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GENERALIZED KURATOWSKI CLOSURE OPERATORS IN THE BIPOLAR METRIC SETTING

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Abstract. We initiate the investigation of the topological aspects of bipolar metric spaces. In this context, some concepts that generalize open and closed sets, accumulation points, closure and interior operators for bipolar metric spaces, of which little is known about their topological behaviors, are discussed. In addition, some essential properties regarding these notions were obtained, and counterexamples were provided for some expected but not satisfied properties.

1. Introduction

As a natural extension of the concept of length, one of the oldest quantitative concepts, the concept of distance can be thought of as the length of a gap. One of the most celebrated tools that enable the concept of distance to be considered theoretically is metric spaces being neither too restrictive nor too general. Since they are intuitive and simple structures, they have been the subject of many generalizations, abstractions, and variations since the first day, they were defined in Fréchet's doctoral thesis [12].

Although it may be helpful in many situations to define distances between different kinds of objects, substances, people, phenomena, or concepts, it is very recently that bipolar metric spaces have been presented in the literature [32]. While these spaces were mainly studied within the framework of fixed point theory [13, 14, 15, 19, 20, 23, 24, 31, 32, 33, 34, 35, 36, 37, 43], also some applications [21, 26, 27, 28, 29, 39, 44, 46], generalizations [1, 2, 3, 5, 6, 7, 16, 22, 25, 26, 30, 42, 45, 46], and special cases [10] are studied. However, the topological characteristics of bipolar metric spaces constitute a nearly untouched area full of mysteries yet to be explored.

In this study, some topological concepts concerning bipolar metric spaces are examined. Closure operators are one of the significant classes of mappings, studied particularly within the context of the lattice theory from many various perspectives [4, 8, 9, 11, 18, 38, 40, 41]

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and proven to be useful in defining and analyzing generalized topological structures. It turns out that the similar generalized operators in bipolar metric spaces provide less well-behaved properties than their counterparts in metric spaces. Moreover, some expected properties that the aforementioned concepts or operators do not generally provide are explained with illustrative counterexamples, which also contribute to a better understanding of bipolar metric spaces.

2. Preliminaries

In order to fix the notations and inform about the topic, some basic terminology about bipolar metric spaces is given here. Some details on the information presented in this section can be found in [32] and [14].

Definition 2.1. *A bipolar set is a pair (X, Y) of sets. In this case, each of the sets is called a pole. Inspired by the writing order of the sets, the terms "left pole" and "right pole" are used, with their obvious meanings. The intersection of the two poles is referred to as the center of the bipolar set. A point of the left (right) pole is called a left (right) point. For points of the center, the term "central point" is used. Accordingly, points in $X \setminus Y$ are referred as noncentral left points, and points belonging $Y \setminus X$ are called noncentral right points.*

Although bipolar sets seems to be nothing other than ordinary pairs of sets, what makes them more interesting is the category **Bip**. This category has bipolar sets as objects and covariant mappings as morphisms, and more information about it can be found in [14].

Definition 2.2. *Given two bipolar sets (X, Y) and (X', Y'). Then a function $\varphi : X \cup Y \to X' \cup Y'$ is called a covariant mapping (or mapping, for short) from (X, Y) to (X', Y'), provided that $\varphi(X) \subseteq X'$, $\varphi(Y) \subseteq Y'$. This case is written as $\varphi : (X, Y) \rightrightarrows (X', Y')$.*

In addition to mappings, another tool to relate bipolar sets is contravariant mappings, whose usefulness has been tested in many applications, particularly on the fixed point theory.

Definition 2.3. *Let (X, Y) and (X', Y') be bipolar sets. A function $\varphi : X \cup Y \to X' \cup Y'$ is called a contravariant mapping from (X, Y) to (X', Y'), if $\varphi(X) \subseteq Y'$, $\varphi(Y) \subseteq X'$, and this case is denoted by $\varphi : (X, Y) \looparrowright (X', Y')$.*

Since bipolar metric spaces defined on bipolar sets are structures that can be explained with the help of sequences, just like the case of metric spaces, sequences defined on bipolar sets have particular importance. However, since sequences with mixed noncentral left and right points on bipolar metric spaces are essentially useless, the concept of sequence on a bipolar set is defined in terms of arrays consisting of either only left points or only right points. Of course, sequences consisting only of central points are also possible, and these are the most similar ones to sequences in the classical sense.

