http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 74, Number 1, Pages 138–149 (2025)
https://doi.org/10.31801/cfsuasmas.1448993
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: March 8, 2024; Accepted: December 14, 2024

New integral inequalities involving p-convex and s-p-convex functions

Sercan TURHAN,[1] Aykut KILIÇ[2] and İmdat İŞCAN[3]

[1,3]Department of Mathematics, Giresun University, Giresun, TÜRKİYE
[2]Institute of Science, Giresun University, Giresun, TÜRKİYE

Abstract. In this study, new lemmas on p-convex and s-p-convex functions were derived utilizing the integral $\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx$. Through this equality, new integral inequalities were established, and novel upper bounds were obtained with the aid of Euler's beta and hypergeometric functions. The results provided new inequalities for the class of classical convex functions and the class of harmonic convex functions.

2020 Mathematics Subject Classification. 26A51, 26D15.
Keywords. P-convex function, s-p-convex function, harmonically convex function, hypergeometric function.

1. Introduction

Recently, new and innovative approaches to classical convexity principles have been integrated to develop extended and generalised ideas in various fields. These developments include $p-$convex functions and s-p-second kind convex functions.

Definition 1. *A function $m : I \subset (0,\infty) \to \mathbb{R}$ is said to be $p-$convex, if*

$$m\left([uj^p+(1-u)k^p]^{1/p}\right) \le um(j)+(1-u)m(k), \tag{1}$$

for all $j,k \in I$ and $u \in [0,1]$(see [8]).

For some new research, results and generalisations for the $p-$convex function (see [5], [6], [8], [9], [11], [12]). In definition 1, for $p=1$, a $p-$ convex function reduces to a convex function, and for $p=-1$, a $p-$convex function reduces to a harmonically convex (HA) function.

Definition 2. *Let $s \in [0,1]$ and $p \in \mathbb{R}\backslash\{0\}$. A function $m : I \subset (0,\infty) \to [0,\infty)$ is said to be the $s-p$-convex function in second kind, if*

$$m\left([uj^p+(1-u)k^p]^{\frac{1}{p}}\right) \le u^s m(j)+(1-u)^s m(k), \tag{2}$$

for all $j,k \in I$ and $u \in [0,1]$ (see [2]).

In inequality 2, for $s \in [0,1]$, if $p=1$ and $s=1$, it corresponds to the definition of convexity; if $p=-1$ and $s=1$, it corresponds to the definition of harmonically convex function; if $p=1$, it corresponds to the definition of $s-$convexity in the second kind; if $s=1$, it corresponds to the definition of p-convexity. For some new research, results, and generalizations for the $s-p$-convex function. (see [2], [3].)

The Gauss-Jacobi typical generalised quadrature formula is a well-known mathematical inequality with an significant position in the literature and is defined as follows:

$$\int_j^k (x-j)^p(k-x)^q m(x)dx = \sum_{j=0}^{m} B_{m,j} m(\delta_j) + \Re_m[m] \tag{3}$$

[1] sercan.turhan@giresun.edu.tr-Corresponding author; 0000-0002-4392-2182
[2] aykut.90@windowslive.com; 0009-0007-2478-841X
[3] imdat.iscan@giresun.edu.tr; 0000-0001-6749-0591.

for certain $B_{m,j}$, δ_j and rest term $\Re_m[m]$ (see [4]).
In (see [7], [10], [13], [14]), the authors established several new integral inequalities concerning the left-hand side of equality 3 via several kinds of convexity.

Recall the following special functions, called beta functions and hypergeometric functions:
For $\mathrm{Re}(j), \mathrm{Re}(k) > 0$

$$\beta(j,k) = \int_0^1 u^{j-1}(1-u)^{k-1}du$$

The function is defined as the beta function. This integral is convergent for $j > 0$ and $k > 0$ (see [1]).
For $g > k > 0, |z| < 1$,

$${}_2F_1(j,k;g;z) = \frac{1}{\beta(k,g-k)} \int_0^1 u^{k-1}(1-u)^{g-k-1}(1-zu)^{-j}du$$

The function defined in the form of is called Hypergeometric function [1].

In their study, İ. İşcan et al. extended the generalized quadrature formula known in the literature as the Gauss-Jacobi integral equality to harmonic convex functions. Utilizing this lemma, they produced new integral inequalities and findings for harmonic convex functions. The fundamental lemma upon which their work is based is stated as follows:

Lemma 1. *[7] Let $m : [j,k] \subseteq (0,\infty) \to \mathbb{R}$ be a function integrable on the interval $[j,k]$ for fixed $f, g > 0$, then*

$$\int_j^k (x-j)^f (k-x)^g m(x)dx = j^{f+1}k^{g+1}(k-j)^{f+g+1} \int_0^1 \frac{t^f(1-t)^g}{A_t^{f+g+2}} m\left(\frac{jk}{A_t}\right) dt, \tag{4}$$

where $A_t = tj + (1-t)k$. Specifically, if $f = g$, the following equation is obtained:

$$\int_j^k (x-j)^f (k-x)^f m(x)dx = (jk)^{f+1}(k-j)^{2f+1} \int_0^1 \frac{t^f(1-t)^f}{A_t^{2f+2}} m\left(\frac{jk}{A_t}\right) dt.$$

The primary aim of this article was to consider the integral expression $\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}} dx$ as a new lemma for $p-$convex and $s-p-$convex functions. Subsequently, new theorems were contemplated in light of this lemma, leading to novel upper bounds for different classes of convex functions based on the Gauss-Jacobi expression. These upper bounds revealed new limits within the classes of classical convex functions and harmonic convex functions for varying values.

