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3D Chaotic Nonlinear Dynamic Population-Growing Mathematical System Modeling with Multiple Controllers

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ABSTRACT Modeling, stabilization, and identification processes are significant stages in the process of developing knowledge about chaotic dynamical systems which entail the effective prediction depending on the degree of uncertainty toleration in the forecast, accuracy of the current state to be measured as well as a time scale resting on the dynamics of the system. Control of under-activated dynamical systems has been considered substantially, and it is for periods and is currently developing in various domains such as biology, data analysis, computing systems, and so forth. Dynamic systems of growing population signifies a model describing the way a population evolves over time during which population goes through major life events, split into discrete time periods. The size of the population at a given time period is determined by the rate of growth as well as other related factors. Most progress has been made in model-based control theory, which has drawbacks when the system under consideration is exceedingly complicated, and no model can be constructed. Accordingly, a 3D-discrete and dynamic human population growth system with many controllers is proposed by examining the stability and symmetry of controller system clarifications. The symmetric stability control results are presented by considering a special parametric dynamic system in its coefficients besides suggesting periodic functional coefficients in terms of sin and cos functions. The controllers have the ability to reduce population growth rate unpredictability or enhance system stability under various external conditions. The unique and very effective strategies in relevant domains could provide a deeper understanding of their impact as well as the theoretical or technological innovations thereof. These controllers are capable of reducing population growth rate unpredictability or improving system stability under various external conditions, and applicable strategies in the relevant domains can provide profound comprehension over the impact along with the theoretical as well as technological advancements.

KEYWORDS
Control system
Dynamic system
Difference system
Stability analysis
Growing human population
Stabilization
Mathematical modeling
3D-discrete chaotic systems
Kendall coefficient
Discrete systems
Difference equation
Multiple controllers
Jacobian matrix model

INTRODUCTION

A difference equation is a type of mathematical equation that describes the relationship between a function and its differences (or "deltas"). The general form of a difference equation is:

$$h(n) = H(h(n-1), h(n-2), ..., h(n-k))$$

where $h(n)$ is the function being studied, H is some function of the previous values of y, and n is the independent variable (often thought of as time). The theory of difference equations involves the study of properties and solutions of equations of this form, including stability, existence and uniqueness of solutions, and methods for finding explicit solutions.
Difference equations are used to model a wide range of phenomena in fields such as mathematics, physics, engineering, economics and many others. There are different methods to solve difference equations such as Z-transform, Laplace transform, generating function, and more.

A discrete and dynamic system of growing population refers to a model that describes how a population changes over time. In

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this type of system, the population going through major life events is divided into discrete time periods, and the size of the population at each time period is determined by the rate of growth and other factors. The population may change due to various factors such as births, deaths, migration, and changes in reproductive rates (Keyfitz 2005; Schoen 2013). One common model used to describe the growth of a population over time is the logistic growth model. This model takes into account the carrying capacity of the environment, which is the maximum number of individuals that can be supported by the available resources. The logistic growth model predicts that the population will grow at a faster rate until it reaches the carrying capacity, at which point the growth rate will start to decrease (Iannelli and Milner 2005; Salih and Al-Saidi 2022).

There are many different factors that can affect the growth of a population, including environmental conditions, resource availability, and interactions with other species. Understanding how these factors influence population growth can help us better predict and manage the population of a given species (N. M. Al-Saidi 2023; Shaw and Neubert 2018).

If a growing population has several controllers, it means that there are multiple factors or mechanisms that can influence the rate of population growth (Li and Ma 2022; Rending L. and P. 2022; Dhinakaran V. and H. 2021; Yellin and Samuelson 1974). Some common controllers of population growth going through major life events include:

- Birth rate: The number of births in a population over a given period of time can influence population growth. It can be represented by:
  $P_{birth} = \beta * P$,
  where P_{birth} refers to the increase in population based on births, β is the birth rate, and P is the initial population.
- Death rate: The number of deaths in a population over a given period of time can also influence population growth. It can be represented by:
  $P_{death} = \lambda * P$,
  where P_{death} refers to the decrease in population based on death, λ is the death rate.
- Migration: The movement of individuals into or out of a population can affect its size, such that, $P_M = M$,
  This represents the change in population based on migration.
- Reproductive rates: The number of offspring produced by individuals in a population can impact the population growth. It can be represented by:
  $P_R = R * P$,
  where P_R refers to the change in population based on reproductive, and R is the rate of reproductive.
- Environmental conditions: The availability of resources, such as food and water, as well as the presence of predators or other environmental factors, can affect population growth. It can be represented by:
  $P_E = f(E, P)$,
  where P_E refers to the change in population based on environmental conditions, and f is the impact rate of the environmental impact on the population.
- Human activity: Human actions, such as habitat destruction or the introduction of invasive species, can also influence population growth.
  $P_H = f(H, P)$,
  where P_H refers to the change in population based on human activity, and f is the impact of the human activity on the population.

  Therefore, the total population dynamics after considering all the influence factors can be represented by:
  $P_{total} = P + P_{birth}\text{-}P_{death} + P_M + P_R + P_E + P_H$

Understanding the various controllers of population growth can help us better predict and manage the size of a population over time.

For a long time, control of under-activated dynamical systems has been considered. The majority of development has been made in model-based control theory, which has limitations when the system under examination is extremely complex and no model can be built. This needs data-driven control approaches like machine learning, which has now spread to many disciplines, including control theory.

Control of under-activated growth systems refers to the process of regulating the growth of a system, such as a cell or organism, when it is not growing at its optimal rate. This can be achieved through a variety of methods; such as manipulating the levels of hormones or other signaling molecules, changing the environment that system is growing in, or applying genetic modifications. Hormones play a crucial role in controlling growth and development, and a balance of hormones is essential for normal growth. For example, the hormone insulin promotes cell growth and division, while the growth hormone stimulates the growth of bones and muscles. Manipulating the levels of these hormones can help to regulate growth in under-activated systems.

Environmental factors such as temperature, light, and nutrient availability can also affect growth. By controlling these factors, it is possible to regulate the growth of under-activated systems. Genetic modifications can also be used to control growth. For example, knocking out or over-expressing certain genes can affect the rate of growth, and can be used to regulate growth in under-activated systems. It is also important to note that in some cases under activation could be a symptom of a disease or malfunction of some internal process, in that case a medical or biological approach should be taken.

In this paper, a 3D-discrete and dynamic human population growth system with many controllers is proposed by examining the stability and symmetry of controller system clarifications. The symmetric stability control results are presented by considering a special parametric dynamic system in its coefficients besides suggesting periodic functional coefficients in terms of *sin* and *cos* functions. The controller laws for one, two and three dimensions are addressed, while numerical simulations are provided for supporting the preliminary findings of the study.

