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Taylor's Power Law and Packing Circles

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Abstract
The famous Taylor Power Law is in general observed in ecology and relates the variance of the population of a certain species in a unit area while Circle Packing is an arrangement of circles in a given area. We show that the circle packing problem in $\mathbb{R}^2$ satisfies the Taylor power law formula for $b = 2$.

Keywords: *Density, Distance, Packing circles, Population, Taylor's Law*

AMS Subject Classification (2020): *60C05; 62-07; 65K05*

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1. Introduction

In [1], the author presented a linear relationship between the expectation and the variance of a population size in a complex system. Since then, this relation stated explicitly as

$$\text{variance} = a(\text{mean})^b \text{ with } \ a, b > 0.$$

is called Taylor's Power Law (abbreviated as TPL) and has been observed in various ecological and biological systems, including populations of animals, plants, and microorganisms. The exponent b in TPL for the majority of these analyzed systems ranges from 1 to 2, with a clustering around $b = 2$. Different models have been investigated thus far, but no clear cause for this occurrence has yet been found. Our approach here may be a reference to that phenomenon. Note that when $b = 1$ the population is distributed homogeneously across space. In order to predict how populations will behave over time or in determining the spatial distribution of populations TPL is helpful.

In this study, which aims to address the spatial distribution of individuals within a population, we associate TPL with another important concept *the circle packing problem*(abbreviated as CPP), which is about optimizing the maximum radius of n ($n \geq 1$) identical circles placed in a closed region in $\mathbb{R}^d$ ($d \geq 2$) such that none of the circles in the region overlap. There are several variations of CPP, including the problem where circles must be placed within a specific shape or the sizes of the circles are not all equal. The reader can find various packing representations of circles in [2] when $d = 2$; for example, if the closed region is a square in $\mathbb{R}^2$, in the case where $n = 1$ there exists a unique circle in the packing and the radius of the circle is 0.5. In the case where $n = 7$, the best packing is given in Figure 1 below. The best packing means that the region contains the largest number of non-overlapping identical

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circles. By [2], the radius r of each circle in the packing is approximately 0.1744576302 and the greatest distance between two centers is approximately 0.535898384. In [2], the CPP is solved up to $n = 10000$ circles inside the different shapes in $\mathbb{R}^2$.

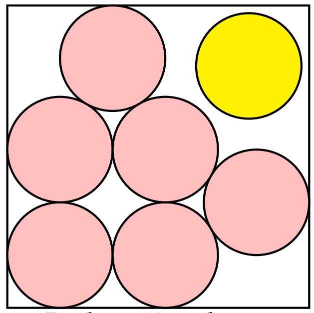


Figure 1. Packing 7 circles in a square

In dimension 3, the circle packing problem becomes the sphere packing problem which begins with a conjecture of Kepler and solved in [3]. So far, the problem has been solved up to the case where $d = 24$.

Here we will see the circle packing problem as the distribution of points in a closed region. More precisely, suppose we are trying to place n distinct points in a closed region in $\mathbb{R}^d$ such that the minimum distance between any two points is as large as possible. Assuming each of these points to be the center of a circle, the distribution of points in this closed region coincides with the problem of finding the radius of circles in the circle packing problem. we show that the distance between the centers of two randomly chosen circles in a packing obeys TPL.

Here, the TPL formula, which has been applied to explain the demographic structure of a living species (insects, microorganisms, humans) is actually thought to be related to the CPP. Our result mainly based on [4] in which the author established TPL as an important tool for understanding population dynamics and spatial patterns in different fields. In the next section, we study the probability distribution of the distance between two randomly chosen points on a line, on a circle and also on a square in $\mathbb{R}^2$. We assume that the distribution of distances between randomly chosen points is independent and uniformly distributed in the fixed region. In the rest of the work, we present the relationship between the expectation and the variance of the distance between the centers of the circles placed in a square with respect to the optimization of the packing and, we show that CPP satisfies TPL.