2. Main Result

We will use Lemmas 2 and 3 to obtain some new integral inequalities for $p-$convex and $s-p-$convex functions.

Lemma 2. *$m : [j,k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$. For $f, g > 0, p < 0$ the following equality holds.*

$$\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx = \frac{(k^p-j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f(1-u)^g m\left((uk^p+(1-u)j^p)^{1/p}\right)}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}} du. \tag{5}$$

Proof. The intended result is easily calculated by taking $x = (uk^p + (1-u)j^p)^{1/p}$ and changing the variable. □

Conclusion 1. *If $p = -1$ is taken in Lemma 2, [7, Lemma 1] is obtained.*

Lemma 3. *$m : [j,k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$. For $f, g > 0, p > 0$ the following equality holds*

$$\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx = \frac{(k^p-j^p)^{f+g+1}}{p} \int_0^1 \frac{(1-u)^f u^g m\left((uj^p+(1-u)k^p)^{1/p}\right)}{(uj^p+(1-u)k^p)^{f+g+\frac{(p-1)}{p}}} du. \tag{6}$$

Proof. The intended result is easily calculated by taking $x = (uj^p + (1-u)k^p)^{1/p}$ and changing the variable. □

Conclusion 2. *If $p = 1$ then equality 6 in Lemma 3, then we get:*

$$\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{(f+g)}} dx = (k-j)^{f+g+1} \int_0^1 \frac{(1-u)^f u^g m(uj + (1-u)k)}{(uj + (1-u)k)^{f+g}} du$$

Theorem 1. *Let $m : [j,\ k] \subseteq [0, \infty) \to R$ be a function such that $m \in L[j, k]$. If m is $p-convex$ on $[j, k]$ for some fixed $f, g > 0$ then :*

a) *For $p < 0$*

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{7}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left[m(k)\beta(f+2, g+1)_2F_1 \left(f + g + \frac{p-1}{p}, f+2; g+f+3; 1 - \frac{k^p}{j^p} \right) \right.$$
$$\left. + m(j)\beta(f+1, g+2)_2F_1 \left(f + g + \frac{p-1}{p}, f+1; g+f+3; 1 - \frac{k^p}{j^p} \right) \right]$$

b) *For $p > 0$*

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{8}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{pk^{(f+g+1)p-1}} \left[m(j)\beta(g+2, f+1)_2F_1 \left(f + g + \frac{(p-1)}{p}, g+2; g+f+3; 1 - \frac{j^p}{k^p} \right) \right.$$
$$\left. + m(k)\beta(g+1, f+2)_2F_1 \left(f + g + \frac{(p-1)}{p}, g+1; g+f+3; 1 - \frac{j^p}{k^p} \right) \right]$$

Proof. Since m is $p-convex$ on $[j,\ k]$,using the lemma 2 for all $u \in [0, 1]$ we have

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g m\left((uk^p + (1-u)j^p)^{1/p} \right)}{(uk^p + (1-u)j^p)^{f+g+\frac{p-1}{p}}} du \tag{9}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g [um(k) + (1-u)m(j)]}{(uk^p + (1-u)j^p)^{f+g+\frac{p-1}{p}}} du$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \int_0^1 \frac{u^f (1-u)^g [um(k) + (1-u)m(j)]}{\left(j^p \left(1 - \left(1 - \frac{k^p}{j^p} \right) u \right) \right)^{f+g+\frac{p-1}{p}}} du$$
$$= \frac{(k^p - j^p)^{f+g+1}}{p} \left(m(k) \int_0^1 \frac{u^{f+1}(1-u)^g}{\left(j^p \left(1 - \left(1 - \frac{k^p}{j^p} \right) u \right) \right)^{f+g+\frac{p-1}{p}}} du + m(j) \int_0^1 \frac{u^f (1-u)^{g+1}}{\left(j^p \left(1 - \left(1 - \frac{k^p}{j^p} \right) u \right) \right)^{f+g+\frac{p-1}{p}}} du \right)$$
$$= \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(m(k) \int_0^1 \frac{u^{f+1}(1-u)^g}{\left(1 - \left(1 - \frac{k^p}{j^p} \right) u \right)^{f+g+\frac{p-1}{p}}} du + m(j) \int_0^1 \frac{u^f (1-u)^{g+1}}{\left(1 - \left(1 - \frac{k^p}{j^p} \right) u \right)^{f+g+\frac{p-1}{p}}} du \right)$$