THE GROWING HUMAN POPULATION SYSTEMS

In this part, some of the 3D-dynamic and discrete systems of the growing human population (SGHP) P_1, P_2, P_3 is formulated. In Shaw and Neubert (2018), Joeland Samuelson presented the 3D-SGHP, as follows:

$$\begin{aligned}\frac{dP_1}{d\tau} &= -\lambda_1 P_1 + \beta_1 P_3 \\ \frac{dP_2}{d\tau} &= -\lambda_2 P_2 + \beta_2 P_3 \\ \frac{dP_3}{d\tau} &= -\lambda_3 P_3,\end{aligned} \tag{1}$$

where λ_i, $i = 1,2,3$ are the population rate and β_j, $j = 1,2$ are the connections of the population, which are admitted positive values. System (1) was extended into the following structure by Waldstatter (Waldstatter 1989)

$$\begin{aligned} \frac{dP_1}{d\tau} &= -\lambda_1 P_1 + \beta_1 P_3 \\ \frac{dP_2}{d\tau} &= -\lambda_2 P_2 + \beta_2 P_3 \\ \frac{dP_3}{d\tau} &= -\lambda_3 P_3 + \beta_3, \end{aligned} \tag{2}$$

where β_3 is a realistic constant. Pollard (1997) generated System (2) as follows

$$\begin{aligned} \frac{dP_1}{d\tau} &= -\lambda_1 P_1 + \beta_1 P_3 \\ \frac{dP_2}{d\tau} &= -\lambda_2 P_2 + \beta_2 P_3 \\ \frac{dP_3}{d\tau} &= -(\lambda_3 + \beta_3) P_3. \end{aligned} \tag{3}$$

Later, the author considered the Kendall observation to the system to get non-linearity system. Kendall discovered that the differential equations could describe a population of single males, single females, and couples, with the following system

$$\begin{aligned} \frac{dP_1}{d\tau} &= -\lambda_1 P_1 + \beta_1 P_3 - K_1 \\ \frac{dP_2}{d\tau} &= -\lambda_2 P_2 + \beta_2 P_3 - K_2 \\ \frac{dP_3}{d\tau} &= -(\lambda_3 + \beta_3) P_3 + K_3. \end{aligned} \tag{4}$$

The Kendall coefficient of concordance is a measure of the strength and direction of association between two variables in a population. It can be used to determine whether there is a statistically significant relationship between the variables, and if so, whether the relationship is positive or negative. The Kendall coefficient is often used in studies of population growth, as it can help researchers understand the factors that influence population size and change over time. It is calculated by comparing the ranks of the values of the two variables in a sample, and it can range from -1 (perfect negative association) to +1 (perfect positive association). A value of 0 indicates no association between the variables (Kendall 1997). Hadeler (2012) suggested an extension of System (4) arithmetically by adding the separation rate of pairs σ as follows:

$$\begin{aligned} \frac{dP_1}{d\tau} &= -\lambda_1 P_1 + (\beta_1 + \sigma) P_3 - K_1 \\ \frac{dP_2}{d\tau} &= -\lambda_2 P_2 + (\beta_2 + \sigma) P_3 - K_2 \\ \frac{dP_3}{d\tau} &= -(\lambda_3 + \beta_3 + \sigma) P_3 + K_3. \end{aligned} \tag{5}$$

In this effort, we consider the system of the structure

$$\begin{aligned} \frac{dP_1}{d\tau} &= -\lambda_1 P_1 + \sigma_1 P_3 - K_1 \\ \frac{dP_2}{d\tau} &= -\lambda_2 P_2 + \sigma_2 P_3 - K_2 \\ \frac{dP_3}{d\tau} &= -(\lambda_3 + \sigma_3) P_3 + K_3, \end{aligned} \tag{6}$$

where $\sigma_i = \beta_i + \sigma$, $i = 1, 2, 3$.

A 3D dynamic system of rising population could refer to a mathematical model that mimics the three-dimensional growth of a population through time. This could be used to investigate issues such as food availability, sickness, and social and environmental situations that influence population growth. Typically, the model would include variables that reflect these components as well as equations that describe how they interact and change over time. It could also incorporate three-dimensional population growth visualizations or simulations.

System (6) can be viewed as 3D discrete system of growing population, using the information

$$\Delta^n P_i = P_i^{n+1} - P_i^n, \quad i = 1, 2, 3$$

thus, we have 3D-SGHP

$$\begin{aligned} P_1^{n+1} &= (1 - \lambda_1) P_1^n + \sigma_1 P_3^n - K_1 \\ P_2^{n+1} &= (1 - \lambda_2) P_2^n + \sigma_2 P_3^n - K_2 \\ P_3^{n+1} &= -(\lambda_3 + \sigma_3 - 1) P_3^n + K_3. \end{aligned} \tag{7}$$

In terms of a matrix formula, System (7) can be viewed as follows:

$$P(n+1) = \Lambda(n) P(n) \Xi(n) + \Sigma(n), \tag{8}$$

where Λ, Ξ and Σ are square matrices of the same order. The solution of System (8) can be established by using the concept of the technique of variation of parameters.

Proposition 1.

Let $M_1(n)$ and $M_2(n)$ be fundamental matrix solution of the systems (Murty and Prasannam 1997)

$$P(n+1) = \Lambda(n) P(n) \Xi(n)$$

and

$$P(n+1) = \Xi^*(n) P(n) \Xi(n),$$

respectively. Then the solution of the of the homogeneous matrix difference system

$$P(n+1) = \Lambda(n) P(n) \Xi(n). \tag{9}$$

is given by the formula

$$P(n) = M_1(n) \Pi M_2^*(n)$$

where Π is an arbitrary constant square matrix of the same order. Moreover, any the solution of System (8) is formulated by

$$P(n) = M_1(n) \Pi M_2^*(n) + \bar{P}(n),$$

where $\bar{P}(n)$ is a particular solution of System (8).

A 3D discrete system of growing population refers to a mathematical model that represents the growth of a population in three dimensions over discrete intervals of time, rather than continuously (H. Natiq 2022). This means that the population size and other variables in the model are updated at specific points in time, rather than constantly changing. A 3D discrete system could be used to study the same types of factors that affect population growth as a continuous model, but the equations and approach may be different. Discrete models can be useful for understanding how a system changes over time in a more granular way, as the model is updated at specific points rather than continuously.

Example 1.

