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Research Article

On generalized distributions associated with singular partial differential operators

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Abstract

In this paper, we discuss various properties of the Riemann-Liouville operator over the generalized distributions $\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$ and other spaces. Next, we examine some properties of the convolution of the generalized distributions on the space $\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$.

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1. Introduction

In this paper, we consider the singular partial differential operators defined on $]0,+\infty[\times\mathbb{R}$ by [1]

$$\begin{cases} \Delta = \frac{\partial}{\partial x}, \\ \mathcal{D} = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \ \alpha \geqslant 0. \end{cases}$$

The following integral transform associated with Δ and $\mathcal{D}$ is called the Riemann-Liouville operator defined on the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable, by

$$\mathcal{R}(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\frac{dt}{\sqrt{1-t^2}}; \qquad \text{if } \alpha = 0. \end{cases}$$

Many harmonic analysis results related to the Riemann-Liouville operator have been established see for example [2, 3, 7–10, 16].

The theory of the Fourier transform play an important role in several fields such as mathematical, physical and engineering sciences. One of the most important concepts in Fourier theory, is that of a convolution. In this context, the convolution theory has many applications as signal processing, the theory of linear differential equations and quantum mechanics. The theory of distributions allows, by placing itself in a broader framework

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than, classical, the ordinary differential equations, to solve many equations from physics, fluid mechanics or signal processing. One of the fundamental ideas of this theory consists of defining the distributions through their action on a space of functions, called functions tests. This theory has gained considerable attention and has been extended to a wide class of integral transforms, (see for example [4, 5, 11, 15]).
The ultradistributions have been introduced by Beurling [4], Björck [5] and independently by Roumieu [14]. Ultradistribution theory is a natural generalization of the distribution theory. A unification of Beurling-Björck theory and Roumieu theory has been given by Komatsu [12]. The Hankel transform of ultradistributions in Roumieu setting has been given by Pathak and Pandey [13].

Based on the ideas of Björck [5] and Hörmander [11], we discuss various properties of the Riemann-Liouville operator over the generalized distributions $\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$ and other spaces.
This work is organized as follows. In the next section, we give a brief background of some harmonic analysis results related to the Riemann-Liouville operator. In section 3, we define and study the spaces of type $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$. In the last section, using the theory of the Riemann-Liouville operator, we examine several properties of convolutions of generalized distributions.

2. Preliminaries

In this section, we recall some harmonic analysis results related to the Riemann-Liouville operator. For more details, see [1].
We denote by
• ν the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr \otimes \frac{dx}{(2\pi)^{\frac{1}{2}}}.$$

• $L^p(d\nu)$, $p \in [1,+\infty]$ the Lebesgue space of measurable functions f on $[0,+\infty[\times\mathbb{R}$, such that $\|f\|_{p,\nu} < +\infty$.
• Υ the set given by

$$\Upsilon = \mathbb{R}\times\mathbb{R} \cup \big\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R},\ |r| \leqslant |x|\big\}.$$

• $\mathcal{B}_{\Upsilon_+}$ the σ-algebra defined on Υ_+ by,

$$\mathcal{B}_{\Upsilon_+} = \{\theta^{-1}(B)\ ,\ B \in \mathcal{B}_{Bor}([0,+\infty[\times\mathbb{R})\},$$

where θ is the bijective function, defined on the set

$$\Upsilon_+ = [0,+\infty[\times\mathbb{R} \cup \big\{(is,y)\ ;\ (s,y)\in[0,+\infty[\times\mathbb{R};\ s \leqslant |y|\big\},$$

by

$$\theta(s,y) = (\sqrt{s^2+y^2}, y). \tag{2.1}$$

• γ the measure defined on $\mathcal{B}_{\Upsilon_+}$ by, $\gamma(B) = \nu(\theta(B))$.
• $L^p(d\gamma)$, $p \in [1,+\infty]$ the Lebesgue space of measurable functions f on Υ_+, such that $\|f\|_{p,\gamma} < +\infty$.
• $\mathcal{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect the first variable.
The space $\mathcal{S}_e(\mathbb{R}^2)$ is equipped with the topology associated to the countable family of norms

$$\forall m\in\mathbb{N},\ \mathcal{N}_m(\phi) = \sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R},\\ k+|\beta|\leqslant m}} (1+r^2+x^2)^k \Big|\frac{\partial^{\beta_1+\beta_2}}{\partial r^{\beta_1}\partial x^{\beta_2}}\phi(r,x)\Big|.$$

If f is a non-negative measurable function on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu$), then $fo\theta$ is a measurable non-negative function on Υ_+, (respectively integrable on Υ_+ with respect to the measure $d\gamma$) and we have

$$\iint_{\Upsilon_+}(fo\theta)(\mu,\lambda)\,d\gamma(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\,d\nu(r,x). \tag{2.2}$$

For every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}$, the system

$$\begin{cases}\Delta_j u(r,x)=-i\lambda_j u(r,x),\\ \mathcal{D}u(r,x)=-\mu^2u(r,x),\\ u(0,0)=1,\\ \\ \dfrac{\partial u}{\partial r}(0,x)=0,\quad x\in\mathbb{R},\end{cases}$$

admits a unique solution $\varphi_{(\mu,\lambda)}$ given by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R},\quad \varphi_{(\mu,\lambda)}(r,x)=j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x},$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+1+k)}(\frac{z}{2})^{2k},\ z\in\mathbb{C}.$$

The function $\varphi_{(\mu,\lambda)}$ is bounded on $\mathbb{R}\times\mathbb{R}$ if and only if (μ,λ) belongs to the set Υ and in this case

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}}\left|\varphi_{(\mu,\lambda)}(r,x)\right|=1. \tag{2.3}$$

To define the translation operator associated with the Riemann-Liouville operator, we use the product formula for the eigenfunction $\varphi_{(\mu,\lambda)}$, that is for (r,x) and $(s,y)\in[0,+\infty[\times\mathbb{R}$,

$$\begin{aligned}\varphi_{(\mu,\lambda)}(r,x)\varphi_{(\mu,\lambda)}(s,y)=&\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac12)}\int_0^\pi\varphi_{(\mu,\lambda)}(\sqrt{r^2+s^2+2rs\cos\theta},x+y)\\ &\times\sin^{2\alpha}(\theta)d\theta. \end{aligned}\tag{2.4}$$

Definition 2.1. For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the Riemann-Liouville operator is defined on $L^p(d\nu)$, $p\in[1,+\infty]$, by

$$\mathcal{T}_{(r,x)}(f)(s,y)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac12)}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta},x+y)\sin^{2\alpha}(\theta)d\theta.$$