2. Distance between points in a fixed region and TPL

Let ℓ be a line in $\mathbb{R}^2$ of length $L > 0$. The choice of a randomly chosen point on ℓ is given by a random variable X_1 with the probability density function

$$f_{X_1}(x) = \begin{cases} \frac{1}{L} & \text{if } x \in [0, L] \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Now let us choose a second point on ℓ. It gives the random variable X_2. Obviously, the distance $Y = |X_1 - X_2|$ between the points will also be a random variable. The probability density function of Y is known to be

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = \frac{1}{L^2}$$

Proposition 2.1. *With preceding notation, the random variable Y obeys TPL.*

Proof. Consider

$$\varphi(x_1, x_2) = |x_1 - x_2| = \begin{cases} x_1 - x_2, & \text{if } x_1 \geq x_2 \\ x_2 - x_1, & \text{if } x_2 \geq x_1 \end{cases} \tag{2.2}$$

The expected value of the distance between two randomly chosen points is

$$E(Y) = E(\varphi(x_1, x_2)) = \int_0^L \int_0^L \varphi(x_1, x_2) f_{X_1X_2}(x_1, x_2)dx_1dx_2$$

$$= \frac{1}{L^2}\int_0^L\int_0^L |x_1 - x_2| dx_2 dx_1 = \frac{L}{3}$$

so the variance is

$$Var(Y) = \frac{1}{L^2}\int_0^L\int_0^L |x_1 - x_2|^2 dx_2 dx_1 - \frac{L^2}{9} = \frac{L^2}{18}$$

Hence Y obeys TPL with the values $b = 2$ and $a = \frac{1}{2}$. □

Let $\mathcal{C}$ be a circle with radius $r > 0$ in $\mathbb{R}^2$. By [4, 5], the probability density function of the distance between two randomly selected points on $\mathcal{C}$ is

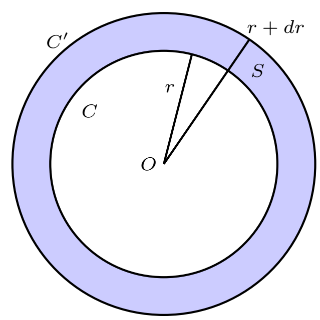
$$f(x) = \frac{4x}{\pi r^2}\left(\arccos\Big(\frac{x}{2r}\Big) - \frac{x}{2r}\Big(1 - \frac{x^2}{4r^2}\Big)^{\frac{1}{2}}\right).$$

Proposition 2.2. *The distance between two random points on $\mathcal{C}$ obeys TPL.*

Proof. Let us choose two points $P_1 = (T_1, \Theta_1)$ and $P_2 = (T_2, \Theta_2)$ (in polar coordinates) on $\mathcal{C}$. The randomness of the selection tells us that the probability of one of the points lying in the area dA is proportional to dA:

$$\mathbb{P}\{T_i \in (r_i, r_i + dr_i), \Theta_i \in (\theta_i, \theta_i + d\theta_i)\} = \frac{r_i dr_i d\theta_i}{\pi r^2}, i = 1, 2.$$

Let Y be the distance between P_1 and P_2 which belongs to the interval $(x, x + dx)$. Consider another circle $\mathcal{C}'$ with the same center as $\mathcal{C}$. So, its radius is $r + dr$. Denote by S the annulus between two circles. If two points are in C' we have one of the following cases:
(i) Both points are in C,
(ii) At least one point is in S.



The probability that two points are in C is

$$\mathbb{P}\{r + dr\} = \mathbb{P}\{r + dr|\text{ case}(i)\ \} \times \mathbb{P}\{\text{ case}(i)\ \} + \mathbb{P}\{r + dr|\text{ case}(ii)\ \} \times \mathbb{P}\{\text{ case}(ii)\ \} \tag{2.3}$$

Let us consider each point separately to compute $\mathbb{P}\{r + dr|\text{ case}(i)\ \}$:

$$\mathbb{P}\{P_1 \text{ is in } C\} = \frac{\text{area(C)}}{\text{area(C')}} = \frac{\pi r^2}{\pi(r + dr)^2} == \frac{1}{1 + 2dr/r + dr^2/r^2} = 1 - \frac{2dr}{r} + o(dr)$$

Since the cases (i) and (ii) are independent we obtain

$$\mathbb{P}\{\text{case}(i)\} = (1 - \frac{2dr}{r} + o(dr))^2 = 1 - \frac{4dr}{r} + o(dr)$$

Hence $\mathbb{P}\{r + dr \mid \text{ case}(ii)\} = \dfrac{2xdx}{\pi r^2} \arccos\dfrac{x}{2r}$ and $\mathbb{P}\{\text{ case}(ii)\} = \frac{4dr}{r} + o(dr)$.