Definition 2.4. *A right (left) sequence on a bipolar set, is a sequence consisting solely of right (left) points. In the context of bipolar sets, when the generic term "sequence" is used, it is understood that either a right sequence, or a left sequence is meant. If all terms of (u_n) are central points, then it is called a central sequence.*

When a bipolar metric space is given over a bipolar set, there is a notion of convergence for sequences. However, to generalize concepts, such as Cauchy sequences, to bipolar metric spaces, the following additional tool is needed:

Definition 2.5. *A bisequence on a bipolar set (X, Y) is defined to be an ordinary sequence on the product set $X \times Y$.*

Now, as introduced at [32], we present a definition for the notion of a bipolar metric space, which will hereafter be shortly referred as BMS.

Definition 2.6. *Given a bipolar set (X, Y) and let $b : X \times Y \to \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, \infty)$. If b satisfies the following, then it is called a bipolar metric on (X, Y), and in this case (X, Y, b) is called a bipolar metric space.*

(B0) $b(x, y) = 0$ *implies* $x = y$, *for each* $x \in X$ *and* $y \in Y$.
(B1) $x = y$ *implies* $b(x, y) = 0$, *for each* $x \in X$ *and* $y \in Y$.
(B2) $b(u, v) = b(v, u)$, *for each* $u, v \in X \cap Y$.
(B3) $b(x, y) \leq b(x, y') + b(x', y') + b(x', y)$, *for each* $x, x' \in X$ *and* $y, y' \in Y$.

The inequality (B3) is known as the quadrilateral inequality. If (B0) is dropped from the definition, then b is called a bipolar pseudo-metric.

The concept of bipolar metric space was introduced to deal with distances defined between separate kinds of objects frequently occur in both mathematical and applied sciences. Examples include distances between curves and points in $\mathbb{R}^n$, distances between sets and points in a pseudometric space, and distances between arbitrarily chosen points and sites in a Voronoi diagram. Moreover, the value of the characteristic function χ_{A^c} associated with the complement of a crisp or fuzzy set A at a point x also defines a distance between sets and points. Another list of examples includes the distances between branches of a company and delivery addresses, the distance between pairs from sets of stars and planetary bodies based on the observable luminosities, the distances between pairs of some suitable bases and acids based on their reaction rate, and the distance of a group of children and a set of abilities based on test scores. It is possible to examine whether such distances conform more or less to the bipolar metric space structure or their generalizations.

Example 2.1. ***(i)*** *If (M, d) is a metric space, then (M, M, d) is a BMS. Conversely, if (X, X, b) is a BMS, then (X, b) is a metric space.*

(ii) *If (Q, d) is a quasi-metric space* [47], *and $\tilde{Q} = \{\tilde{q} : q \in Q\}$ be a disjoint copy of Q, that is $\tilde{Q}$ is any set of same cardinality with Q, such that $\tilde{Q} \cap Q = \varnothing$ and the mapping $q \mapsto \tilde{q}$ is a bijection. Then $\left(Q, \tilde{Q}, b\right)$ is a bipolar pseudo-metric space, where b is given by $b(q_1, \tilde{q}_2) := d(q_1, q_2)$, for all $q_1, q_2 \in Q$.*

(iii) *Let (X, δ) be a dislocated metric space* [17]. *Define the set $U = \{x \in X : \delta(x, x) = 0\}$ and $\tilde{U}^c$ be the disjoint copy of $X \setminus U$. Say $Y = U \cup \tilde{U}^c$. There is a unique function $b : X \times Y \to \mathbb{R}_0^+$ satisfying*

$$\delta(x, y) = \begin{cases} b(x, y), \text{ if } y \in U \\ b(x, \tilde{y}), \text{ if } \tilde{y} \in \tilde{U}^c \text{ for } y \in U^c \end{cases}$$

for each $(x, y) \in X \times Y$. In this case, (X, Y, b) becomes a BMS, and U becomes the center of (X, Y, b)

(iv) *Consider the set C of all functions from $\mathbb{R}$ to the interval $[1, 3]$. Define the function $b : C \times \mathbb{R} \to \mathbb{R}_0^+$ by $b(f, x) = f(x)$. Then $(C, \mathbb{R}, b)$ is a BMS and its center is the empty set.*

Example 2.1 (i) clearly states that BMSs generalize metric spaces. As a result, it can be said that every metric space is a BMS, as formalized in the following proposition. As a principle, each definition given in BMSs must be given in a way that it generalizes its namesake in metric spaces in this context.