where a simple calculation gives

$$\int_0^1 \frac{u^{f+1}(1-u)^g}{\left(1 - \left(1 - \frac{k^p}{j^p} \right) u \right)^{\frac{p-1}{p}}} du = \beta(f+2, g+1)_2F_1 \left(f + g + \frac{p-1}{p}, f+2, g+f+3; 1 - \frac{k^p}{j^p} \right) \tag{10}$$

and

$$\int_0^1 \frac{u^f (1-u)^{g+1}}{\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{\frac{p-1}{p}}} du = \beta(f+1, g+2)_2F_1\left(f+g+\frac{p-1}{p}, f+1, g+f+3; 1-\frac{k^p}{j^p}\right) \tag{11}$$

Substituting equations 10 and 11 into the inequality 9, we obtain the required result. The proof is thus complete.

b) Using Lemma 3, inequality 8 is obtained by applying a similar proof method. □

Conclusion 3. *If $p = -1$ is taken in inequality 7, [7, Inequality of (2.2)] is obtained.*

Conclusion 4. *If it is taken $p = 1$ in inequality 8, then the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{f+g}} dx \le \frac{(k-j)^{f+g+1}}{k^{f+g}} [m(j)\beta(g+2, f+1) + m(k)\beta(g+1, f+2)]$$

Theorem 2. *Let $m : [j,\ k] \subseteq [0,\infty) \to R$ be a function such that $m \in L[j,k]$ and $\alpha \ge 1$ .If $|m|^\alpha$ is $p-convex$ on $[j,k]$ for some fixed $f, g > 0$ then:*
a) *For $p < 0$*

$$\begin{aligned} &\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\ &\le \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\beta(f+1, g+1)_2F_1\left(f+g+\frac{p-1}{p}, f+1; f+g+2; 1-\frac{k^p}{j^p}\right)\right)^{1-\frac{1}{\alpha}} \\ &\times \begin{pmatrix} |m(k)|^\alpha \beta(f+2, g+1)_2F_1\left(f+g+\frac{p-1}{p}, f+2; f+g+3; 1-\frac{k^p}{j^p}\right) \\ + \\ |m(j)|^\alpha \beta(f+1, g+2)_2F_1\left(f+g+\frac{p-1}{p}, f+1; f+g+3; 1-\frac{k^p}{j^p}\right) \end{pmatrix}^{\frac{1}{\alpha}} \end{aligned} \tag{12}$$

b) *For $p > 0$*

$$\begin{aligned} &\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \\ &\le \frac{(k^p - j^p)^{f+g+1}}{pk^{(f+g)p-1}} \left(\beta(g+1, f+1)_2F_1\left(f+g+\frac{p-1}{p}, g+1; f+g+2; 1-\frac{j^p}{k^p}\right)\right)^{1-\frac{1}{\alpha}} \\ &\times \begin{pmatrix} |m(j)|^\alpha \beta(g+2, f+1)_2F_1\left(f+g+\frac{p-1}{p}, g+2; g+f+3; 1-\frac{j^p}{k^p}\right) \\ + \\ |m(k)|^\alpha \beta(g+1, f+2)_2F_1\left(f+g+\frac{p-1}{p}, g+1; g+f+3; 1-\frac{j^p}{k^p}\right) \end{pmatrix}^{\frac{1}{\alpha}} \end{aligned} \tag{13}$$

Proof. Since $|m|^\alpha$ is $p-convex$ on $[j,k]$, using Lemma 2, by the power mean integral inequality for all $u\in[0,1]$ we have

$$\begin{aligned}
&\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx\\
&\le \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\\
&\le \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{1-\frac{1}{\alpha}}\left(\int_0^1 \frac{u^f(1-u)^g\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|^\alpha}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{\frac{1}{\alpha}}\\
&\le \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{1-\frac{1}{\alpha}}\left(\int_0^1 \frac{(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}du\right)^{\frac{1}{\alpha}}\\
&= \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^f(1-u)^g}{j^{(g+f+1)p-1}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{p}}}du\right)^{1-\frac{1}{\alpha}}\left(\int_0^1 \frac{u^f(1-u)^g(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{j^{(g+f+1)p-1}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{p}}}du\right)^{\frac{1}{\alpha}}\\
&= \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\beta(f+1,g+1)F_1\left(f+g+\frac{p-1}{p},f+1;f+g+2;1-\frac{k^p}{j^p}\right)\right)^{1-\frac{1}{\alpha}}\\
&\times\begin{pmatrix} |m(k)|^\alpha\beta(f+2,g+1)_2F_1\left(f+g+\frac{p-1}{p},f+2;f+g+3;1-\frac{k^p}{j^p}\right)\\ + \\ |m(j)|^\alpha\beta(f+1,g+2)_2F_1\left(f+g+\frac{p-1}{p},f+1;f+g+3;1-\frac{k^p}{j^p}\right)\end{pmatrix}^{\frac{1}{\alpha}}
\end{aligned}$$

which completes the proof.