Remark 2.2. For every $(r,x)\in]0,+\infty[\times\mathbb{R}$, and by a standard change of variables, we have

$$\forall (s,y)\in]0,+\infty[\times\mathbb{R},\ \mathcal{T}_{(r,x)}(f)(s,y)=\frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty}f(t,x+y)\mathcal{W}_\alpha(r,s,t)t^{2\alpha+1}dt,$$

where the kernel $\mathcal{W}_\alpha$, is given by

$$\mathcal{W}_\alpha(r,s,t)=\frac{\Gamma(\alpha+1)^2}{2^{\alpha-1}\Gamma(\alpha+\frac12)\sqrt{\pi}}\frac{\left((r+s)^2-t^2\right)^{\alpha-\frac12}\left(t^2-(r-s)^2\right)^{\alpha-\frac12}}{(rst)^{2\alpha}}\mathbf{1}_{]|r-s|,r+s[}(t).$$

Proposition 2.3. *(1) For every $f\in L^p(d\nu)$, $p\in[1,+\infty]$, and $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu)$ and we have*

$$\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu}\leqslant\|f\|_{p,\nu}.$$

(2) The product formula (2.4) can be written

$$\mathcal{T}_{(r,x)}(\varphi_{(\mu,\lambda)})(s,y)=\varphi_{(\mu,\lambda)}(r,x)\varphi_{(\mu,\lambda)}(s,y).$$

(3) The kernel $\mathcal{W}_\alpha$ is symmetric in the variables r, s, t and

$$\frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty}\mathcal{W}_\alpha(r,s,t)t^{2\alpha+1}dt = 1. \tag{2.5}$$

Definition 2.4. The convolution product of $f, g \in L^1(d\nu)$ is defined by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R};\ f*g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n}\mathcal{T}_{(r,x)}(f)(s,y)g(s,y)d\nu(s,y). \tag{2.6}$$

Let $p, q, r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. Then for every $f \in L^p(d\nu)$ and $g \in L^q(d\nu)$, the function $f*g$ belongs to the space $L^r(d\nu)$, and we have the following Young's inequality

$$||f*g||_{r,\nu} \leqslant ||f||_{p,\nu}||g||_{q,\nu}.$$

Definition 2.5. The Fourier transform $\mathcal{F}$ associated with the Riemann-Liouville operator is defined on $L^1(d\nu)$ by

$$\forall (\mu,\lambda) \in \Upsilon\ ,\ \mathcal{F}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{(\mu,\lambda)}(r,x)d\nu(r,x).$$

Proposition 2.6. *(1) For $f \in L^1(d\nu)$, we have*

$$(\mu,\lambda) \in \Upsilon, \quad \mathcal{F}(f)(\mu,\lambda) = \tilde{\mathcal{F}}(f)\circ\theta(\mu,\lambda),$$

where for every $(\mu,\lambda) \in \mathbb{R}^2$,

$$\tilde{\mathcal{F}}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\mu)e^{-i\lambda x}d\nu(r,x) \tag{2.7}$$

and θ is the function defined by relation (2.1).
(2) For every $f \in L^1(d\nu)$ and $(r,x) \in [0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^1(d\nu)$ and we have

$$\forall (\mu,\lambda) \in [0,+\infty[\times\mathbb{R},\ \tilde{\mathcal{F}}(\mathcal{T}_{(r,x)}(f))(\mu,\lambda) = j_\alpha(r\mu)e^{-ix\lambda}\tilde{\mathcal{F}}(f)(\mu,\lambda). \tag{2.8}$$

(3) The Fourier transform $\mathcal{F}$ is a bounded linear operator from $L^1(d\nu)$ into $L^\infty(d\gamma)$ and that for every $f \in L^1(d\nu)$, we have

$$\|\mathcal{F}(f)\|_{\infty,\gamma} \leqslant \|f\|_{1,\nu}. \tag{2.9}$$

(4) For all $f, g \in L^1(d\nu)$, we have

$$\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g), \tag{2.10}$$

and

$$\tilde{\mathcal{F}}(f*g) = \tilde{\mathcal{F}}(f)\tilde{\mathcal{F}}(g).$$

(5) For all $f, g \in L^1(d\nu)$, we have

$$\tilde{\mathcal{F}}(fg) = \tilde{\mathcal{F}}(f)*\tilde{\mathcal{F}}(g). \tag{2.11}$$

Theorem 2.7 (Inversion formula)**.** *Let $f \in L^1(d\nu)$ such that $\mathcal{F}(f) \in L^1(d\gamma)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$*

$$\begin{aligned} f(r,x) &= \iint_{\Upsilon_+}\mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}d\gamma(\mu,\lambda) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}\tilde{\mathcal{F}}(f)(s,y)j_\alpha(rs)e^{i\lambda y}d\nu(s,y). \end{aligned} \tag{2.12}$$

Theorem 2.8 (Plancherel theorem)**.** *The Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$. In particular, for every $f \in L^2(d\nu)$*

$$\|\mathcal{F}(f)\|_{2,\gamma} = \|f\|_{2,\nu} = \|\tilde{\mathcal{F}}(f)\|_{2,\nu}.$$

Corollary 2.9. *For all functions f and g in $L^2(d\nu)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}d\nu(r,x) = \iint_{\Upsilon_+} \mathcal{F}(f)(\mu,\lambda)\overline{\mathcal{F}(g)(\mu,\lambda)}d\gamma(\mu,\lambda)$$
$$= \int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(f)(s,y)\overline{\tilde{\mathcal{F}}(g)(s,y)}d\nu(s,y). \tag{2.13}$$

Remark 2.10. Let $f,g \in L^2(d\nu)$, the function $f*g$ belongs to $L^2(d\nu)$ if and only if $\mathcal{F}(f)\mathcal{F}(g)$ belongs to $L^2(d\gamma)$ and we have

$$\|\mathcal{F}(f)\mathcal{F}(g)\|_{2,\gamma} = \|f*g\|_{2,\nu}.$$

We define a basic function for the Riemann-Liouville operator by

$$\begin{aligned}\mathcal{D}_\alpha(r,x,s,y,u,v) &= \int_0^{+\infty}\int_{\mathbb{R}} j_\alpha(rz)e^{-ixt}\\ &\quad\times j_\alpha(sz)e^{iyt}j_\alpha(uz)e^{-ivt}d\nu(z,t)\\ &= \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty}\Big(\int_{\mathbb{R}} e^{-i(x-y+v)t}\frac{dt}{(2\pi)^{\frac{1}{2}}}\Big)\\ &\quad\times j_\alpha(rz)j_\alpha(sz)j_\alpha(uz)z^{2\alpha+1}dz.\end{aligned} \tag{2.14}$$

Now according to [17], we have

$$\int_0^{+\infty} j_\alpha(ut)\mathcal{W}_\alpha(u,r,s)\frac{u^{2\alpha+1}du}{2^\alpha\Gamma(\alpha+1)} = j_\alpha(rt)j_\alpha(st).$$