Proposition 2.1. *(X, X, b) is a BMS iff (X, b) is a metric space.*

Notation. *Throughout the remainder of this section,* (X, Y, b) *and* (Ξ, Υ, β) *will always denote BMSs, while A and B will represent arbitrary subsets of* $X \cup Y$*. Moreover, for simplicity, we do not distinguish the function b and its restrictions to subsets, in notation.*

Definition 2.7. *If* $P \subseteq X$*, and* $Q \subseteq Y$ *are arbitrary subsets, then* (P, Q, b) *is called a bipolar subspace of* (X, Y, b)*. As a special case, if there exists a set C such that* $P = X \cap C$*,* $Q = Y \cap C$*, then* (P, Q, b) *is called a subspace.*

Clearly, subspaces and bipolar subspaces of a BMS correspond to different concepts. While every subspace is also a bipolar subspace, the converse is not generally true, and subspaces are helpful in most cases because they preserve the balance of the structure to some extent. In contrast, bipolar subspaces can arise in a more chaotic sense. However, they have an instrumental role, especially in constructing examples. For example in the light of Proposition 2.1, a metric space (M, d) can be viewed as a BMS (M, M, d), bipolar subspaces of (M, M, d) will provide plenty of examples of BMS that are not metric spaces. This situation raises the question of whether all BMSs arise this way. Mutlu and Gürdal showed that the answer is negative [32], they nevertheless obtained a partially affirmative result for a generalized type of metric space by utilizing the following tools.

Proposition 2.2. *The function* $b_X : X \times X \to \mathbb{R}_0^+$*,*

$$b_X(x_1, x_2) = \sup_{y \in Y} |b(x_1, y) - b(x_2, y)|,$$

is a pseudo-metric on X, for every $x_1, x_2 \in X$*. Similarly,* $b_Y : Y \times Y \to \mathbb{R}_0^+$*, defined by*

$$b_Y(y_1, y_2) = \sup_{x \in X} |b(x, y_1) - b(x, y_2)|,$$

is a pseudo-metric on Y, for every $y_1, y_2 \in Y$*.*

The approach in Proposition 2.1, which connects BMSs to classical metric spaces, can be taken one step further with the help of the concept of the center of a BMS consisting of central points, which will naturally be a metric space.

Definition 2.8. *For any BMS* (X, Y, b)*, the metric space* $(X \cap Y, b)$ *is called the center metric space, and it is denoted by* $\mathcal{Z}(X, Y, b)$*.*

Definition 2.9. *Let* (X, Y, b) *be a BMS. The function* $\bar{b} : Y \times X \to \mathbb{R}^+$ *defined by* $\bar{b}(y, x) = b(x, y)$ *for every* $(y, x) \in Y \times X$*, is also a bipolar metric on* (Y, X) *and* $(Y, X, \bar{b})$ *is called the opposite of* (X, Y, b)*, denoted by* $\overleftrightarrow{(X, Y, b)} = (Y, X, \bar{b})$*.*

It is obvious from the definition that, one always has $\overleftrightarrow{\overleftrightarrow{(X, Y, b)}} = (X, Y, b)$.

Definition 2.10. ***(i)*** *A mapping* $f : (X, Y, b) \rightrightarrows (Z, W, b')$ *is continuous at a left point* $x_0 \in X$*, if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in Y,\ b(x_0, y) < \delta \Rightarrow b'(f(x_0), f(y)) < \varepsilon,$$

and it is continuous at a right point $y_0 \in Y$*, if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in X,\ b(x, y_0) < \delta \Rightarrow b'(f(x), f(y_0)) < \varepsilon.$$

(ii) *Similarly, a contravariant mapping* $f : (X, Y, b) \leftrightarrow (Z, W, b')$ *is continuous at a left point* $x_0 \in X$*, if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in Y,\ b(x_0, y) < \delta \Rightarrow b'(f(y), f(x_0)) < \varepsilon,$$

and it is continuous at a right point $y_0 \in Y$*, if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in X,\ b(x, y_0) < \delta \Rightarrow b'(f(y_0), f(x)) < \varepsilon.$$

Convergence of sequences on a BMSs is defined as follows.