Let $K_i = 0$ in System (7), then the general solution becomes

$$P_1^n = \frac{(\sigma_1 c_3((\lambda_1-1)^n - (-\lambda_3-\sigma_3+1)^n))}{(\lambda_1+\lambda_3+\sigma_3-2)} + c_1(\lambda_1-1)^n$$
$$P_2^n = \frac{(c_3\sigma_2((\lambda_2-1)^n - (-(\lambda_3+\sigma_3)+1)^n))}{(\lambda_2+\lambda_3+\sigma_3-2)} + c_2(\lambda_2-1)^n$$
$$P_3^n = c_3(-(\lambda_3+\sigma_3)+1)^n, \quad (c_1,c_2,c_3) \in \mathbb{Z}^3$$

In general, when $K_i \neq 0$, we have

$$P_1^n = \frac{(\sigma_1 c_3((\lambda_1-1)^n - (-\lambda_3-\sigma_3+1)^n))}{(\lambda_1+\lambda_3+\sigma_3-2)} + c_1(\lambda_1-1)^n - \\ - \frac{\sigma_1 K_3 + K_1(\lambda_3+\sigma_3)}{(\lambda_1-2)(\lambda_3+\sigma_3)}$$
$$P_2^n = \frac{(c_3\sigma_2((\lambda_2-1)^n - (-(\lambda_3+\sigma_3)+1)^n))}{(\lambda_2+\lambda_3+\sigma_3-2)} + c_2(\lambda_2-1)^n - \\ - \frac{\sigma_2 K_3 + K_2(\lambda_3+\sigma_3)}{(\lambda_2-2)(\lambda_3+\sigma_3)}$$
$$P_3^n = c_3(-(\lambda_3+\sigma_3)+1)^n + \frac{K_3}{\lambda_3+\sigma_3}, \quad (c_1,c_2,c_3) \in \mathbb{Z}^3,$$

where $\lambda_3+\sigma_3 \neq 0$, $\lambda_1 \neq 2$ and $\lambda_2 \neq 2$.

Example 2.
Suppose the following data:

- $\lambda_1 = \lambda_2 = \sigma_1 = \sigma_2 = 0.5$ and $\lambda_3+\sigma_3 = 0.5$ then we obtain the following numerical solution of System (7) (see Fig.1-A)
$$(P_1^n, P_2^n, P_3^n) = ((-1/2)^n(-2c+(-1)^n-1), \\ (-1/2)^n(-2c+(-1)^n-1), 2^{1-n})$$
where c is a constant.
- $\lambda_1 = \lambda_2 = 0.5, \sigma_1 = 0.6, \sigma_2 = 0.9$ and $\lambda_3+\sigma_3 = 0.75$ then we get the following numerical solution of System (7) (see Fig.1-B)
$$(P_1^n, P_2^n, P_3^n) = \left(\frac{1}{5}(-1)^{n+1}2^{1-2n}((5c_1+8)2^n - 8(-1)^n), \\ \frac{1}{5}(-1)^n 2^{1-2n}(12(-1)^n - (5c_2+12)2^n), 4^{1-n}\right)$$

Negative values of growing population

The suggested system may have negative values for the growing population. There are several potential negative consequences associated with a growing population, including:

- Strain on Resources: As the population grows, there is an increased demand for natural resources such as food, water, and energy. This can lead to depletion of resources and increased pollution, which can have negative impacts on the environment and public health.
- Overcrowding: A growing population can lead to overcrowding in cities and other areas, which can contribute to a range of problems such as increased crime, congestion, and a lack of affordable housing.
- Unequal Distribution: Even though the population may be growing overall, the distribution of people is not always equal. This can lead to disparities in access to resources and opportunities, which can contribute to social and economic inequalities.
- Strain on Social Services: As the population grows, there can be increased demand for social services such as healthcare, education, and welfare programs. This can strain government budgets and resources and can contribute to political and social unrest.
- Environmental Impact: A growing population can have significant impacts on the environment, including deforestation, loss of biodiversity, and climate change. These negative impacts can have long-term consequences for future generations.

STABILITY ANALYSIS

In this part, we analyze the suggested 3D-SGHP in (6) and its discrete form (7). The Jacobian matrix of Model (6) is given by

$$\mathbf{J} = \begin{pmatrix} -\lambda_1 & 0 & \sigma_1 \\ 0 & -\lambda_2 & \sigma_2 \\ 0 & 0 & -\lambda_3-\sigma_3 \end{pmatrix}$$

where $|\mathbf{J}| = -\lambda_1\lambda_2(\lambda_3+\sigma_3)$. Therefore, the set of eigenvalues of $\mathbf{J}$ is

$$\rho_1 = -\lambda_1, \quad \rho_2 = -\lambda_2, \quad \rho_3 = -(\lambda_3+\sigma_3).$$

System (6) is asymptotically stable whenever $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3+\sigma_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1,0,0),\; v_2 = (0,1,0),\; v_3 = \left(\frac{\sigma_1}{\lambda_1-\lambda_3-\sigma_3}, \frac{\sigma_2}{\lambda_2-\lambda_3-\sigma_3}, 1\right),$$

where $\lambda_1 \neq \lambda_3+\sigma_3$ and $\lambda_2 \neq \lambda_3+\sigma_3$.

Model (6) has the following set of non-vanishing fixed points for all axis

$$S = \Big\{(P_1,P_2,P_3) : \Big(\frac{(\sigma_1K_3 - K_1(\sigma_3+\lambda_3+1))}{((\lambda_1+1)(\sigma_3+\lambda_3+1))}, \\ \frac{(\sigma_2K_3 - K_2(\sigma_3+\lambda_3+1))}{((\lambda_2+1)(\sigma_3+\lambda_3+1))}, \frac{K_3}{(\sigma_3+\lambda_3+1)}\Big) \\ \Big(-\frac{K_1}{\lambda_1+1}, \frac{(\sigma_2K_3 - K_2(\sigma_3+\lambda_3+1))}{((\lambda_2+1)(\sigma_3+\lambda_3+1))}, \frac{K_3}{(\sigma_3+\lambda_3+1)}\Big) \\ \Big(-\frac{K_1}{\lambda_1+1}, \frac{-K_2}{\lambda_2+1}, \frac{K_3}{(\sigma_3+\lambda_3+1)}\Big)\Big\}$$

whenever $\lambda_1 \neq -1, \lambda_2 \neq -1$ and $\lambda_3+\sigma_3 \neq -1$, and $K_i \in [-1,1]$, $i = 1,2,3$.
The set of the non-vanishing equilibrium points corresponding to System (6) is

$$E = \Big\{(P_1,P_2,P_3) : \Big(\frac{(\sigma_1K_3 - K_1(\sigma_3+\lambda_3))}{(\lambda_1(\sigma_3+\lambda_3))}, \frac{(\sigma_2K_3 - K_2(\sigma_3+\lambda_3))}{(\lambda_2(\sigma_3+\lambda_3))}, \\ \frac{K_3}{(\sigma_3+\lambda_3)}\Big) \\ \Big(-\frac{K_1}{\lambda_1}, \frac{(\sigma_2K_3 - K_2(\sigma_3+\lambda_3))}{(\lambda_2(\sigma_3+\lambda_3))}, \frac{K_3}{(\sigma_3+\lambda_3)}\Big) \\ \Big(-\frac{K_1}{\lambda_1}, -\frac{K_2}{\lambda_2}, \frac{K_3}{(\sigma_3+\lambda_3)}\Big)\Big\},$$