Also, by the inversion formula for the Fourier-Bessel transform ([9], p. 125), we get

$$\mathcal{D}_\alpha(r,x,s,y,u,v) = \mathcal{W}_\alpha(r,s,u)\delta(x-y+v),$$

where δ is the Dirac delta function.
Hence, from (2.5), we obtain

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v) &= \int_0^{+\infty} \mathcal{W}_\alpha(r,s,u)\frac{u^{2\alpha+1}du}{2^\alpha\Gamma(\alpha+1)}\\ &\quad\times\int_{\mathbb{R}}\delta(x-y+v)\frac{dv}{(2\pi)^{\frac{1}{2}}}\\ &= 1.\end{aligned} \tag{2.15}$$

Lemma 2.11. *Let $f \in \mathcal{S}_e(\mathbb{R}^2)$. The translation operator $\mathcal{T}_{(r,x)}$ associated with the Riemann-Liouville operator can be expressed as*

$$\mathcal{T}_{(r,x)}(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(u,v)\mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v). \tag{2.16}$$

Proof. Let $f \in \mathcal{S}_e(\mathbb{R}^2)$. Then by (2.8) and (2.12), we have

$$\mathcal{T}_{(r,x)}(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(f)(\mu,\lambda)j_\alpha(r\mu)e^{-ix\lambda}j_\alpha(\mu s)e^{iy\lambda}d\nu(\mu,\lambda).$$

Then, the result follows from (2.7) and (2.14). □

We denote by Λ_α the partial differential operator defined by

$$\Lambda_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} = \ell_\alpha + \frac{\partial^2}{\partial x^2},$$

where ℓ_α is the Bessel operator.
The differential operator Λ_α is continuous from $\mathcal{S}_e(\mathbb{R}^2)$ into itself, and that for every $f \in \mathcal{S}_e(\mathbb{R}^2)$, we have

$$\forall (s,y) \in \Upsilon, \mathcal{F}(\Lambda_\alpha(f))(s,y) = -|\theta(s,y)|^2\mathcal{F}(f)(s,y). \tag{2.17}$$

Proposition 2.12. *Let $\phi, \psi \in \mathcal{S}_e(\mathbb{R}^2)$ and let $\beta \in \mathbb{N}$. Then*

$$(\Lambda_\alpha)^\beta(\phi(r,x)\psi(r,x)) = \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} \times r^{p-\beta}\frac{\partial^{\ell+q}}{\partial x^\ell \partial r^q}(\phi(r,x))\frac{\partial^{2\beta-2i+p-q-\ell}}{\partial r^{p-q}\partial x^{2\beta-2i-\ell}}(\psi(r,x)). \tag{2.18}$$

Proof. Let $\phi, \psi \in \mathcal{S}_e(\mathbb{R}^2)$. Then according to ([6], p. 14), there is a constant $\aleph_{\beta,p}$ for $p \in \{0, 1, ..., \beta\}$ depending only on α satisfying

$$(\Lambda_\alpha)^\beta(\phi(r,x)\psi(r,x)) = \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\binom{i}{\beta}\aleph_{\beta,p} r^{p-\beta}\frac{\partial^{2\beta-2i}}{\partial x^{2\beta-2i}}\frac{\partial^p}{\partial r^p}(\phi(r,x)\psi(r,x)).$$

Now, by Leibnitz formula we have

$$(\Lambda_\alpha)^\beta(\phi(r,x)\psi(r,x)) = \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p} C_\beta^i C_p^q \aleph_{\beta,p} r^{p-\beta}\frac{\partial^{2\beta-2i}}{\partial x^{2\beta-2i}}\frac{\partial^q}{\partial r^q}(\phi(r,x)) \times \frac{\partial^{p-q}}{\partial r^{p-q}}(\psi(r,x)).$$

Again, from Leibnitz formula we get

$$(\Lambda_\alpha)^\beta(\phi(r,x)\psi(r,x)) = \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} \times r^{p-\beta}\frac{\partial^{\ell+q}}{\partial x^\ell \partial r^q}(\phi(r,x))\frac{\partial^{2\beta-2i+p-q-\ell}}{\partial r^{p-q}\partial x^{2\beta-2i-\ell}}(\psi(r,x)).$$

□

3. Properties of spaces of type $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$

In this section, we define and study the spaces of type $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$. For this we begin by the following definitions and notations.

Definition 3.1. A real valued function w on $[0,+\infty[\times\mathbb{R}$ is called subadditive function if it satisfies

$$\forall (r,x),(s,y) \in [0,+\infty[\times\mathbb{R},\ 0 = w(0,0) = \lim_{(u,v)\longrightarrow(0,0)} w(u,v) \leqslant w(r+s,x+y) \leqslant w(r,x)+w(s,y). \tag{3.1}$$

We denote by $\mathcal{M}_0$ be the set of all continuous real valued functions w on $[0,+\infty[\times\mathbb{R}$ satisfying the condition (3.1) and

$$\mathcal{J}(w) = \int\int_{|\theta(s,y)|\geqslant 1}\frac{w\circ\theta(s,y)}{|\theta(s,y)|^2}d\gamma(s,y) < \infty,$$

where θ is the function defined by relation (2.1).
Let w satisfy (3.1). If $\phi \in L^1(d\nu)$ and if σ is a real number, we write

$$\|\phi\|_{\sigma,w} = \int\int_{\Upsilon_+}|\mathcal{F}(\phi)(s,y)|e^{\sigma w\circ\theta(s,y)}d\gamma(s,y). \tag{3.2}$$

We denote by $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ the set of all $\phi \in L^1(d\nu)$ such that ϕ has compact support and $\|\phi\|_{\sigma,w} < \infty$. The elements $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ will be called test functions.

Definition 3.2. Let Σ is an open subset of $[0,+\infty[\times\mathbb{R}$, then

$$\mathcal{D}_w(\Sigma) = \{\phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R}),\ supp(\phi) \subset \Sigma\}.$$

Theorem 3.3. *Let $w \in \mathcal{M}_0$ and $\phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$. If $g \in L^1(d\nu)$ with compact support, then $g * \phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and*

$$\|\phi * g\|_{\sigma,w} \leqslant \|g\|_{1,\nu}\|\phi\|_{\sigma,w}.$$

Proof. It is clear that the function $g*\phi$ belongs to $L^1(d\nu)$, then from (3.2), (2.10) and (2.9), we get

$$\begin{aligned}\|\phi * g\|_{\sigma,w} &= \iint_{\Upsilon_+} |\mathcal{F}(\phi * g)(s,y)| e^{\sigma w\circ\theta(s,y)} d\gamma(s,y)\\ &= \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)||\mathcal{F}(g)(s,y)| e^{\sigma w\circ\theta(s,y)} d\gamma(s,y)\\ &\leqslant \|\mathcal{F}(g)\|_{\infty,\gamma} \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{\sigma w\circ\theta(s,y)} d\gamma(s,y)\\ &\leqslant \|g\|_{1,\nu}\|\phi\|_{\sigma,w}.\end{aligned}$$

Our demonstration of Theorem 3.3 is thus completed. □

Theorem 3.4. *Let $w_1, w_2 \in \mathcal{M}_0$ satisfies for some real number η and positive C*

$$w_2 \circ \theta(s,y) \leqslant \eta + C(w_1 \circ \theta(s,y)). \tag{3.3}$$

Then $\mathcal{D}_{w_1}([0,+\infty[\times\mathbb{R})$ is dense subset of $\mathcal{D}_{w_2}([0,+\infty[\times\mathbb{R})$.

