m) + \frac{log\left(1/\sqrt{5}\right)}{log|\beta|} \right| < \left| \frac{16}{\alpha^{2m} log|\beta|} \right| < \frac{34}{\alpha^{2m}}.$$

When applying Dujella-Pethő lemma into the last inequality by considering $M = 9.1 \times 10^{27}$ ($M > 4n > k$) and $\tau = \frac{log\alpha}{log|\beta|}$, computing the continued fraction expansions of τ yields.

$$\frac{p_{47}}{q_{47}} = \frac{1394991136110806534618331l454}{921342236120432337936155l6979}.$$

This means that $6M < q_{47} = 9213422361204323379361551 6979$. As a result, we obtain

$$\varepsilon := \|\mu q_{47}\| - M\|\tau q_{47}\|, \varepsilon > 0.486, \text{ and } \mu = \frac{log\left(1/\sqrt{5}\right)}{log|\beta|}.$$

In this case, taking $A := 34$, $B := \alpha^2$, and $z := m$ in Lemma 2, we conclude that $m \leq 73$.

Case II: Assume that $5 < m \leq 73$. Considering

$$\Gamma_2 \coloneqq -k log\alpha + n log|\beta| - log\left(\frac{1}{\sqrt{5}L_m}\right),$$

we have

$$\Lambda_2 = |exp(\Gamma_2) - 1| < \frac{33}{\alpha^n}.$$

From Lemma 4, we can write

$$0 < \left| k\frac{log\alpha}{log|\beta|} - n + \frac{log\left(\sqrt{5}L_m\right)}{log|\beta|} \right| < \left| \frac{66}{\alpha^n log|\beta|} \right| < \frac{138}{\alpha^n}.$$

For the case where $M = 9.1 \times 10^{27}$ ($M > 4n > k$) and $\tau = \frac{log\alpha}{log|\beta|}$, computing the continued fraction expansions of τ gives.

$$\frac{p_{47}}{q_{47}} = \frac{13949911361108065346183311454}{92134223612043233793615516979}.$$

This means that $6M < q_{47} = 92134223612043233793615516979$. In this case

$$\varepsilon_m := \|\mu_m q_{47}\| - M\|\tau_m q_{47}\|, \varepsilon > 0.034, \text{ and } \mu_m = \frac{log\left(\sqrt{5}L_m\right)}{log|\beta|},$$

As a result, taking where $A := 138$, $B \coloneqq \alpha$, and $z := n$ in Lemma 2, we obtain that $n \leq 156$.

It should be noted that applying a similar investigation into Equation (3), we obtain the bounds in which $m \leq 75$ and $n \leq 153$. Then, we can compose a unique looping in Mathematica© over the range $m \leq 75$ and $n \leq 156$ to determine all possible solutions to both Equations (2) and (3). So, running our Pc algorithm validates Theorem 5. This exhausts the proof.

A simple observation of the outcomes of Theorem 5 reveals the following inferences.

Corollary 7 *All common terms of the Fibonacci and Lucas numbers are 1 and 3.*

Proof. For the case where $m = 1$, Equation (2) is reduced to $F_k = L_n$. In this case, the result follows from Theorem 5.

Corollary 8 *The only Lucas-square Fibonacci numbers are* $F_1 = L_1^2 = 1$ *and* $F_2 = L_1^2 = 1$.

Proof. When $m = n$, Equation (2) is reduced to $F_k = L_n^2$. From Theorem 2, the result can be drawn.

Corollary 9 *The only Fibonacci-square Lucas numbers are* $L_1 = F_1^2 = 1$, $L_1 = F_2^2 = 1$ *and* $L_3 = F_3^2 = 4$.

Proof. Taking $m = n$ in Equation (3) into account, the proof is easily obtained.

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CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Ahmet Daşdemir: Formal analysis, Writing - original draft, Investigation, Conceptualization, Software.
Ahmet Emin: Formal analysis, Writing – Review & Editing, Investigation, Conceptualization, Software.

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