b) Using Lemma 3, inequality 13 is obtained by applying a similar proof method. □

Conclusion 5. *If $p=-1$ is taken in inequality 12, [7, Inequality of (2.5)] is obtained.*

Conclusion 6. *If $p=1$ in inequality 13, the following inequality is obtained:*

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{f+g}}dx \le (k-j)^{f+g+1}\beta(g+1,f+1)^{1-\frac{1}{\alpha}}\begin{bmatrix}|m(j)|^\alpha\beta(g+2,f+1)\\+\\|m(k)|^\alpha\beta(g+1,f+2)\end{bmatrix}^{\frac{1}{\alpha}}$$

Theorem 3. *Let $m:[j,\ k]\subseteq[0,\infty)\to R$ be a function such that $m\in L[j,k]$ and $\alpha>1$ .If $|m|^\alpha$ is $p-convex$ on $[j,k]$ for some fixed $f,g>0$ then :*

a) *For $p<0$*

$$\begin{aligned}
&\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \qquad (14)\\
&\le \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,f\delta+1;(g+f)\delta+2;1-\frac{k^p}{j^p}\right)\\
&\times\left(\frac{|m(k)|^\alpha+|m(j)|^\alpha}{2}\right)^{1/\alpha}
\end{aligned}$$

a) *For $p>0$*

$$\begin{aligned}
&\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \qquad (15)\\
&\le \frac{(k^p-j^p)^{f+g+1}}{pk^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,g\delta+1;(f+g)\delta+2;1-\frac{j^p}{k^p}\right)\\
&\times\left(\frac{|m(j)|^\alpha+|m(k)|^\alpha}{2}\right)^{1/\alpha}
\end{aligned}$$

where $1/\alpha+1/\delta=1$.

Proof.

a) Since $|m|^{\alpha}$ is $p-convex$ on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u\in[0,1]$ we have

$$\int_j^k \frac{(x^p-j^p)^f\,(k^p-x^p)^g\,m(x)}{x^{(f+g)p}}dx$$

$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|du$$

$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^{f\delta}(1-u)^{g\delta}}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\delta}}du\right)^{\frac{1}{\delta}}\left(\int_0^1\left|m\left((uk^p+(1-u)j^p)^{\frac{1}{p}}\right)\right|^{\alpha}du\right)^{\frac{1}{\alpha}}$$

$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^{f\delta}(1-u)^{g\delta}}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\delta}}du\right)^{\frac{1}{\delta}}\left(\int_0^1 (u|m(k)|^{\alpha}+(1-u)|m(j)|^{\alpha})\,dt\right)^{\frac{1}{\alpha}}$$

$$=\frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^{f\delta}(1-u)^{g\delta}}{j^{((f+g+1)p-1)\delta}\left(1-\left(1-\frac{k^P}{j^P}\right)u\right)^{\left(f+g+\frac{p-1}{p}\right)\delta}}du\right)^{\frac{1}{\delta}}\left(\int_0^1 (u|m(k)|^{\alpha}+(1-u)|m(j)|^{\alpha})\,d\boldsymbol{u}\right)^{\frac{1}{\alpha}}$$

$$=\frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,f\delta+1;(g+f)\delta+2;1-\frac{k^p}{j^p}\right)$$

$$\times\left(\frac{|m(k)|^{\alpha}+|m(j)|^{\alpha}}{2}\right)^{1/\alpha}$$

which completes the proof.

b) Using Lemma 3, inequality 15 is obtained by applying a similar proof method. □

Conclusion 7. *If* $p=-1$ *is taken in inequality 14, [7, Inequality of (2.6)] is obtained.*

Conclusion 8. *If* $p=1$ *in inequality 15, the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f(k-x)^g m(x)}{x^{f+g}}dx$$

$$\leq \frac{(k-j)^{f+g+1}}{k^{f+g}}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)_2F_1^{\frac{1}{\delta}}\left((f+g)\delta,g\delta+1;(f+g)\delta+2;1-\frac{j}{k}\right)^{1/\alpha}$$

$$\times\left(\frac{|m(j)|^{\alpha}+|m(k)|^{\alpha}}{2}\right)^{1/\alpha}$$

Theorem 4. *Let* $m:[j,\ k]\subseteq[0,\infty)\to R$ *be a function such that* $m\in L[j,k]$ *and* $\alpha>1$ *.If* $|m|^{\alpha}$ *is* $p-convex$ *on* $[j,k]$ *for some fixed* $f,g>0$ *then :*