Proof. Let $\phi \in \mathcal{D}_{w_1}([0,+\infty[\times\mathbb{R})$. Then we have $\|\phi\|_{\sigma,w_1} < \infty$. Now, by (3.3), we obtain

$$\begin{aligned}\|\phi\|_{\sigma,w_2} &\leqslant \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{\sigma(\eta + Cw_1\circ\theta(s,y))} d\gamma(s,y)\\ &= e^{\sigma\eta} \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{C\sigma w_1\circ\theta(s,y)} d\gamma(s,y)\\ &= e^{\sigma\eta}\|\phi\|_{C\sigma,w_1} < \infty.\end{aligned}$$

This show that $\phi \in \mathcal{D}_{w_2}([0,+\infty[\times\mathbb{R})$.
Now, let $g \in \mathcal{D}_{w_2}([0,+\infty[\times\mathbb{R})$, then by Theorem 3.3, the function $g*\phi_\varepsilon \in \mathcal{D}_{w_1}([0,+\infty[\times\mathbb{R})$, where φ_ε is the function defined by

$$\forall \varepsilon > 0,\ \forall (r,x) \in [0,+\infty[\times\mathbb{R},\ \varphi_\varepsilon(r,x) = \frac{\varphi(\frac{r}{\varepsilon}, \frac{x}{\varepsilon})}{\varepsilon^{2\alpha+3}}.$$

We have

$$\begin{aligned}\|g - g*\phi_\varepsilon\|_{\sigma,w_2} &= \iint_{\Upsilon_+} |\mathcal{F}(g - g*\phi_\varepsilon)(s,y)| e^{\sigma w_2\circ\theta(s,y)} d\gamma(s,y)\\ &= \iint_{\Upsilon_+} |\mathcal{F}(g)(s,y)(1 - \mathcal{F}(\phi_\varepsilon)(s,y))| e^{\sigma w_2\circ\theta(s,y)} d\gamma(s,y)\\ &= \iint_{\Upsilon_+} |\mathcal{F}(g)(s,y)||1 - \mathcal{F}(\phi)(\varepsilon s, \varepsilon y)| e^{\sigma w_2\circ\theta(s,y)} d\gamma(s,y).\end{aligned}$$

Then, by using the Dominated Convergence Theorem, we get

$$\lim_{\varepsilon \longrightarrow 0^+} \|g - g*\phi_\varepsilon\|_{\sigma,w_2} = 0.$$

Hence $\mathcal{D}_{w_1}([0,+\infty[\times\mathbb{R})$ is dense in $\mathcal{D}_{w_2}([0,+\infty[\times\mathbb{R})$. □

We denote by $\mathcal{S}$ the set of all continuous real valued functions w satisfying the conditions (3.1) and (3.2) and such that

$$\forall a \in \mathbb{R}, b > 0, \forall (s,y) \in \Upsilon, w\circ\theta(s,y) \geqslant a + b\ln(1 + |\theta(s,y)|). \tag{3.4}$$

Definition 3.5. Let $w \in \mathcal{S}$ and let $\phi \in L^1(d\nu)$. For every real number σ, we define

$$\|\phi\|_\sigma = \|\phi\|_{\sigma,w} = \sup_{(s,y)\in\Upsilon} |\mathcal{F}(\phi)(s,y)| e^{\sigma(w\circ\theta(s,y))}.$$

Proposition 3.6. *Let $w \in \mathcal{S}$ and $\phi \in L^1(d\nu)$. Then there exists a positive constant C_ρ such that*

$$\forall \sigma \in \mathbb{R},\ \|\phi\|_\sigma \leqslant C_\rho \|\phi\|_{\sigma+\rho},$$

where

$$C_\rho = \iint_{\Upsilon_+} e^{-\rho(w\circ\theta(s,y))} d\gamma(s,y).$$

Proof. Let $w \in \mathcal{S}$ and $\phi \in L^1(d\nu)$. We have

$$\begin{aligned}
C_\rho \|\phi\|_{\sigma+\rho} &= \sup_{(s,y)\in\Upsilon} |\mathcal{F}(\phi)(s,y)| e^{(\sigma+\rho)(w\circ\theta(s,y))} \iint_{\Upsilon_+} e^{-\rho(w\circ\theta(s,y))} d\gamma(s,y) \\
&\geqslant \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{(\sigma+\rho)(w\circ\theta(s,y))} e^{-\rho(w\circ\theta(s,y))} d\gamma(s,y) \\
&= \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{\sigma(w\circ\theta(s,y))} d\gamma(s,y) = \|\phi\|_\sigma.
\end{aligned}$$

Hence,

$$\|\phi\|_\sigma \leqslant C_\rho \|\phi\|_{\sigma+\rho}.$$

□

Theorem 3.7. *Let $w \in \mathcal{S}$ and let $\phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$. Then, for every multi-index β, the function $(\Lambda_\alpha)^\beta(\phi)$ belongs to $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$.*

Proof. Let $w \in \mathcal{S}$. From (3.4), we have

$$w \circ \theta(s,y) \geqslant a + b\ln(1 + |\theta(s,y)|).$$

On other words, we obtain

$$e^{\frac{w\circ\theta(s,y)-a}{b}} \geqslant 1 + |\theta(s,y)| > |\theta(s,y)|. \tag{3.5}$$

Now, by (2.17) and (3.5), we get

$$\begin{aligned}
\|(\Lambda_\alpha)^\beta(\phi)\|_{\sigma,w} &= \iint_{\Upsilon_+} |\mathcal{F}((\Lambda_\alpha)^\beta(\phi))(s,y)| e^{\sigma(w\circ\theta(s,y))} d\gamma(s,y) \\
&= \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)||\theta(s,y)|^{2\beta} e^{\sigma(w\circ\theta(s,y))} d\gamma(s,y) \\
&\leqslant \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{2\beta\frac{w\circ\theta(s,y)-a}{b}} e^{\sigma(w\circ\theta(s,y))} d\gamma(s,y) \\
&= e^{\frac{-2\beta a}{b}} \iint_{\Upsilon_+} |\mathcal{F}(\phi)(s,y)| e^{(\frac{2\beta}{b}+\sigma)(w\circ\theta(s,y))} d\gamma(s,y) \\
&= e^{\frac{-2\beta a}{b}} \|\phi\|_{\sigma+\frac{2\beta}{b},w}.
\end{aligned}$$

This show that $(\Lambda_\alpha)^\beta(\phi) \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$. □

Definition 3.8. Let $N = (N_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers and let Σ be an open subset of $[0,+\infty[\times\mathbb{R}$. Then $C^N(\Sigma)$ is the set of all $g \in C^\infty(\Sigma)$ such that, for each compact subset K of Σ, there exists a constant C such that

$$\sup_K |(\Lambda_\alpha)^\beta| \leqslant C^{k+1} N_k^k,$$

where β is multi-index with $|\beta| = k$, $k = 0, 1, \ldots$.