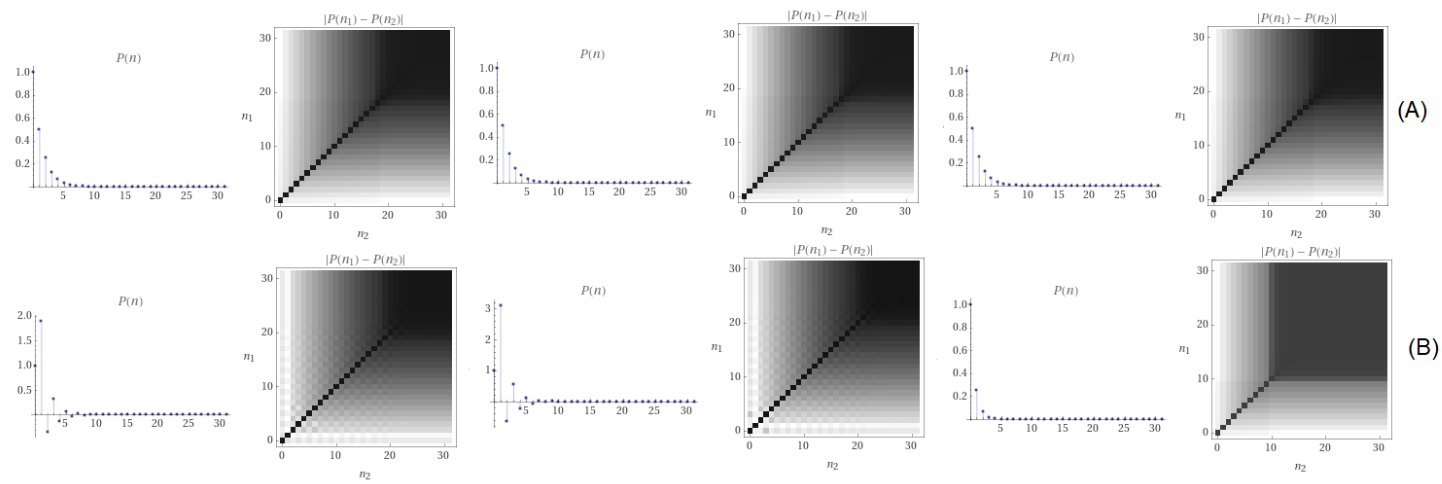

Figure 1 The plot of solutions (P_1, P_2, P_3) of System (7) in Example, respectively.

whenever $\lambda_1 \neq 0, \lambda_2 \neq 0$ and $\lambda_3 + \sigma_3 \neq 0$, and $K_i \in [-1, 1]$, $i = 1, 2, 3$. Note that when $K_i = 0$, $i = 1, 2, 3$, then the origin is the only equilibrium point and fixed point of System (6). Hence, $(0, 0, 0)$ is the unique solution for the system.

Special case: $\lambda_1 = \lambda_2$

In this case System (6) reduces into 2D-system, as follows:

$$\begin{aligned} \frac{dP_4}{d\tau} &= -\lambda P_4 + (\sigma_1 - \sigma_2)P_3 - (K_1 - K_2) \\ \frac{dP_3}{d\tau} &= -(\lambda_3 + \sigma_3)P_3 + K_3, \end{aligned} \tag{10}$$

where $P_4 = P_1 - P_2$, and $\lambda_1 = \lambda_2 = \lambda$.

$$\mathbf{J} = \begin{pmatrix} -\lambda & \sigma_1 - \sigma_2 \\ 0 & -\lambda_3 - \sigma_3 \end{pmatrix}$$

where $|\mathbf{J}| = \lambda(\lambda_3 + \sigma_3)$. Therefore, the set of eigenvalues of $\mathbf{J}$ is

$$\varrho_1 = -\lambda, \quad \varrho_2 = -(\lambda_3 + \sigma_3).$$

System (10) is asymptotically stable whenever $\lambda > 0$ and $\lambda_3 + \sigma_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1, 0), \quad v_2 = \left(-\frac{-\sigma_1 + \sigma_2}{\lambda - \lambda_3 - \sigma_3}, 1\right),$$

where $\lambda \neq -(\lambda_3 + \sigma_3)$.

Yields the following general solution;

$$\begin{aligned} P_4(\tau) &= \frac{-(\sigma_1 \alpha_1 e^{\tau(-(\lambda_3+\sigma_3))}))}{\lambda_3 + \sigma_3 - \lambda} + \frac{\sigma_2 \alpha_1 e^{\tau(-(\lambda_3+\sigma_3))}}{\lambda_3 + \sigma_3 - \lambda} + \frac{\sigma_1 K_3}{\lambda(\lambda_3 + \sigma_3)} - \\ &- \frac{\sigma_2 K_3}{\lambda(\lambda_3 + \sigma_3)} - \frac{\sigma_3 K_1}{\lambda(\lambda_3 + \sigma_3)} - \frac{\lambda_3 K_1}{\lambda(\lambda_3 + \sigma_3)} + \frac{\sigma_3 K_2}{\lambda(\lambda_3 + \sigma_3)} + \\ &+ \frac{\lambda_3 K_2}{\lambda(\lambda_3 + \sigma_3)} + \alpha_2 e^{\lambda \tau} \\ P_3(\tau) &= \alpha_1 e^{-\lambda_3 \tau - \sigma_3 \tau} + \frac{K_3}{(\lambda_3 + \sigma_3)}, \end{aligned}$$

where α_1 and α_2 are fixed constants.

Example 3.