a) *For* $p<0$

$$\int_j^k \frac{(x^p-j^p)^f\,(k^p-x^p)^g\,m(x)}{x^{(f+g)p}}dx \tag{16}$$

$$\leq \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\begin{array}{l}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,2;3;1-\frac{k^p}{j^p}\right)\frac{|m(k)|^{\alpha}}{2}\right.\\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,1;3;1-\frac{k^p}{j^p}\right)\frac{|m(j)|^{\alpha}}{2}\right]\end{array}\right)$$

b) *For* $p>0$

$$\int_j^k \frac{(x^p-j^p)^f\,(k^p-x^p)^g\,m(x)}{x^{(f+g)p}}dx \tag{17}$$

$$\leq \frac{(k^p-j^p)^{f+g+1}}{pk^{((f+g+1)p-1)}}\left(\begin{array}{c}\beta^{\frac{1}{f}}(g\delta+1,f\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,2;3;1-\frac{j^p}{k^p}\right)\frac{|m(j)|^\alpha}{2}\right.\\ \left.+{}_2F_1\left((f+g+\frac{p-1}{p})\alpha,1;3;1-\frac{j^p}{k^p}\right)\frac{|m(k)|^\alpha}{2}\right]^{\frac{1}{\alpha}}\end{array}\right)$$

where $1/\alpha+1/\delta=1$.

Proof.

a) Since $|m|^\alpha$ is $p-convex$ on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u\in[0,1]$ we have

$$\int_j^k\frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx$$

$$\leq\frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1\frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|du$$

$$\leq\frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}$$

$$\times\left(\int_0^1\frac{1}{(uk^p+(1-u)j^p)^{(f+g+\frac{p-1}{p})\alpha}}\left|m\left(uk^p+(1-u)j^p\right)^{1/p}\right|^\alpha du\right)^{\frac{1}{\alpha}}$$

$$=\frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1\frac{(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{(f+g+\frac{p-1}{p})\alpha}}du\right)^{\frac{1}{\alpha}}\right]$$

$$=\frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1\frac{(u|m(k)|^\alpha+(1-u)|m(j)|^\alpha)}{j^{((f+g+1)p-1)\alpha}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{(f+g+\frac{p-1}{p})\alpha}}du\right)^{\frac{1}{\alpha}}\right]$$

$$=\frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\begin{array}{l}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,2;3;1-\frac{k^p}{j^p}\right)\frac{|m(k)|^\alpha}{2}\right.\\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,1;3;1-\frac{k^p}{j^p}\right)\frac{|m(j)|^\alpha}{2}\right]\end{array}\right)$$

which completes the proof.

b) Using Lemma 3, inequality 17 is obtained by applying a similar proof method. □

Conclusion 9. *If* $p=-1$ *is taken in inequality 16, [7, Inequality of (2.7)] is obtained.*

Conclusion 10. *If* $p=1$ *in inequality 17, the following inequality is obtained:*

$$\int_j^k\frac{(x-j)^f(k-x)^g m(x)}{x^{f+g}}dx\leq\frac{(k-j)^{f+g+1}}{k^{f+g}}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)\left[{}_2F_1\left((f+g)\alpha,2;3;1-\frac{j}{k}\right)\frac{|m(j)|^\alpha}{2}\right.\\ \left.+{}_2F_1\left((f+g)\alpha,1;3;1-\frac{j}{k}\right)\frac{|m(k)|^\alpha}{2}\right]^{\frac{1}{\alpha}}\end{array}\right)$$

Theorem 5. *Let* $m:[j,k]\subseteq[0,\infty)\to R$ *be a function such that* $m\in L[j,k]$ *and* $\alpha\geq1$*.If* $|m|^\alpha$ *is* $s-p-convex$ *in the second kind on* $[j,k]$ *for some fixed* $f,g>0$, $s\in[0,1]$ *then :*

a) *For* $p<0$

$$\int_j^k\frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \tag{18}$$

$$\leq\frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\beta(f+1,g+1)_2F_1\left(f+g+\frac{p-1}{p},f+1;f+g+2;1-\frac{k^p}{j^p}\right)\right)^{1-\frac{1}{\alpha}}$$

$$\times\left(\begin{array}{c}|m(k)|^\alpha\beta(f+s+1,g+1)_2F_1\left(f+g+\frac{p-1}{p},f+s+1;f+g+s+2;1-\frac{k^p}{j^p}\right)\\ +\\ |m(j)|^\alpha\beta(f+1,g+s+1)_2F_1\left(f+g+\frac{p-1}{p},f+1;f+g+s+2;1-\frac{k^p}{k^p}\right)\end{array}\right)^{\frac{1}{\alpha}}$$

b) *For* $p > 0$

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx \tag{19}$$
$$\leq \frac{(k^p - j^p)^{f+g+1}}{pk^{((f+g+1)p-1)}} \left(\beta(g+1, f+1) F_1 \left(f + g + \frac{p-1}{p}, g+1; f+g+2; 1 - \frac{j^p}{k^p}\right)\right)^{1-\frac{1}{\alpha}}$$
$$\times \left(\begin{array}{c} |m(j)|^\alpha \beta(g+s+1, f+1)_2 F_1 \left(f+g+\frac{p-1}{p}, g+s+1; f+g+s+2; 1-\frac{j^p}{k^p}\right) \\ + \\ |m(k)|^\alpha \beta(g+1, f+s+1)_2 F_1 \left(f+g+\frac{p-1}{p}, g+1; g+f+s+2; 1-\frac{j^p}{k^p}\right) \end{array} \right)^{\frac{1}{\alpha}}$$