Theorem 3.9. *Let $g \in \mathcal{D}_w(\Sigma)$ and suppose that*

$$|\mathcal{F}(g)(s,y)| \leqslant \frac{C}{p_N(as,ay)(1+|\theta(s,y)|)^2}, \tag{3.6}$$

where C and a are positive numbers with

$$p_N(s,y)=\sum_{k=0}^{+\infty}\Big(\frac{|\theta(s,y)|^2}{N_k}\Big)^k,\ (s,y)\in\Upsilon. \tag{3.7}$$

Then $g\in C^N(\Sigma)$.

Proof. Let $g\in\mathcal{D}_w(\Sigma)$. Then from (2.12), we get

$$\begin{aligned}(\Lambda_\alpha)^\beta g(r,x)&=\iint_{\Upsilon_+}\mathcal{F}(g)(s,y)(\Lambda_\alpha)^\beta\Big(\varphi_{(s,y)}(r,x)\Big)d\gamma(s,y)\\&=\iint_{\Upsilon_+}\mathcal{F}(g)(s,y)(\Lambda_\alpha)^\beta\Big(j_\alpha(r\sqrt{s^2+y^2})e^{i\langle x,y\rangle}\Big)d\gamma(s,y)\\&=(-1)^\beta\iint_{\Upsilon_+}\mathcal{F}(g)(s,y)|\theta(s,y)|^{2\beta}\varphi_{(s,y)}(r,x)d\gamma(s,y)\end{aligned}$$

Now, by using (2.3), (3.6) and (3.7), we obtain

$$\begin{aligned}&\max_{|\beta|=k}\sup_{(r,x)\in\Sigma}|(\Lambda_\alpha)^\beta g(r,x)|\\&\leqslant\max_{|\beta|=k}\iint_{\Upsilon_+}|\mathcal{F}(g)(s,y)||\theta(s,y)|^{2\beta}d\gamma(s,y)\\&\leqslant\max_{|\beta|=k}\iint_{\Upsilon_+}\frac{C}{p_N(as,ay)(1+|\theta(s,y)|)^2}|\theta(s,y)|^{2\beta}d\gamma(s,y)\\&\leqslant\max_{|\beta|=k}\iint_{\Upsilon_+}\frac{CN_k^k}{a^k|\theta(s,y)|^{2k}(1+|\theta(s,y)|)^2}|\theta(s,y)|^{2\beta}d\gamma(s,y)\\&\leqslant a^{-k}CN_k^k\iint_{\Upsilon_+}\frac{d\gamma(s,y)}{(1+|\theta(s,y)|)^2}\\&=C_1a^{-k}N_k^k,\ k=0,1,2,....\end{aligned}$$

Hence, $g\in C^N(\Sigma)$. □

4. Convolutions of the generalized distributions

In this section, by using some harmonic analysis related to the Riemann-Liouville operator, we discuss various properties of convolutions of generalized distributions. We start by the following definition.

Definition 4.1. Let $w\in\mathcal{S}$ and let Σ be the open subset of $[0,+\infty[\times\mathbb{R}$. We denote by $\mathcal{D}'_w(\Sigma)$ the space of all linear functionals g on $\mathcal{D}_w(\Sigma)$.

Theorem 4.2. *Let $g\in\mathcal{D}'_w(\Sigma)$, then*

$$\forall\phi\in\mathcal{D}_w(\Sigma),\quad\int_0^{+\infty}\int_{\mathbb{R}}g(r,x)\phi(r,x)d\nu(r,x)=g*\check{\phi}(0,0),$$

where $\check{\phi}(r,x)=\phi(r,-x)$.

Proof. Let $g\in\mathcal{D}'_w(\Sigma)$ and $\phi\in\mathcal{D}_w(\Sigma)$. Then from (2.6) and (2.16), we get

$$\begin{aligned}g*\phi(r,x)&=\int_0^{+\infty}\int_{\mathbb{R}}\mathcal{T}_{(r,x)}(g)(s,y)\phi(s,y)d\nu(s,y)\\&=\int_0^{+\infty}\int_{\mathbb{R}}\phi(s,y)\\&\times\Big(\int_0^{+\infty}\int_{\mathbb{R}}g(u,v)\mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big)d\nu(s,y).\end{aligned}$$

Now, by using (2.14), (2.12) and Fubini's theorem, we obtain

$$\begin{aligned} g*\phi(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}}\Big(\int_0^{+\infty}\int_{\mathbb{R}}\phi(s,y)j_\alpha(s\mu)e^{iy,\lambda}d\nu(s,y)\Big) \\ &\quad\times\Big(\int_0^{+\infty}\int_{\mathbb{R}}g(u,v)j_\alpha(u\mu)e^{-iv,\lambda}d\nu(u,v)\Big)j_\alpha(r\mu)e^{-ix,\lambda}d\nu(\mu,\lambda) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}\tilde{\mathcal{F}}(g)(\mu,\lambda)\tilde{\mathcal{F}}(\phi)(\mu,-\lambda)j_\alpha(r\mu)e^{-ix,\lambda}d\nu(\mu,\lambda). \end{aligned}$$

Putting $(r,x)=(0,0)$, we get

$$\begin{aligned} g*\phi(0,0) &= \int_0^{+\infty}\int_{\mathbb{R}}\tilde{\mathcal{F}}(g)(\mu,\lambda)\tilde{\mathcal{F}}(\phi)(\mu,-\lambda)d\nu(\mu,\lambda) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}\tilde{\mathcal{F}}(g)(\mu,\lambda)\tilde{\mathcal{F}}(\check{\phi})(\mu,\lambda)d\nu(\mu,\lambda). \end{aligned}$$

Then the desired result follows from the Parseval formula (2.13). $\square$

Definition 4.3. Let $g\in L^1_{loc}(\Sigma)$, then we identify g with the element in $\mathcal{D}'_w(\Sigma)$ which is defined by

$$\forall\phi\in\mathcal{D}_w(\Sigma),\ \langle g,\phi\rangle=\int_0^{+\infty}\int_{\mathbb{R}}g(r,x)\phi(r,x)d\nu(r,x). \tag{4.1}$$

Definition 4.4. Let $w\in\mathcal{S}$. If $g\in\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$ and $\phi\in\mathcal{D}_w([0,+\infty[\times\mathbb{R})$, then the convolution $g*\phi$ is defined by

$$g*\phi(r,x)=\langle g(s,y),\phi(r,x,s,y)\rangle=\langle g,\mathcal{T}_{(r,x)}(\phi)\rangle. \tag{4.2}$$