Definition 2.11. *A left sequence (x_n) converges to a right point $y \in Y$, if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ (which may depend upon ε) such that $b(x_n, y) < \varepsilon$ whenever $n \geq n_0$, and this case is denoted by $(x_n) \rightharpoondown y$ or $\lim_{n\to\infty} x_n = y$.*

A right sequence (y_n) converges to a left point x, if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $n \geq n_0 \implies b(x, y_n) < \varepsilon$, and this is denoted by $(y_n) \rightharpoonup x$ or $\lim_{n\to\infty} y_n = x$.

If a central sequence (u_n) converges to a central point u, such that $(u_n) \rightharpoondown u$ and $(u_n) \rightharpoonup u$, then it is said that (u_n) converges to u and this is denoted by $(u_n) \to u$.

In a BMS, convergence to noncentral left points is not defined for noncentral left sequences, and convergence to noncentral right points is not defined for noncentral right sequences. So, when it is given, for example, that $(u_n) \rightharpoondown v$, then v and (u_n) are readily understood to be a right point and a left sequence, respectively.

Proposition 2.3. *$(x_n) \rightharpoondown y$ on (X, Y, b) iff $(x_n) \rightharpoonup y$ on $\overleftrightarrow{(X, Y, b)}$.*

Remark. *In the light of Proposition 2.3, it is often convenient to consider only left sequences stating and proving general results on convergence in BMSs unless otherwise needed. Similar results for right sequences will readily follow by the duality between a BMS, and its opposite.*

Proposition 2.4. *$(x_n) \rightharpoondown y$ iff $(b(x_n, y)) \to 0$ on $\mathbb{R}$, and $(y_n) \rightharpoonup x$ iff $(b(x, y_n)) \to 0$, on $\mathbb{R}$.*

It is often desirable for convergent sequences to have only one limit. BMSs generally do not have this property, but the uniqueness of the limit can be guaranteed under additional conditions.

Theorem 2.5. [14] *If (X, Y, b) can be embedded as a bipolar subspace into any metric space, then each convergent sequence has a unique limit in (X, Y, b).*

Theorem 2.6. [14] *If a sequence converges to a central point, then this limit is unique.*

Definition 2.12. *A bisequence (x_n, y_n) is called convergent, if there exist points x and y such that $(x_n) \rightharpoondown y$ and $(y_n) \rightharpoonup x$. Moreover, if $x = y$, then (x_n, y_n) is said to be biconvergent to that point.*

Definition 2.13. *A bisequence (x_n, y_n) is called a Cauchy bisequence, if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n, y_m) < \varepsilon$ whenever $m, n \geq n_0$.*

The following proposition is a concise statement that relates the concepts of Cauchyness, convergence, and biconvergence for bisequences on a BMS.

Proposition 2.7. *Every biconvergent bisequence is Cauchy, and every convergent Cauchy bisequence is biconvergent.*

3. Some Topological Notions on Bipolar Metric Spaces

Throughout the section, except for examples, it has been assumed that a fixed BMS (X, Y, b) is given, and A and B are subsets of $X \cup Y$.

Definition 3.1. *Let $x_0 \in X$, $y_0 \in Y$ and $r > 0$. Then, the set*

$$D_X(x_0; r) = \{y \in Y : b(x_0, y) < r\}$$

is called the left-centric open ball with radius r and center x_0, and the set

$$D_Y(y_0; r) = \{x \in X : b(x, y_0) < r\}$$

is called the right-centric open ball with radius r and center y_0. Similarly, the set

$$\bar{D}_X(x_0; r) = \{y \in Y : b(x_0, y) \leq r\}$$

is called the left-centric closed ball with radius r and center x_0. Similarly, the set

$$\bar{D}_Y(y_0; r) = \{x \in X : b(x, y_0) \leq r\}$$

is called the right-centric closed ball with radius r and center y_0.

An interesting aspect of the definitions given above is the fact that if a ball has a left point as its center, then the ball consists of some right points, and vice versa. Thus, as an extreme case, if x_0 is a noncentral left point, then x_0 will not be an element of the balls accepting it as the center.