Proof. Since $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$, using Lemma 2, by the power mean integral inequality for all $u \in [0,1]$ we have

$$\int_j^k \frac{(x^p - j^p)^f (k^p - x^p)^g m(x)}{x^{(f+g)p}} dx$$
$$\leq \frac{(k^p - j^p)^{g+f+1}}{p} \int_0^1 \frac{u^f (1-u)^g \left| m\left((uk^p + (1-u)j^p)^{1/p} \right) \right|}{(uk^p + (1-u)j^p)^{f+g+\frac{p-1}{p}}} du$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}} du \right)^{1-\frac{1}{\alpha}} \left(\int_0^1 \frac{u^f(1-u)^g \left| m\left((uk^p+(1-u)j^p)^{1/p}\right) \right|^\alpha}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}} du \right)^{\frac{1}{\alpha}}$$
$$\leq \frac{(k^p-j^p)^{g+f+1}}{p} \left(\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}} du \right)^{1-\frac{1}{\alpha}} \left(\int_0^1 \frac{u^f(1-u)^g (u^s|m(k)|^\alpha + (1-u)^s |m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}} du \right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p-j^p)^{f+g+1}}{p} \left(\int_0^1 \frac{u^f(1-u)^g}{j^{(g+f+1)p-1}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{p-1}{P}}} du \right)^{1-\frac{1}{\alpha}} \left(\int_0^1 \frac{u^f(1-u)^g (u^s|m(k)|^\alpha + (1-u)^s|m(j)|^\alpha}{j^{(g+f+1)p-1}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{f+g+\frac{(p-1)}{p}}} \right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p - j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\beta(f+1, g+1) F_1 \left(f+g+\frac{p-1}{p}, f+1; f+g+2; 1-\frac{k^p}{j^p} \right) \right)^{1-\frac{1}{\alpha}}$$
$$\times \left(\begin{array}{c} |m(k)|^\alpha \beta(f+s+1, g+1)_2 F_1 \left(f+g+\frac{p-1}{p}, f+s+1; f+g+s+2; 1-\frac{k^p}{j^p}\right) \\ + \\ |m(j)|^\alpha \beta(f+1, g+s+1)_2 F_1 \left(f+g+\frac{p-1}{p}, f+1; f+g+s+2; 1-\frac{k^p}{j^p}\right) \end{array} \right)^{\frac{1}{\alpha}}$$

which completes the proof.

b) Using Lemma 3, inequality 19 is obtained by applying a similar proof method. □

Conclusion 11. *If* $p = -1$ *in inequality 18, the following inequality is obtained:*

$$\int_j^k (x-j)^f (k-x)^g m(x) dx$$
$$\leq \left(\frac{j}{k}\right)^{f+1} (k-j)^{f+g+1} \left(\beta(f+1, g+1)_2 F_1 \left(f+g+2, f+1; f+g+2; 1-\frac{j}{k} \right) \right)^{1-\frac{1}{\alpha}} \tag{20}$$
$$\times \left(\begin{array}{c} |m(k)|^\alpha \beta(f+s+1, g+1)_2 F_1 \left(f+g+2, f+s+1; f+g+s+2; 1-\frac{j}{k}\right) \\ + \\ |m(j)|^\alpha \beta(f+1, g+s+1)_2 F_1 \left(f+g+2, f+1; f+g+s+2; 1-\frac{j}{k}\right) \end{array} \right)^{\frac{1}{\alpha}}$$

Conclusion 12. *If $p=1$ in inequality 19, the following inequality is obtained:*

$$\int_j^k \frac{(x-j)^f(k-x)^g m(x)}{x^{f+g}}dx$$
$$\leq \frac{(k-j)^{f+g+1}}{k^{f+g}}\beta(g+1,f+1)_2F_1\left(f+g,g+1;f+g+2;1-\frac{j}{k}\right)^{1-\frac{1}{\alpha}} \tag{21}$$
$$\times\left(\begin{array}{c}|m(j)|^\alpha\beta(g+s+1,f+1)_2F_1\left(f+g,g+s+1;f+g+s+2;1-\frac{j}{k}\right)\\+|m(k)|^\alpha\beta(g+1,f+s+1)_2F_1\left(f+g,g+1;f+g+s+2;1-\frac{j}{k}\right)\end{array}\right)^{\frac{1}{\alpha}}$$