Theorem 4.5. *Let $w\in\mathcal{S}$. If $\phi,\psi\in\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $g\in\mathcal{D}'_w([0,+\infty[\times\mathbb{R})$, then*

$$(g*\phi)*\psi=g*(\phi*\psi).$$

Proof. Let $w\in\mathcal{S}$ and $\phi,\psi\in\mathcal{D}_w([0,+\infty[\times\mathbb{R})$. For $\varepsilon>0$, we form the Riemann sum

$$f_\varepsilon(r,x)=\varepsilon^{2\alpha+3}\sum_{(s,y)}\mathcal{T}_{(\varepsilon s,\varepsilon y)}(\phi)(r,x)\psi(\varepsilon s,\varepsilon y), \tag{4.3}$$

where (s,y) denotes the integer co-ordinates. We claim that $f_\varepsilon(r,x)$ converges to $\phi*\psi(r,x)$ in $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$ as $\varepsilon\longrightarrow 0$.
Now, by using (2.8), we obtain

$$\begin{aligned} \tilde{\mathcal{F}}(f_\varepsilon)(\mu,\lambda) &= \varepsilon^{2\alpha+3}\sum_{(s,y)}\psi(\varepsilon s,\varepsilon y)j_{\frac{n-1}{2}}(\varepsilon\mu s)e^{-i\langle\lambda,\varepsilon y\rangle}\tilde{\mathcal{F}}(\phi)(\mu,\lambda) \\ &= \tilde{\mathcal{F}}(\phi)(\mu,\lambda)\mathcal{F}_\varepsilon(\mu,\lambda), \end{aligned}$$

where $\mathcal{F}_\varepsilon(\mu,\lambda)=\varepsilon^{2\alpha+3}\sum_{(s,y)}\psi(\varepsilon s,\varepsilon y)j_{\frac{n-1}{2}}(\varepsilon\mu s)e^{-i\langle\lambda,\varepsilon y\rangle}$ is the Riemann sum for the integral defining $\tilde{\mathcal{F}}(\psi)$, which tends to 0 as $\varepsilon\longrightarrow 0$.
On the other hand, by (2.2) we have

$$\begin{aligned} \|f_\varepsilon-\phi*\psi\|_{\sigma,w} &= \iint_{\Upsilon_+}|\mathcal{F}(f_\varepsilon-\phi*\psi)(s,y)|e^{\sigma(w\circ\theta(s,y))}d\gamma(s,y) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}|\tilde{\mathcal{F}}(f_\varepsilon-\phi*\psi)(s,y)|e^{\sigma w(s,y)}d\nu(s,y) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}|\tilde{\mathcal{F}}(f_\varepsilon)(s,y)-\tilde{\mathcal{F}}(\phi)(s,y)\tilde{\mathcal{F}}(\psi)(s,y)|e^{\sigma w(s,y)}d\nu(s,y) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}|\tilde{\mathcal{F}}(\phi)(s,y)||\tilde{\mathcal{F}}(\psi)(s,y)-\mathcal{F}_\varepsilon(s,y)|e^{\sigma w(s,y)}d\nu(s,y). \end{aligned}$$

Since $\tilde{\mathcal{F}}(\phi)(s,y) \longrightarrow 0$ when $|(s,y)| \longrightarrow \infty$, then

$$f_\varepsilon \longrightarrow \phi * \psi, \text{ as } \varepsilon \longrightarrow 0. \tag{4.4}$$

Now from (4.2), (4.4) and (4.3), we have

$$\begin{aligned}
g*(\phi*\psi)(r,x) &= \Big\langle g(s,y), \phi*\psi(r,x,s,y)\Big\rangle \\
&= \Big\langle g(s,y), \lim_{\varepsilon\longrightarrow 0} f_\varepsilon(r,x,s,y)\Big\rangle \\
&= \Big\langle g(s,y), \lim_{\varepsilon\longrightarrow 0} \varepsilon^{2\alpha+3}\sum_{(t,z)} \mathcal{T}_{(\varepsilon t,\varepsilon z)}(\phi)(r,x,s,y)\psi(\varepsilon t,\varepsilon z)\Big\rangle \\
&= \Big\langle g(s,y), \lim_{\varepsilon\longrightarrow 0} \varepsilon^{2\alpha+3}\sum_{(t,z)} \phi(\varepsilon t,\varepsilon z,r,x,s,y)\psi(\varepsilon t,\varepsilon z)\Big\rangle \\
&= \lim_{\varepsilon\longrightarrow 0} \varepsilon^{2\alpha+3}\Big\langle g(s,y), \sum_{(t,z)} \phi(\varepsilon t,\varepsilon z,r,x,s,y)\Big\rangle\psi(\varepsilon t,\varepsilon z) \\
&= \lim_{\varepsilon\longrightarrow 0} \varepsilon^{2\alpha+3}\sum_{(t,z)} g*\phi(\varepsilon t,\varepsilon z,r,x)\psi(\varepsilon t,\varepsilon z) \\
&= \lim_{\varepsilon\longrightarrow 0} \varepsilon^{2\alpha+3}\sum_{(t,z)} \mathcal{T}_{(\varepsilon t,\varepsilon z)}(g*\phi)(r,x)\psi(\varepsilon t,\varepsilon z).
\end{aligned}$$

Therefore the desired result follows from (4.3). □

Lemma 4.6. *Let $w \in \mathcal{S}$. If $\phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $g \in \mathcal{D}'_w([0,+\infty[\times\mathbb{R})$, then*

$$\tilde{\mathcal{F}}(\psi(g*\phi))(s,y) = (g*\phi)*\check{\psi}_{(\mu,\lambda)}(0,0),$$

where $\psi_{(\mu,\lambda)}(r,x) = j_\alpha(r\mu)e^{-i\lambda x}\psi(r,x)$.