Substitution of these values in (3) gives

$$\mathbb{P}\{r+dr\} = \mathbb{P}\{r\}\left(1-\frac{4dr}{r}\right) + \frac{2xdx}{\pi r^2}\arccos\left(\frac{x}{2r}\right)\left(\frac{4dr}{r}\right) + o(dr)$$

Denote by P. We then get

$$dP = \mathbb{P}\{r+dr\} - \mathbb{P}\{r\} = \left[\frac{-4P}{r} + \frac{8x\,dx}{\pi r^3}\arccos\biggl(\frac{x}{2r}\biggr)\right]dr + o(dr)$$

$$r^4 dP + 4r^3 P dr = \frac{8x\,dx\,r}{\pi}\arccos\biggl(\frac{x}{2r}\biggr)dr + o(dr)$$

$$\frac{d}{dr}(Pr^4) = \frac{8x\,dx\,r}{\pi}\arccos\biggl(\frac{x}{2r}\biggr)$$

The integration of both sides gives

$$Pr^4 = \frac{4x^2dx}{\pi}\int\frac{2r}{x}\arccos\biggl(\frac{x}{2r}\biggr)dr + C$$

Therefore,

$$Pr^4 = \frac{4x^2dx}{\pi}\left(\frac{\arccos\left(\frac{x}{2r}\right)r^2}{x} - \frac{\sqrt{4r^2-x^2}}{4}\right)$$

$$P = \frac{4xdx}{\pi r^2}\left(\arccos\biggl(\frac{x}{2r}\biggr) - \frac{x}{2r}\left(1-\frac{x^2}{4r^2}\right)^{\frac{1}{2}}\right)$$

Now let us compute $E(Y)$ where Y is the distance between the points.

$$E(Y) = \int_0^{2R} x\frac{4x}{\pi R^2}\left(\arccos\biggl(\frac{x}{2R}\biggr) - \frac{x}{2R}\left(1-\frac{x^2}{4R^2}\right)^{\frac{1}{2}}\right)dx$$

First, replace $\frac{x}{2R} = u$ for computing $I_1 = \dfrac{4}{\pi R^2}\int x^2\arccos\left(\dfrac{x}{2R}\right)dx$, :

$$I_1 = \frac{4}{\pi R^2}8R^3\int u^2\arccos(u)du$$

Integration by parts with $f = arccos(u), g' = u^2$ gives

$$I_1 = \frac{4}{\pi R^2}8R^3\left(\frac{u^3 arccos(u)}{3} + \frac{(1-u^2)^{\frac{3}{2}}}{9} - \frac{\sqrt{1-u^2}}{3}\right)$$

$$I_1 = \frac{4}{\pi R^2}\left(\frac{x^3\arccos(\frac{x}{2R})}{3} + \frac{8R^3(1-(\frac{x}{2R})^2)^{\frac{3}{2}}}{9} - \frac{8R^3\sqrt{1-(\frac{x}{2R})^2)^2}}{3}\right)$$

$$I_1 = -\frac{4\left(\sqrt{4R^2-x^2}\left(Rx^2+8R^3\right) - 3\left|R\right|x^3\arccos\left(\frac{x}{2R}\right)\right)}{9\pi R^2\left|R(\frac{x}{2R})\right|}$$

Substitute $u = 4R^2 - x^2$. We get

$$I_2 = -\int\frac{2x^3\sqrt{1-\frac{x^2}{4R^2}}}{\pi R^3}dx = -\frac{1}{\pi R^4}\int x^3\sqrt{4R^2-x^2}$$

$$= -\frac{1}{\pi R^4}\left(\frac{1}{2}\int u^{\frac{3}{2}} - 4R^2\sqrt{u}\,du\right) = \frac{\left(24x^2+64R^2\right)\left(1-\frac{x^2}{4R^2}\right)^{\frac{3}{2}}}{15\pi R}$$