Theorem 6. *Let $m:[j,k]\subseteq[0,\infty)\to R$ be a function such that $m\in L[j,k]$ and $\alpha>1$. If $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$ for some fixed $f,g>0$, $s\in[0,1]$ then:*

a) *For $p<0$*

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \tag{22}$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,f\delta+1;(f+g)\delta+2;1-\frac{k^p}{j^p}\right)$$
$$\times\left(\frac{|m(k)|^\alpha+|m(j)|^\alpha}{s+1}\right)^{1/\alpha}$$

b) *For $p>0$*

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \tag{23}$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{pk^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(\delta g+1,f\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,g\delta+1;(f+g)\delta+2;1-\frac{j^p}{k^p}\right)$$
$$\times\left(\frac{|m(j)|^\alpha+|m(k)|^\alpha}{s+1}\right)^{1/\alpha}$$

where $1/\alpha+1/\delta=1$.

Proof.

a) Since $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u\in[0,1]$ we have

$$\int_j^k \frac{(x^p-j^p)^f(k^p-x^p)^g m(x)}{x^{(f+g)p}}dx$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)}}\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|du$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^{f\delta}(1-u)^{\delta g}}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)}}du\right)^{\frac{1}{\delta}}\left(\int_0^1\left|m\left((uk^p+(1-u)j^p)^{\frac{1}{p}}\right)\right|^\alpha dt\right)^{\frac{1}{\alpha}}$$
$$\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^{f\delta}(1-u)^{\delta g}}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\delta}}du\right)^{\frac{1}{\delta}}\left(\int_0^1 (u^s|m(k)|^\alpha+(1-u)^s|m(j)|^\alpha)\,dt\right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 \frac{u^{f\delta}(1-u)^{\delta g}}{j^{((f+g+1)p-1)\delta}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{\left(f+g+\frac{p-1}{p}\right)\delta}}du\right)^{\frac{1}{\delta}}\left(\int_0^1 (u^s|m(k)|^\alpha+(1-u)^s|m(j)|^\alpha)\,dt\right)^{\frac{1}{\alpha}}$$
$$= \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)_2F_1^{\frac{1}{\delta}}\left(\left(f+g+\frac{p-1}{p}\right)\delta,f\delta+1;(f+g)\delta+2;1-\frac{k^p}{j^p}\right)$$
$$\times\left(\frac{|m(k)|^\alpha+|m(j)|^\alpha}{s+1}\right)^{1/\alpha}$$

which completes the proof.
b) Using Lemma 3, inequality 23 is obtained by applying a similar proof method. □

Conclusion 13. *If $p=-1$ in inequality 22, the following inequality is obtained:*

$$\begin{aligned}&\int_j^k (x-j)^f (k-x)^g m(x)dx\\ &\leq \left(\frac{j}{k}\right)^{f+1} (k-j)^{f+g+1} \beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1) {}_2F_1^{\frac{1}{\delta}}\left((f+g+2)\delta, f\delta+1; (g+f)\delta+2; 1-\frac{j}{k}\right)\\ &\times \left(\frac{|m(k)|^\alpha + |m(j)|^\alpha}{s+1}\right)^{1/\alpha}\end{aligned}$$

Conclusion 14. *If $p=1$ in inequality 23, the following inequality is obtained:*

$$\begin{aligned}&\int_j^k \frac{(x-j)^f (k-x)^g m(x)}{x^{(f+g)}} dx\\ &\leq \frac{(k-j)^{f+g+1}}{k^{(f+g)}} \beta^{\frac{1}{\delta}}(g\delta+1, f\delta+1) {}_2F_1^{\frac{1}{\delta}}\left((f+g)\delta, g\delta+1; (f+g)\delta+2; 1-\frac{j}{k}\right)\\ &\times \left(\frac{|m(j)|^\alpha + |m(k)|^\alpha}{s+1}\right)^{1/\alpha}\end{aligned}$$

Theorem 7. *Let $m:[j,k]\subseteq[0,\infty)\to R$ be a function such that $m\in L[j,k]$ and $\alpha>1$. If $|m|^\alpha$ is $s-p-$convex in the second kind on $[j,k]$ for some fixed $f,g>0$, $s\in[0,1]$ then:*
a) *For $p<0$*

$$\begin{aligned}&\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx\\ &\leq \frac{(k^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}} \left(\begin{array}{c}\beta^{\frac{1}{\delta}}(f\delta+1, g\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, s+1; s+2; 1-\frac{k^p}{j^p}\right)\frac{|m(k)|^\alpha}{s+1}\right.\\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, 1; s+1; 1-\frac{k^p}{j^p}\right)\frac{|m(j)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)\end{aligned} \tag{24}$$

b) *For $p>0$*

$$\begin{aligned}&\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}} dx\\ &\leq \frac{(k^p-j^p)^{f+g+1}}{pk^{(f+g+1)p-1}} \left(\begin{array}{c}\beta^{\frac{1}{\delta}}(g\delta+1, f\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, s+1; s+2; 1-\frac{j^p}{k^p}\right)\frac{|m(j)|^\alpha}{s+1}\right.\\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha, 1; s+1; 1-\frac{j^p}{k^p}\right)\frac{|m(k)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)\end{aligned} \tag{25}$$

where $1/\alpha+1/\delta=1$.