For a constant a, we suggest the parametric connections system corresponding to (10), as follows

$$\begin{aligned} \frac{dP_4}{d\tau} &= -\cos(a)P_4 + (\sigma_1 - \sigma_2)P_3 - (K_1 - K_2) \\ \frac{dP_3}{d\tau} &= -\sin(a)P_3 + K_3. \end{aligned} \tag{11}$$

Then the solution becomes

$$\begin{aligned} P_4(\tau) &= e^{-1/2\tau \sin(2a)\csc(a)} \int_1^\tau \Big(-1/2 e^{1/2\xi \csc(a)\sin(2a) - \xi \sin(a)} * \\ &\csc(a)(-2e^{\xi \sin(a)}\sigma_1 - 2\sin(a)\sigma_1 + 2e^{\xi \sin(a)}\sigma_2 + 2\sin(a)\sigma_2 + \\ &+ 2e^{\xi \sin(a)} \sin(a) K_1 - 2e^{\xi \sin(a)} \sin(a) K_2 \Big) d\xi + \alpha_1 e^{-1/2\tau \sin(2a) K_3 \csc(a)} \\ P_3(\tau) &= \alpha_2 e^{-\tau \sin(a)} + K_3 \csc(a), \end{aligned}$$

where α_1 and α_2 are constants. Fig. 2 shows the symmetric behavior of the solution when $\sigma_1 = \sigma_2$ and $K_3 = \pm 1$. In this case, we obtain the solution

$$\begin{aligned} P_4(\tau) &= \alpha_1 e^{-\tau \cos(a)} - K_1 \sec(a) + K_2 \sec(a) \\ P_3(\tau) &= \alpha_2 e^{-\tau \sin(a)} \pm \csc(a). \end{aligned}$$

A parametric dynamic system is a mathematical model that describes the behavior of a system over time. It is defined by a set of differential equations, which are functions that describe how the system changes with respect to time. The parameters of the system are variables that can be adjusted to change the behavior of the system. These equations can be used to predict the future behavior of the system, given the current state and the parameters. These systems are widely used in fields such as physics, engineering, and economics to model and analyze real-world systems.

A parametric dynamic system with periodic coefficients is a type of mathematical model that describes the behavior of a sys- tem over time, where the parameters of the system are functions that vary periodically with time. These systems are often used to model physical systems that exhibit periodic behavior, such as oscillations or waves. The mathematical equations that define the system

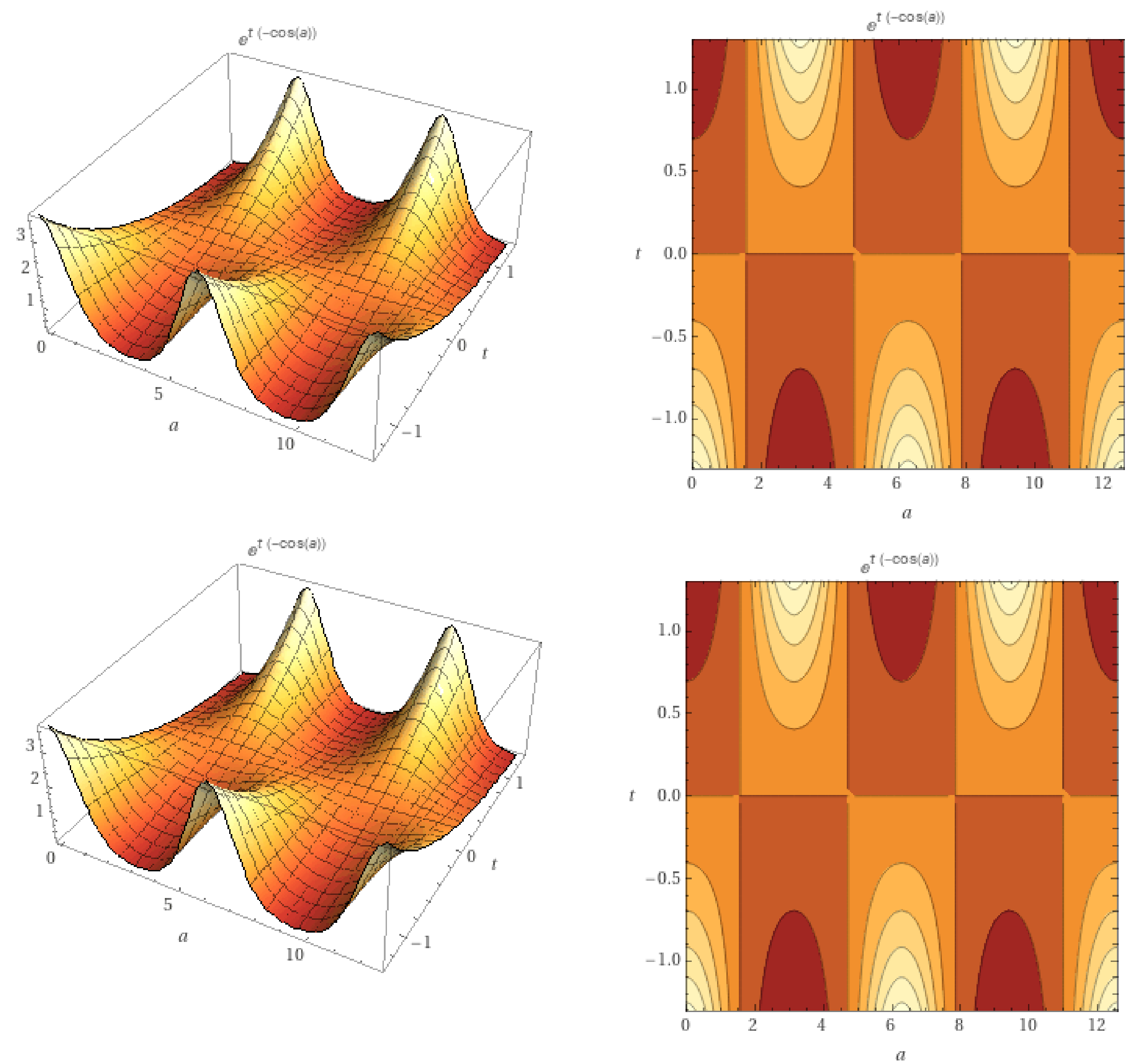

Figure 2 The plot of solutions (P_4, P_3) of System (11) in Example, when $\sigma_1 = \sigma_2$ and $K_3 = \pm 1$, respectively.

include terms that represent the periodic variations of the parameters, and the solution of these equations will also exhibit periodic behavior. These types of systems can be analyzed using techniques from the field of dynamic systems, including frequency analysis and bifurcation theory, to understand the behavior of the system and how it responds to changes in the parameters.