Proof. By using (2.6) and (2.16), we get

$$\begin{aligned}
(g*\phi)*\check{\psi}_{(\mu,\lambda)}(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,x)}(g*\phi)(s,y)\check{\psi}_{(\mu,\lambda)}(s,y)d\nu(s,y) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \check{\psi}_{(\mu,\lambda)}(s,y)\Big(\int_0^{+\infty}\int_{\mathbb{R}} g*\phi(u,v) \\
&\quad\times \mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big)d\nu(s,y).
\end{aligned}$$

Now, from (2.14), (2.12) and Fubini's theorem, we obtain

$$\begin{aligned}
(g*\phi)*\check{\psi}_{(\mu,\lambda)}(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}}\Big(\int_0^{+\infty}\int_{\mathbb{R}} g*\phi(u,v)j_\alpha(uz)e^{-ivt}d\nu(u,v)\Big) \\
&\quad\times\Big(\int_0^{+\infty}\int_{\mathbb{R}} \check{\psi}_{(\mu,\lambda)}(s,y)j_\alpha(sz)e^{iyt}d\nu(s,y)\Big) \\
&\quad\times j_\alpha(rz)e^{-ixt}d\nu(z,t) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(g*\phi)(z,t)\tilde{\mathcal{F}}(\psi_{(\mu,\lambda)})(z,t)j_\alpha(rz)e^{-ixt}d\nu(z,t).
\end{aligned}$$

Putting $(r,x) = (0,0)$, we get

$$(g*\phi)*\check{\psi}_{(\mu,\lambda)}(0,0) = \int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(g*\phi)(z,t)\tilde{\mathcal{F}}(\psi_{(\mu,\lambda)}(z,t)d\nu(z,t).$$

Parseval formula (2.13) and the fact that $\psi_{(\mu,\lambda)}(r,x) = j_\alpha(r\mu)e^{-i\lambda x}\psi(r,x)$, completes the proof of this lemma. □

Proposition 4.7. *Let $\phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $g \in \mathcal{D}'_w([0,+\infty[\times\mathbb{R})$. If supp($\phi$) is contained in fixed compact set then there exists a constant C such that*

$$|\langle g, \phi\rangle| \leqslant C\|\phi\|_{\sigma,w}.$$

Proof. Let $\phi \in \mathcal{D}_w([0,+\infty[\times\mathbb{R})$ and $g \in \mathcal{D}'_w([0,+\infty[\times\mathbb{R})$. From (4.1), (2.13) and (2.2), we get

$$\begin{aligned}\langle g, \phi\rangle &= \int_0^{+\infty}\int_{\mathbb{R}} g(r,x)\phi(r,x)d\nu(r,x)\\ &= \iint_{\Upsilon_+} \mathcal{F}(g)(\mu,\lambda)\mathcal{F}(\phi)(\mu,\lambda)d\gamma(\mu,\lambda).\end{aligned}$$

Then

$$\begin{aligned}|\langle g, \phi\rangle| &\leqslant \iint_{\Upsilon_+} |\mathcal{F}(g)(\mu,\lambda)||\mathcal{F}(\phi)(\mu,\lambda)|e^{\sigma(w\circ\theta(\mu,\lambda))}e^{-\sigma(w\circ\theta(\mu,\lambda))}d\gamma(\mu,\lambda)\\ &\leqslant \iint_{\Upsilon_+} |\mathcal{F}(g)(\mu,\lambda)||\mathcal{F}(\phi)(\mu,\lambda)|e^{\sigma(w\circ\theta(\mu,\lambda))}e^{-\sigma(w\circ\theta(\mu,\lambda))}d\gamma(\mu,\lambda)\\ &\leqslant \sup_{(\mu,\lambda)\in\Upsilon} |\mathcal{F}(\phi)(\mu,\lambda)|e^{\sigma(w\circ\theta(\mu,\lambda))}\iint_{\Upsilon_+} |\mathcal{F}(g)(\mu,\lambda)|e^{-\sigma(w\circ\theta(\mu,\lambda))}d\gamma(\mu,\lambda)\\ &= C\|\phi\|_{\sigma,w},\end{aligned}$$

where $C = \iint_{\Upsilon_+} |\mathcal{F}(g)(\mu,\lambda)|e^{-\sigma(w\circ\theta(\mu,\lambda))}d\gamma(\mu,\lambda)$. □

Lemma 4.8. *Let ϕ and ψ in $\mathcal{D}_w([0,+\infty[\times\mathbb{R})$. Then*
(1)

$$\phi * \psi = \psi * \phi.$$

(2) for every $\beta \in \mathbb{N}$, we have

$$(\Lambda_\alpha)^\beta(\phi) * \psi = \phi * (\Lambda_\alpha)^\beta(\psi) = (\Lambda_\alpha)^\beta(\phi * \psi). \tag{4.5}$$

Definition 4.9. Let $\mathcal{S}_c$ be the set of all $w \in \mathcal{S}$ such that $w(r,x) = f(|(r,x)|)$, where f is an increasing continuous concave function on $[0,+\infty[$.

Definition 4.10. Let $w \in \mathcal{S}_c$ and $\mathcal{S}_w([0,+\infty[\times\mathbb{R})$ is defined to be the set of all functions $\phi \in L^1(d\nu)$ with the property that $\phi, \tilde{\mathcal{F}}(\phi) \in C^\infty([0,+\infty[\times\mathbb{R})$ such that, for each $\beta \in \mathbb{N}$ and for each non-negative number σ, we have

$$\mathcal{P}_{\beta,\sigma}(\phi) = \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(r,x)}|(\Lambda_\alpha)^\beta\phi(r,x)| < \infty, \tag{4.6}$$

and

$$\Pi_{\beta,\sigma}(\phi) = \sup_{(s,y)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(s,y)}|(\Lambda_\alpha)^\beta\tilde{\mathcal{F}}(\phi)(s,y)| < \infty. \tag{4.7}$$

Theorem 4.11. *Let $w \in \mathcal{S}_c$, then $\mathcal{S}_w([0,+\infty[\times\mathbb{R})$ is topological algebra under point-wise multiplication.*

Proof. Let $\phi, \psi \in \mathcal{S}_w([0,+\infty[\times\mathbb{R})$. Then by (2.18), we have

$$\begin{aligned}(\Lambda_\alpha)^\beta(\phi(r,x)\psi(r,x)) &= \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} r^{p-\beta}\\ &\quad\times \frac{\partial^{\ell+q}}{\partial x^\ell \partial r^q}(\phi(r,x))\frac{\partial^{2\beta-2i+p-q-\ell}}{\partial r^{p-q}\partial x^{2\beta-2i-\ell}}(\psi(r,x)).\end{aligned}$$

Again, by (4.6), we get

$$
\begin{aligned}
\mathcal{P}_{\beta,\sigma}(\phi\psi) &= \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(r,x)}\Big|\sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} r^{p-\beta} \\
&\times \frac{\partial^{\ell+q}}{\partial x^\ell \partial r^q}(\phi(r,x))\frac{\partial^{2\beta-2i+p-q-\ell}}{\partial r^{p-q}\partial x^{2\beta-2i-\ell}}(\psi(r,x))\Big| \\
&\leqslant \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} \\
&\times e^{\frac{\sigma w(r,x)}{2}}|\frac{\partial^{\ell+q}}{\partial x^\ell \partial r^q}(\phi(r,x))|e^{\frac{\sigma+2k}{2}w(r,x)}|\frac{\partial^{2\beta-2i+p-q-\ell}}{\partial r^{p-q}\partial x^{2\beta-2i-\ell}}(\psi(r,x))|, k\in\mathbb{N} \\
&\leqslant \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\frac{\sigma w(r,x)}{2}}|\frac{\partial^{\ell+q}}{\partial x^\ell \partial r^q}(\phi(r,x))| \\
&\times \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\frac{\sigma+2k}{2}w(r,x)}|\frac{\partial^{2\beta-2i+p-q-\ell}}{\partial r^{p-q}\partial x^{2\beta-2i-\ell}}(\psi(r,x))|.
\end{aligned}
$$