Proof.
a) Since $|m|^\alpha$ is $s-p-convex$ in the second kind on $[j,k]$, using Lemma 2, by the Hölder integral inequality for all $u\in[0,1]$ we have

$$\begin{aligned}
&\int_j^k \frac{(x^p-j^p)^f (k^p-x^p)^g m(x)}{x^{(f+g)p}}dx \\
&\leq \frac{(k^p-j^p)^{f+g+1}}{p}\int_0^1 \frac{u^f(1-u)^g}{(uk^p+(1-u)j^p)^{f+g+\frac{p-1}{p}}}\left|m\left((uk^p+(1-u)j^p)^{1/p}\right)\right|du \\
&\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{1}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{(p-1)}{p}\right)\alpha}}\left|\, m\left(uk^p+(1-u)j^p\right)^{1/p}\right|^{\alpha}du\right)^{\frac{1}{\alpha}} \\
&\leq \frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{(u^s|m(k)|^\alpha+(1-u)^s|m(j)|^\alpha)}{(uk^p+(1-u)j^p)^{\left(f+g+\frac{p-1}{p}\right)\alpha}}du\right)^{\frac{1}{\alpha}}\right] \\
&= \frac{(k^p-j^p)^{f+g+1}}{p}\left[\left(\int_0^1 u^{f\delta}(1-u)^{g\delta}du\right)^{\frac{1}{\delta}}\left(\int_0^1 \frac{(u^s|m(k)|^\alpha+(1-u)^s|m(j)|^\alpha)}{j^{((g+f+1)p-1)\alpha}\left(1-\left(1-\frac{k^p}{j^p}\right)u\right)^{\left(f+g+\frac{p-1}{p}\right)\alpha}}du\right)^{\frac{1}{\alpha}}\right] \\
&= \frac{(\nu^p-j^p)^{f+g+1}}{pj^{(f+g+1)p-1}}\left(\begin{array}{c}\beta^{\frac{1}{m}}(f\delta+1,g\delta+1)\left[{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,s+1;s+2;1-\frac{k^p}{j^p}\right)\frac{|m(k)|^\alpha}{s+1}\right. \\ \left.+{}_2F_1\left(\left(f+g+\frac{p-1}{p}\right)\alpha,1;s+2;1-\frac{k^p}{j^p}\right)\frac{|m(j)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)
\end{aligned}$$

which completes the proof.

b) Using Lemma 3, inequality 25 is obtained by applying a similar proof method. □

Conclusion 15. *If $p=-1$ in inequality 25, the following inequality is obtained:*

$$\begin{aligned}
&\int_j^k (x-j)^f(k-x)^g m(x)dx \\
&\leq \left(\frac{j}{k}\right)^{f+1}(k-j)^{f+g+1}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(f\delta+1,g\delta+1)\left[{}_2F_1\left((f+g+2)\alpha,s+1;s+2;1-\frac{j}{k}\right)\frac{|m(k)|^\alpha}{s+1}\right. \\ \left.+{}_2F_1\left((f+g+2)\alpha,1;s+1;1-\frac{j}{k}\right)\frac{|m(j)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)
\end{aligned}$$

Conclusion 16. *If $p=1$ in inequality 25, the following inequality is obtained:*

$$\begin{aligned}
&\int_j^k \frac{(x-j)^f(k-x)^g m(x)}{x^{f+g}}dx \\
&\leq \frac{(k-j)^{f+g+1}}{k^{(f+g)}}\left(\begin{array}{c}\beta^{\frac{1}{\delta}}(g\delta+1,f\delta+1)\left[{}_2F_1\left((f+g)\alpha,s+1;s+2;1-\frac{j}{k}\right)\frac{|m(j)|^\alpha}{s+1}\right. \\ \left.+{}_2F_1\left((f+g)\alpha,1;s+1;1-\frac{j^p}{k^p}\right)\frac{|m(k)|^\alpha}{s+1}\right]^{\frac{1}{\alpha}}\end{array}\right)
\end{aligned}$$

Author Contribution Statements All authors have made substantial contributions to the conception, design, execution, and interpretation of this research work. Each author has participated in drafting the article or revising it critically for important intellectual content, and all have given final approval of the version to be published.

Declaration of Competing Interests On behalf of all authors, the corresponding author hereby declares that there are no conflicts of interest to report. This statement is inclusive of any potential conflicts that could be perceived as affecting the research, including financial, consultative, institutional, and other relationships

Acknowledgements We extend our gratitude to the esteemed reviewers for their insightful comments and suggestions, which have significantly enhanced the quality of our manuscript. We also wish to express our appreciation to the journal editors for their meticulous attention to our work and for facilitating a constructive peer-review process.

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