TYPES OF STABILIZATION OF NONLINEAR DYNAMIC SYSTEMS

Stabilization of a dynamic system refers to the process of making a system's behavior more predictable and consistent over time. This can be achieved by various means, such as adjusting the system's parameters, adding control inputs, or implementing a feedback control loop. The specific methods used will depend on the system's characteristics and the desired behavior.

In control theory, the stability of a dynamic system refers to the ability of the system to return to its equilibrium state after being subjected to some disturbance. A system is considered stable if, after a disturbance, the system returns to its equilibrium state or settles into a new equilibrium state that is acceptable. There are several ways to stabilize a dynamic system, including feedback control and feedforward control.

Feedback control involves using the output of the system as input to a controller, which then adjusts the input to the system to bring the output back to the desired equilibrium state. This can be done using a variety of control algorithms, such as PID (proportional-integral-derivative) control or state-space control. Therefore, the controller is a device or algorithm that regulates the behavior of a dynamic system. It compares the system's output (also called the process variable) with the desired output (also called the set-point) and calculates an error signal. The controller then uses this error

signal to adjust the system's inputs (also called the manipulated variables) in order to bring the output closer to the set-point. There are many different types of controllers, such as PID controllers, state-space controllers, and model predictive controllers, each with their own strengths and weaknesses (Hadeler 2012).
Feedforward control involves predicting the effect of a disturbance on the system and applying a counteracting input to the system to prevent the disturbance from affecting the equilibrium state. This can be done using techniques such as model predictive control or adaptive control. In general, the choice of control strategy will depend on the specific characteristics of the system and the requirements of the application.
In mathematics and physics, a chaotic system is a system that exhibits the property of chaos, which is defined as a periodic long-term behavior that is highly sensitive to initial conditions. This means that small differences in initial conditions can lead to drastically different outcomes over time. In other words, the behavior of a chaotic system is seemingly random and unpredictable. Examples of chaotic systems include the weather, the stock market, and some mechanical systems such as the double pendulum.

(1,2,3) D Controllers of System

By adding some control parameters, the chaotic system corresponding to (6) can be reformulated as follows:

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) + \sigma_1 P_3(\tau-1) - K_1\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) + \sigma_2 P_3(\tau-1) - K_2\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3.\end{aligned} \tag{12}$$

This proposition details the natural dynamics of the proposed system with its solution. Based on these results, the manipulation of the system toward desired outcomes using control strategies is given in Proposition 2.
Proposition 2

System (12) can be controlled by 1D-controller

$$U_1(\tau) = -\sigma_1 P_3(\tau),$$

whenever $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 + \sigma_3 > 0$.

Proof.
Consider the system (12). Then under the suggested controller, we have the following system

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) + \sigma_1 P_3(\tau-1) - K_1 + U_1(\tau-1)\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) + \sigma_2 P_3(\tau-1) - K_2\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3.\end{aligned} \tag{13}$$

Consequently, we obtain the difference system

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) + \sigma_1 P_3(\tau-1) - K_1 - \sigma_1 P_3(\tau-1)\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) + \sigma_2 P_3(\tau-1) - K_2\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3,\end{aligned} \tag{14}$$

which is equivalent to

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) - K_1\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) + \sigma_2 P_3(\tau-1) - K_2\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3.\end{aligned} \tag{15}$$

So,

$$\mathbf{J} = \begin{pmatrix} -\lambda_1 & 0 & 0\\ 0 & -\lambda_2 & \sigma_2\\ 0 & 0 & -\lambda_3-\sigma_3 \end{pmatrix}$$

where $|\mathbf{J}| = -\lambda_1\lambda_2(\lambda_3+\sigma_3)$.
Therefore, the set of eigenvalues of $\mathbf{J}$ is

$$v_1 = -\lambda_1, \quad v_2 = -\lambda_2, \quad v_3 = -(\lambda_3+\sigma_3).$$

System (15) is asymptotically stable whenever $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 + \sigma_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = \left(0, \frac{\sigma_2}{\lambda_2-\lambda_3-\sigma_3}, 1\right),$$

where $\lambda_1 \neq \lambda_3 + \sigma_3$ and $\lambda_2 \neq \lambda_3 + \sigma_3$.
In view of the conditions, System (15) is asymptotically stable.
Note that the solution of System (15) is given by the formula

$$\begin{aligned}P_1(n) &= c_1(-\lambda_1)^{n-1} - \frac{(K_1(1-(-\lambda_1)^n))}{(\lambda_1+1)},\\ P_2(n) &= c_2(-\lambda_2)^n - \frac{c_3\sigma_2((-\lambda_2)^n - (-\lambda_3-\sigma_3)^n)}{\lambda_2-\lambda_3-\sigma_3} - \\ &\qquad \frac{\lambda_3 K_2 + K_2}{(1+\lambda_2)(1+\lambda_3+\sigma_3)},\\ P_3(n) &= c_3(-\lambda_3-\sigma_3)^n + \frac{K_3}{\lambda_3+\sigma_3+1},\end{aligned}$$

where $(c_1, c_2, c_3) \in \mathbb{Z}^3$.

Proposition 3 builds on the simpler controller pertaining to Proposition 2 by the incorporation of more extensive or reliable control mechanisms. This progression not only shows how to improve upon simpler models to achieve greater effectiveness, but it also improves comprehension and applicability of controlling complex dynamical systems in real-world situations.
Proposition 3
System (12) can be controlled by 2D-controller

$$U_1(\tau) = -\sigma_1 P_3(\tau), \quad U_2(\tau) = -\sigma_2 P_3(\tau)$$

whenever $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 + \sigma_3 > 0$.
Proof.

Consider the system (12). Then under the recommended controllers, we have the following system

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) + \sigma_1 P_3(\tau-1) - K_1 + U_1(\tau-1)\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) + \sigma_2 P_3(\tau-1) - K_2 + U_2(\tau-1)\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3.\end{aligned} \tag{16}$$

Consequently, we obtain the difference system

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) + \sigma_1 P_3(\tau-1) - K_1 - \sigma_1 P_3(\tau-1)\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) + \sigma_2 P_3(\tau-1) - K_2 - \sigma_2 P_3(\tau-1)\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3,\end{aligned} \tag{17}$$

which is equivalent to

$$\begin{aligned}\Delta P_1(\tau) &= -\lambda_1 P_1(\tau-1) - K_1\\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau-1) - K_2\\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3) P_3(\tau-1) + K_3.\end{aligned} \tag{18}$$

Hence,

$$\mathbf{J} = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 - \sigma_3 \end{pmatrix}$$

where $|\mathbf{J}| = -\lambda_1\lambda_2(\lambda_3 + \sigma_3)$. Therefore, the set of eigenvalues of $\mathbf{J}$ is

$$\varepsilon_1 = -\lambda_1, \quad \varepsilon_2 = -\lambda_2, \quad \varepsilon_3 = -(\lambda_3 + \sigma_3).$$

System (18) is asymptotically stable whenever $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 + \sigma_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1),$$

where $\lambda_1 \neq \lambda_3$ and $\lambda_2 \neq \lambda_3$. In view of the conditions, System (18) is asymptotically stable.