Since $E(Y) = (I_1 + I_2)|_0^{2R}$ we finally obtain

$$E(Y) = \left(-\frac{\sqrt{4R^2 - x^2}\left(9x^4 + 8R^2x^2 + 64R^4\right) - 60R^2x^3 \arccos\left(\frac{x}{2R}\right)}{45\pi R^4} \right)\Bigg|_0^{2R} = \frac{128}{45\pi}R$$

Hence the mean is $E(Y) \approx 0.9054R$. Let us compute the variance of Y: Let $x_1 = (x, y)$ and $x_2 = (x', y')$. So, the square of the distance between x_1 and x_2 is:

$$d^2(x_1, x_2) = (x - x')^2 + (y - y')^2$$

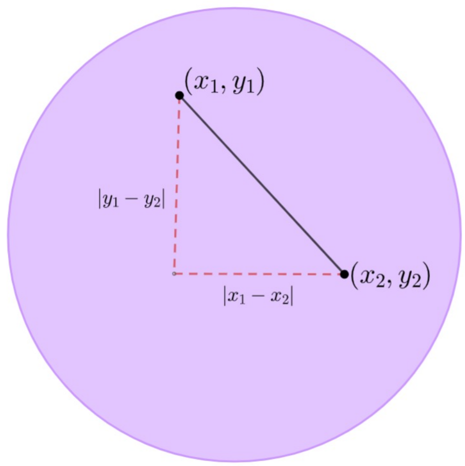


Figure 2. The distance on a circle

Therefore,

$$E(Y^2) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x - x')^2 + (y - y')^2 \, dx \, dy \, dx' \, dy'$$

By the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x' = r'\cos\theta'$, $y' = r'\sin\theta'$ we get

$$E(Y^2) = \frac{1}{\pi^2R^4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{R}\int_0^{R} (r\cos\theta - r'cos\theta')^2 + (r\sin\theta - r'\sin\theta')rr' \, dr \, dr' \, d\theta \, d\theta'$$

$$= \frac{1}{\pi^2R^4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{R}\int_0^{R} r^3r' + (r')^3r \, dr \, dr' \, d\theta \, d\theta' = R^2$$

Therefore $Var(Y) = R^2 - \left(\frac{128R}{45\pi}\right)^2 \approx 0.0934R^2$, which concludes the affirmation of the proposition. □

Proposition 2.3. [4] *Let S be a square of size $R > 0$ in $\mathbb{R}^2$. The distance d between two randomly selected points in S obeys TPL.*

Proof. To evaluate the expectation of the distance $d = \sqrt{(x - x')^2 - (y - y')^2}$, without loss of generality, we assume $R = 1$ and calculate the integral

$$I = \int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{(x - x')^2 + (y - y')^2} \, dx' \, dy' \, dy \, dx$$

By symmetry, we write:

$$I = 4\int_0^1\int_0^1\int_0^y\int_0^x \sqrt{(x - x')^2 + (y - y')^2} \, dx' \, dy' \, dy \, dx$$

First substitute $x' \mapsto xx', \quad y' \mapsto yy'$:

$$I = 4\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2(1-x')^2 + y^2(1-y')^2}yxdx^{'}dy^{'}dydx$$

and then substitute $x' \mapsto 1-x', \quad y' \mapsto 1-y'$ we have:

$$I = 4\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2x'^2 + y^2y'^2}\; yx\, dx'\, dy'\, dydx$$

After another substitution $y^2 = u, \quad x^2 = v$:

$$I = \int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{vx'^2 + uy'^2}\;\; dx'\, dy'\, dudv$$

Finally with $vx'^2 = p, \quad uy'^2 = q$:

$$I = \int_0^1\int_0^1\int_0^{y'^2}\int_0^{x'^2} \sqrt{p+q}\;\; dpdq\frac{dydw}{y^2w^2}$$

$$I = \frac{2}{3}\int_0^1\int_0^1\int_0^{y'^2} \left((q+w^2)^{3/2} - q^{3/2}\right)dq\frac{dy'dx'}{y'^2y^2}$$

$$I = \frac{4}{15}\int_0^1\int_0^1 \left((y'^2+y^2)^{5/2} - y'^5 - y^5\right)\frac{dy'dy}{y'^2y^2}$$