Also, using (4.6), we get

$$
\begin{aligned}
\mathcal{P}_{\beta,\sigma}(\phi\psi) \leqslant & \sum_{i=0}^{\beta}\sum_{p=1}^{2i}\sum_{q=0}^{p}\sum_{\ell=0}^{2(\beta-i)} C_\beta^i C_p^q C_{2(\beta-i)}^\ell \aleph_{\beta,p} \\
&\times \mathcal{P}_{\ell+q,\frac{\sigma}{2}}(\phi)\mathcal{P}_{2\beta-2i+p-q-\ell,\frac{\sigma+2k}{2}}(\psi) < \infty.
\end{aligned}
$$

This show that $\mathcal{P}_{\beta,\sigma}(\phi\psi) < \infty$. Next, we have to prove that $\Pi_{\beta,\sigma}(\phi\psi) < \infty$. From (4.7), (2.11) and (4.5), we have

$$
\begin{aligned}
\Pi_{\beta,\sigma}(\phi\psi) &= \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(r,x)}\Big|\Lambda_\alpha^\beta \tilde{\mathfrak{F}}(\phi\psi)(r,x)\Big| \\
&= \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(r,x)}\Big|\Lambda_\alpha^\beta\Big(\tilde{\mathfrak{F}}(\phi)*\tilde{\mathfrak{F}}(\psi)(r,x)\Big)\Big| \\
&= \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(r,x)}\Big|\tilde{\mathfrak{F}}(\phi)*\Lambda_\alpha^\beta\tilde{\mathfrak{F}}(\psi)(r,x)\Big|.
\end{aligned}
$$

On the other hand, by (2.14), we obtain

$$\begin{aligned}
&\Big|\tilde{\mathcal{F}}(\phi) * \Lambda_\alpha^\beta \tilde{\mathcal{F}}(\psi)(r,x)\Big| \\
&= \Big|\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\phi)(s,y)\mathcal{T}_{(r,x)}(\Lambda_\alpha^\beta \tilde{\mathcal{F}}(\psi))(s,y)d\nu(s,y)\Big| \\
&= \Big|\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\phi)(s,y) \\
&\times \Big(\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\psi)(u,v)\Lambda_\alpha^\beta \mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big)d\nu(s,y)\Big| \\
&= \Big|\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\phi)(s,y)\Big(\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\psi)(u,v) \\
&\times \Lambda_\alpha^\beta\Big(\int_0^{+\infty}\int_{\mathbb{R}} j_\alpha(rz)e^{-ixt}j_\alpha(sz)e^{iyt}j_\alpha(uz) \\
&\times e^{-ivt}d\nu(z,t)\Big)d\nu(u,v)\Big)d\nu(s,y)\Big| \\
&= \Big|\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\phi)(s,y)\Big(\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\psi)(u,v) \\
&\times \Big(\int_0^{+\infty}\int_{\mathbb{R}} j_\alpha(rz)e^{-ixt}(-|(z,t)|^2)^\beta j_\alpha(sz)e^{iyt}j_\alpha(uz) \\
&\times e^{-ivt}d\nu(z,t)\Big)d\nu(u,v)\Big)d\nu(s,y)\Big| \\
&= \Big|\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\phi)(s,y)\Big(\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\psi)(u,v) \\
&\times (\Lambda_\alpha)^\beta_{(u,v)}\mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big)d\nu(s,y)\Big|.
\end{aligned}$$

Now integrating by parts, we get

$$\begin{aligned}
&\Big|\tilde{\mathcal{F}}(\phi) * \Lambda_\alpha^\beta \tilde{\mathcal{F}}(\psi)(r,x)\Big| \\
&= \Big|\int_0^{+\infty}\int_{\mathbb{R}} \tilde{\mathcal{F}}(\phi)(s,y)\Big(\int_0^{+\infty}\int_{\mathbb{R}} (-1)^\beta(\Lambda_\alpha)^\beta_{(u,v)}\tilde{\mathcal{F}}(\psi))(u,v) \\
&\times \mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big)d\nu(s,y)\Big| \\
&\leqslant \int_0^{+\infty}\int_{\mathbb{R}} \Big|\tilde{\mathcal{F}}(\phi)(s,y)\Big|\Big|\Big(\int_0^{+\infty}\int_{\mathbb{R}} (\Lambda_\alpha)^\beta_{(u,v)}\tilde{\mathcal{F}}(\psi))(u,v) \\
&\times \mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big|\Big)d\nu(s,y) \\
&\leqslant \sup_{(u,v)\in[0,+\infty[\times\mathbb{R}} \Big|(\Lambda_\alpha)^\beta_{(u,v)}\tilde{\mathcal{F}}(\psi)(u,v)\Big|\int_0^{+\infty}\int_{\mathbb{R}} \Big|\tilde{\mathcal{F}}(\phi)(s,y)\Big| \\
&\times \Big|\Big(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{D}_\alpha(r,x,s,y,u,v)d\nu(u,v)\Big)\Big|d\nu(s,y).
\end{aligned}$$

Therefore, combining (2.15) and (4.7), we obtain

$$\begin{aligned}
\Pi_{\beta,\sigma}(\phi\psi) &= \sup_{(r,x)\in[0,+\infty[\times\mathbb{R}} e^{\sigma w(r,x)}\left|(\Lambda_\alpha)^\beta\tilde{\mathcal{F}}(\psi))(r,x)\right| \\
&\times \int_0^{+\infty}\int_{\mathbb{R}} e^{\varrho w(s,y)}e^{-\varrho w(s,y)}\left|\tilde{\mathcal{F}}(\phi)(s,y)\right|d\nu(s,y) \\
&\leqslant \Pi_{\beta,\sigma}(\psi)\sup_{(s,y)\in[0,+\infty[\times\mathbb{R}}|\tilde{\mathcal{F}}(\phi)(s,y)|e^{\varrho w(s,y)} \\
&\times \int_0^{+\infty}\int_{\mathbb{R}} e^{-\varrho w(s,y)}d\nu(s,y) \\
&= \Pi_{\beta,\sigma}(\psi)\Pi_{0,\varrho}(\phi)\int_0^{+\infty}\int_{\mathbb{R}} e^{-\varrho w(s,y)}d\nu(s,y) < \infty.
\end{aligned}$$

Which evidently completes the proof of the theorem. □

Theorem 4.12. *Let $w \in \mathcal{S}_c$, then $\mathcal{S}_w([0,+\infty[\times\mathbb{R})$ is topological algebra under convolution.*

Proof. The results can be proved in the same way of Theorem 4.11. □

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