Note that the solution of System (18) is given by the formula

$$\begin{aligned} P_1(n) &= c_1(-\lambda_1)^{n-1} - \frac{(K_1(1 - (-\lambda_1)^n))}{(\lambda_1 + 1)}, \\ P_2(n) &= c_2(-\lambda_2)^{n-1} - \frac{(K_2(1 - (-\lambda_2)^n))}{(\lambda_2 + 1)}, \\ P_3(n) &= c_3(-\lambda_3 - \sigma_3)^{n-1} - \frac{(K_3(\lambda_3 + \sigma_3)(1 - (-\lambda_3 - \sigma_3)^n))}{((-\lambda_3 - \sigma_3)(\lambda_3 + \sigma_3 + 1))}, \end{aligned}$$

where $(c_1, c_2, c_3) \in \mathbb{Z}^3$.

Proposition 4 is essential to the development of the study control strategy narrative because it demonstrates extensive and reliable control capabilities and broadens our understanding of the dynamics and control mechanisms of the system from a theoretical and practical standpoint.

Proposition 4

System (12) can be controlled by 3D-controller

$$U_1(\tau) = -\sigma_1 P_3(\tau), \quad U_2(\tau) = -\sigma_2 P_3(\tau), \quad U_3(\tau) = \sigma_3 P_3(\tau)$$

whenever $\lambda_i > 0, i = 1, 2, 3$.

Proof.

Consider the system (12). Then under the recommended controllers, we have the following system

$$\begin{aligned} \Delta P_1(\tau) &= -\lambda_1 P_1(\tau - 1) + \sigma_1 P_3(\tau - 1) - K_1 + U_1(\tau - 1) \\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau - 1) + \sigma_2 P_3(\tau - 1) - K_2 + U_2(\tau - 1) \\ \Delta P_3(\tau) &= -(\lambda_3 + \sigma_3) P_3(\tau - 1) + K_3 + U_3(\tau - 1). \end{aligned} \tag{19}$$

Consequently, we obtain the difference system

$$\begin{aligned} \Delta P_1(\tau) &= -\lambda_1 P_1(\tau - 1) + \sigma_1 P_3(\tau - 1) - K_1 - \sigma_1 P_3(\tau - 1) \\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau - 1) + \sigma_2 P_3(\tau - 1) - K_2 - \sigma_2 P_3(\tau - 1) \\ \Delta P_3(\tau) &= -(\lambda_3 + \sigma_3) P_3(\tau - 1) + K_3 + \sigma_3 P_3(\tau - 1), \end{aligned} \tag{20}$$

which is equivalent to

$$\begin{aligned} \Delta P_1(\tau) &= -\lambda_1 P_1(\tau - 1) - K_1 \\ \Delta P_2(\tau) &= -\lambda_2 P_2(\tau - 1) - K_2 \\ \Delta P_3(\tau) &= -\lambda_3 P_3(\tau - 1) + K_3. \end{aligned} \tag{21}$$

Then,

$$\mathbf{J} = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{pmatrix}$$

where $|\mathbf{J}| = -\lambda_1\lambda_2\lambda_3$. Therefore, the set of eigenvalues of $\mathbf{J}$ is

$$\vartheta_1 = -\lambda_1, \quad \vartheta_2 = -\lambda_2, \quad \vartheta_3 = -\lambda_3.$$

System (21) is asymptotically stable whenever $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1).$$

In view of the conditions, System (21) is asymptotically stable. Note that the solution of System (21) is given by the formula

$$\begin{aligned} P_1(n) &= c_1(-\lambda_1)^{n-1} - \frac{(K_1(1 - (-\lambda_1)^n))}{\lambda_1 + 1}, \\ P_2(n) &= c_2(-\lambda_2)^{n-1} - \frac{(K_2(1 - (-\lambda_2)^n))}{\lambda_2 + 1}, \\ P_3(n) &= c_3(-\lambda_3)^{n-1} + \frac{K_3(1 - (-\lambda_3)^n)}{\lambda_3 + 1}, \end{aligned}$$

where $(c_1, c_2, c_3) \in \mathbb{Z}^3$.

Stabilization of System (10)

There are several methods for stabilizing a 2D dynamic system, including: feedback control, Lyapunov stability analysis, state-space representation, etc. In our study, the adaptive control is considered; and thus, the related method involves adapting the control input to the system based on the current state, in order to achieve stability. However, to stabilize (10) for the special case, when $\lambda_1 = \lambda_2$, it will perform as follows:

The chaotic system corresponding to (10) can be realized as follows:

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau - 1) + (\sigma_1 - \sigma_2) P_3(\tau - 1) - (K_1 - K_2) \\ \Delta P_3(\tau) &= -(\lambda_3 + \sigma_3) P_3(\tau - 1) + K_3. \end{aligned} \tag{22}$$

Proposition 5

System (22) can be controlled by 1D-controller

$$V_1(\tau) = -(\sigma_1 - \sigma_2) P_3(\tau),$$

whenever $\lambda > 0$ and $\lambda_3 + \sigma_3 > 0$.

Proof.