By symmetry:

$$I = \frac{8}{15}\int_0^1\int_0^y \left((y'^2+y^2)^{5/2} - y'^5 - y^5\right)\frac{dy'dy}{y'^2y^2}$$

Substitute $y' = ys$:

$$I = \frac{8}{15}\int_0^1\int_0^1 y^2\left((1+s^2)^{5/2} - s^5 - 1\right)\frac{dsdy}{s^2}$$

$$I = \frac{8}{45}\int_0^1 \left((1+s^2)^{5/2} - s^5 - 1\right)\frac{ds}{s^2}$$

$$I = \left.\frac{15s\ln\left(\left|\sqrt{s^2+1}+s\right|\right) - 2s^5 + \sqrt{s^2+1}\left(2s^4+9s^2-8\right)+8}{45s}\right|_0^1$$

$$I = \frac{5\,\mathrm{arsinh}\,(1) + \sqrt{2} + 2}{15}$$

This says that the mean of d is $E(d) = \dfrac{R}{15}(\mathrm{arsinh}\,(1) + \sqrt{2} + 2) \approx 0.5214R$. Hence the variance $Var(d)$ is

$$E(d^2) - E(d)^2 = \frac{R^2}{3} - \left(\frac{R}{15}\left(\mathrm{arsinh}\,(1) + \sqrt{2} + 2\right)\right)^2 \approx 0.0615R^2$$

where

$$E(d^2) = \frac{1}{(R^2)^2}\int_0^R\int_0^R\int_0^R\int_0^R (x-x')^2 + (y-y')^2\, dx\, dx'\, dy\, dy'$$

Therefore TPL is $Var(d) = a(E(d))^b$ which is satisfied for $b = 2$. As expectation is a function of R where variance is a function of R^2. $\square$

Listing 1. Python simulation of mean and variance calculation

```
import math
import matplotlib.pyplot as plt
import pandas as pd
import os
from os.path import exists
import glob
from itertools import combinations

# Auxiliar functions

def Read_file(file_name):
    with open(file_name) as file:
        points = [(float(line.split()[-1]), float(line.split()[-2])) for line in
file]
    return points

def Mean_and_Variance(file_name):
    distances = []
    points = Read_file(file_name)
    for p1, p2 in combinations(points, 2):
        distances.append(math.sqrt((p1[0] - p2[0])**2 + (p1[1] - p2[1])**2))
    mean = sum(distances) / len(distances)
    variance = sum(d**2 for d in distances) / len(distances) - mean**2
    return mean, variance

# Execution of the code

file_list = sorted(glob.glob("/path/to/files/*.txt"), key=os.path.getsize)[1:]
means = []
variances = []
for file_name in file_list:
    mean, variance = Mean_and_Variance(file_name)
    means.append(mean)
    variances.append(variance)

coefA = [v / (e**2) for e, v in zip(means, variances)]
```

3. Main result

In this section, we answer the following question:
(*) *Does the distance between the centers of the randomly chosen circles in a best packing in a square obeys TPL?*
Let us consider a square in $\mathbb{R}^2$ and let n be the number of circles in a best packing. Using the data from [2] (In the page, circles in square is used) , we proceed as follows:

1st operation. Assign the coordinates to n points $P_1, P_2 \ldots P_n$ each of which represent the center of a circle in the best packing. For example, for $n = 7$ we list the data as