Consider the system (22). Then under the suggested controller, we have the following system

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau - 1) + (\sigma_1 - \sigma_2) P_3(\tau - 1) - K_1 + V_1(\tau - 1) \\ \Delta P_3(\tau) &= -(\lambda_3 + \sigma_3) P_3(\tau - 1) + K_3. \end{aligned} \tag{23}$$

Consequently, we obtain the difference system

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau - 1) + (\sigma_1 - \sigma_2) P_3(\tau - 1) - K_1 - (\sigma_1 - \sigma_2) P_3(\tau - 1) \\ \Delta P_3(\tau) &= -(\lambda_3 + \sigma_3) P_3(\tau - 1) + K_3, \end{aligned} \tag{24}$$

which is equivalent to

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau-1) - K_1 \\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3)P_3(\tau-1) + K_3. \end{aligned} \tag{25}$$

Thus,

$$\mathbf{J} = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda_3-\sigma_3 \end{pmatrix}$$

where $|\mathbf{J}| = \lambda(\lambda_3+\sigma_3)$. Therefore, the set of eigenvalues of $\mathbf{J}$ is

$$v_1 = -\lambda, \quad v_2 = -(\lambda_3+\sigma_3).$$

System (25) is asymptotically stable whenever $\lambda > 0$ and $\lambda_3+\sigma_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1,0), \quad v_2 = (0,1).$$

In view of the conditions, System (25) is asymptotically stable. Note that the solution of System (25) is given by the formula

$$\begin{aligned} P_4(n) &= c_1(-\lambda)^{n-1} - \frac{(K_1(1-(-\lambda)^n))}{\lambda+1}, \\ P_3(n) &= c_3(-\lambda_3-\sigma_3)^{n-1} + \frac{K_3(\lambda_3+\sigma_3)(1-(-\lambda_3-\sigma_3)^n)}{(-\lambda_3-\sigma_3)(\lambda_3+\sigma_3+1)}, \end{aligned}$$

where $(c_1, c_2) \in \mathbb{Z}^2$.

Proposition 6
System (22) can be controlled by 2D-controller

$$V_1(\tau) = -(\sigma_1-\sigma_2)P_3(\tau), \quad V_2(\tau) = \sigma_3 P_3(\tau)$$

whenever $\lambda > 0$ and $\lambda_3 > 0$.

Proof.

Consider the system (22). Then under the suggested controllers, we have the following system

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau-1) + (\sigma_1-\sigma_2)P_3(\tau-1) - K_1 + V_1(\tau-1) \\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3)P_3(\tau-1) + K_3 + V_2(\tau-1). \end{aligned} \tag{26}$$

Consequently, we obtain the difference system

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau-1) + (\sigma_1-\sigma_2)P_3(\tau-1) - K_1 - (\sigma_1-\sigma_2)P_3(\tau-1) \\ \Delta P_3(\tau) &= -(\lambda_3+\sigma_3)P_3(\tau-1) + K_3 + \sigma_3 P_3(\tau-1), \end{aligned} \tag{27}$$

which is equivalent to

$$\begin{aligned} \Delta P_4(\tau) &= -\lambda P_4(\tau-1) - K_1 \\ \Delta P_3(\tau) &= -\lambda_3 P_3(\tau-1) + K_3. \end{aligned} \tag{28}$$

But,

$$\mathbf{J} = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda_3 \end{pmatrix}$$

where $|\mathbf{J}| = \lambda\lambda_3$; therefore, the set of eigenvalues of $\mathbf{J}$ is

$$v_1 = -\lambda, \quad v_2 = -\lambda_3.$$

System (28) is asymptotically stable whenever $\lambda > 0$ and $\lambda_3 > 0$. The corresponding eigenvectors are

$$v_1 = (1,0), \quad v_2 = (0,1).$$

In view of the conditions, System (28) is asymptotically stable. Note that the solution of System (28) is given by the formula

$$\begin{aligned} P_4(n) &= c_1(-\lambda)^{n-1} - \frac{(K_1(1-(-\lambda)^n))}{\lambda+1}, \\ P_3(n) &= c_3(-\lambda_3)^{n-1} + \frac{K_3(1-(-\lambda_3)^n)}{\lambda_3+1}, \end{aligned}$$

where $(c_1, c_2) \in \mathbb{Z}^2$.

EXEMPLARY APPLICATIONS

An application of the above examples is when dynamic systems with controllers are assumed to be informative systems. The informative problem for control refers to the challenge of ensuring that an autonomous system has enough information to make appropriate decisions and execute its intended actions. This can be especially difficult in complex or dynamic environments where the system may need to process and interpret a large amount of data in real time. It can also be a concern in situations where the system's decision-making process is opaque or difficult to understand. To address the informative problem, researchers may use techniques such as machine learning, computer vision, and sensor fusion to help the system make more accurate and informed decisions. Additionally, they may also implement methods to increase the transparency of the system's decision-making process, such as explainable AI or interpretative machine learning. Note that the data set is informative for the property P (.) if there is if there exists a controller U such that,

$$\Omega_\Delta \subseteq \Omega_{\{P,U\}},$$

where Ω_Δ indicates the set of all systems that are consistent with the data Δ. The growth of population can present a number of challenges, including strain on resources such as food, water, and housing, as well as increased pressure on infrastructure and public services. Therefore, it can be viewed as an informative problem. Additionally, population growth can contribute to environmental degradation and climate change. It can also exacerbate economic and social inequality. It is a complex issue that requires a multifaceted approach to address, involving strategies such as family planning, education and economic development, and sustainable resource management.

The question is how to control growth of population?
There are several strategies that can be implemented to control population growth and address its associated challenges. Some of these include the following points:

- Family planning: Providing access to birth control and education about reproductive health can help individuals and couples make informed decisions about their fertility and family size.
- Education and economic development: Investing in education and economic opportunities for women and girls can lead to lower fertility rates, as women with more education and economic resources tend to have fewer children.
- Sustainable resource management: Managing resources such as water, food, and energy in a sustainable manner can help to mitigate the strain that population growth places on these resources.

- Migration management: Implementing policies to manage migration can help to prevent overpopulation in certain areas and balance the population in a more sustainable way.
- Climate change mitigation: Mitigating climate change and its effects can help to reduce the negative impact of population growth on the environment.

It is important to note that population growth is a complex issue that is influenced by a variety of variables and factors, and addressing it will require a multifaceted approach that concerns diverse sectors and stakeholders across different fields and areas.

CONCLUSION

Control of under-activated dynamical systems refers to the process of manipulating the inputs of a system in order to achieve a desired behavior or output. This can be achieved through a variety of methods, such as feedback control, adaptive control, or optimal control. The specific approach used will depend on the characteristics of the system and the desired outcome. In under-activated systems, the control inputs may have limited effect on the system's behavior, making control more challenging. In these cases, techniques such as input shaping or hybrid control may be used to improve performance. From above, we considered different 3D- systems for growing population of humans. Furthermore, we suggested a set of special cases of the system, including 2D-system and parametric 2D-system. We further discussed the stability of the proposed systems in view of its analysis. Moreover, we gave a set of controllers of chaotic systems. We showed that the proposed system can be controlled by 1D, 2D and 3D controller laws through reverse-engineering efforts extracted from existing systems to be modeled and developed accordingly. For the future efforts, one can generalize the proposed systems using any types of fractional calculus, fractals (locally fractional calculus) and quantum calculus.

Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

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