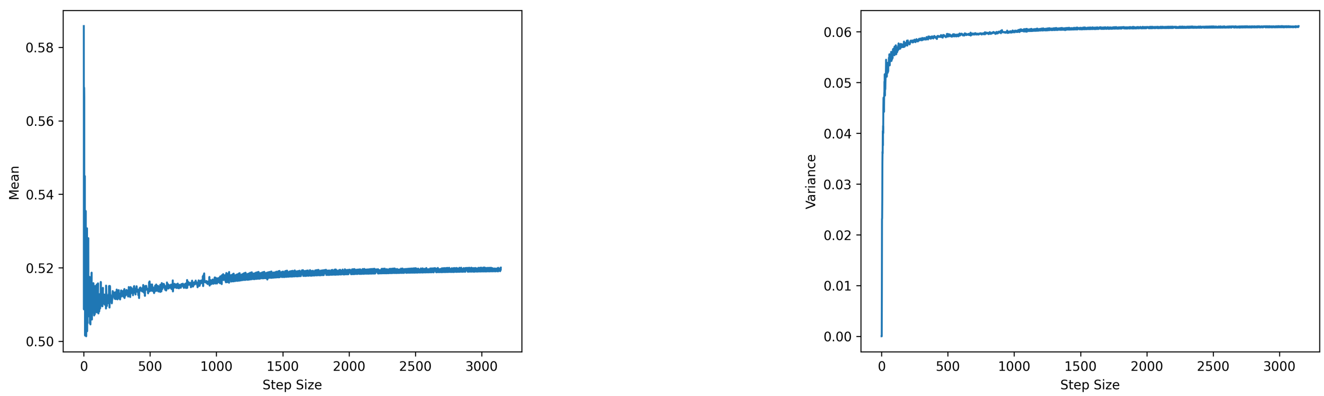
Circle	x-coordinate	y-coordinate
Circle 1	-0.325542369812990561040572795501	-0.325542369812990561040572795501
Circle 2	0.023372890561028316878281613499	-0.325542369812990561040572795500
Circle 3	0.325542369812990561040572795500	-0.151084739625981122081145591001
Circle 4	-0.325542369812990561040572795500	0.023372890561028316878281613499
Circle 5	0.023372890561028316878281613499	0.023372890561028316878281613499
Circle 6	0.300000000000000000000000000000	0.300000000000000000000000000000
Circle 7	-0.151084739625981122081145591001	0.325542369812990561040572795500

2nd operation. Make a list of pairs (P_i, P_j) for all $i \neq j$.
3rd operation. Compute the distance d_{ij} between each pair (P_i, P_j) and transfer the results to the list named as "distances".
4th operation. Compute the mean and the variance using the d_{ij}'s in the list "distances" and transfers the results to the lists named "AllMeans" and "AllVariances" respectively.
5th operation. Store the mean and variance values in the same level in "AllMeans" and "AllVariances" respectively. Then, compute the coefficient a in the formula TPL. Transfer the result to the list named "coefA".
6th operation. Constructing a loop on n. Note that, in [2], the author presents 3146 packings.

Theorem 3.1. *The distance between two randomly chosen centers in a best circle packing satisfies TPL.*

Proof. We will present a visual proof with results that we get from previous simulations. Figure 3a below resulting from our algorithm represent the change of means with respect to the change of number of circles. On the other hand, the Figure 3b the change of variances with respect to the change of number of circles.

Since the data in [2] contains 3146 packings, the step size is not 10000. The graph in Figure 4 shows the change of variance with respect to the change of the number of circles.



(a) Mean change by step

(b) Variance change by step

Figure 3. Mean and variance change by step

Fixing $b = 2$ in the formula TPL, the graph representing the change of coefficient a with respect to the change of number of circles shows that a converges which concludes the empirical proof of TPL in our specific case.

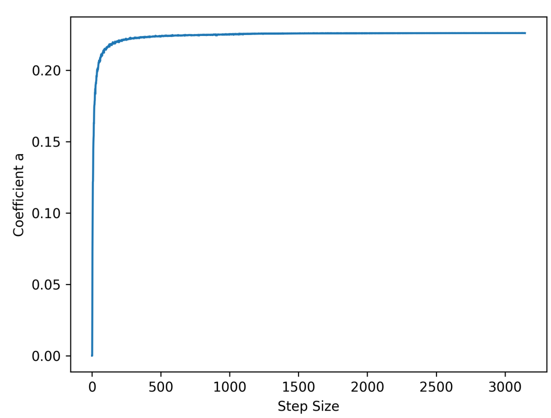


Figure 4. Convergence of *a*

□

Taylor's Power Law can be applied to the population density problems of a city or country ([6]). In this paper, we showed that Circle Packing can be used as another method for population density problems.

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