Definition 3.2. *A is called a left open set, if for each $y \in A \cap Y$ there exists $r > 0$ such that $D_Y(y; r) \subseteq A$, and A is called a right open set, if for each $x \in A \cap X$ there exists $r > 0$ such that $D_X(x; r) \subseteq A$. If A is both right open and left open, then it is called open.*

By definition, each left-centric open ball is a subset of Y, and each right-centric open ball is a subset of X. Hence, we readily have the subsequent proposition, which explains why the adjectives right and left in the definition of open sets are used in contrast with the types of open balls in the definition.

Proposition 3.1. *The left pole (the set X) an its all supersets are always left open and the right pole (the set Y) and its supersets are always right open.*

Proof. For all $y \in Y$ and $r > 0$, $D_Y(y; r) \subseteq X$ by the definition of a right-centric open ball. Thus the set X and all larger sets containing X are left open. The case of right openness of Y and its supersets is similar. □

Remark. *Note that the largest set $X \cup Y$ is always open, as a consequence of Proposition 3.1.*

Theorem 3.2. *Every left-centric open ball is right open.*

Proof. Consider a left-centric open ball $D_X(x_0; r)$. To show that $D_X(x_0; r)$ is right open, we take an element $x \in D_X(x_0; r) \cap X$. Then, from the definiton of a left-centric open ball, $D_X(x_0; r) \subseteq Y$ and thus x is a central point. Now, for $\rho = r - b(x_0, x) > 0$, we claim that $D_X(x; \rho) \subseteq D_X(x_0; r)$. Suppose that $y \in D_X(x; \rho)$. By the definition of $D_X(x; \rho)$, $y \in Y$. Then $b(x, y) < \rho = r - b(x_0, x)$. Therefore, $b(x_0, x) + b(x, y) < r$. From the quadrilateral inequality, and since x is a central point,

$$b(x_0, y) \leq b(x_0, x) + b(x, x) + b(x, y) = b(x_0, x) + b(x, y) < r.$$

Thus, $y \in D_X(x_0; r)$ and hence, $D_X(x; \rho) \subseteq D_X(x_0; r)$. □

The following example illustrates, surprisingly, that while a left-centric open ball is always a right open set, it does not need to be left open.

Example 3.1. *Let $X = (-\infty, 1]$, $Y = [-1, \infty)$, and $b(x, y) = |x-y|$. Consider the left-centric open ball*

$$D_X(0; 3) = \{y \in [-1, \infty) : |0 - y| < 3\} = [-1, 3).$$

We show that $[-1, 3)$ is not left open. For $y = -1 \in Y$ and any $\varepsilon > 0$, observe that

$$D_Y(-1; \varepsilon) = \{x \in X : |x + 1| < \varepsilon\} = (-\varepsilon - 1, \varepsilon - 1) \cap (-\infty, 1].$$

So, $D_Y(-1; \varepsilon) \nsubseteq D_X(0; 3)$, and $D_X(0; 3)$ is not left open.

Remark. *By symmetry, similar results are valid for right-centric open balls. That is, a right-centric open ball is always left open, but does not need to be right open.*

Definition 3.3. *A is called left closed, if for each right sequence in A, $(y_n) \rightharpoonup x$ implies $x \in A$; and it is called right closed, if for each left sequence in A, $(x_n) \rightharpoondown y$ implies $y \in A$. If a set is both left and right closed, then it is called closed.*

We already know that the left pole of a BMS is left open, and the right pole is right open. A similar result for closedness is given by the following straightforward proposition.

Proposition 3.3. *The left pole is left closed, and the right pole is right closed.*

Theorem 3.4. *Every left-centric closed ball is right closed.*

Proof. Let $\bar{D}_X(x_0; r)$ be a left-centric closed ball. Consider a left sequence (x_n) in $\bar{D}_X(x_0; r)$ such that $(x_n) \rightharpoondown y \in Y$. We need to see that $y \in \bar{D}_X(x_0; r)$. Given an $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $b(x_n, y) < \varepsilon$ for $n \geq n_0$. Moreover, since (x_n) is a left sequence in $\bar{D}_X(x_0; r) \subseteq Y$, it is a central sequence, and in particular $x_{n_0} \in X \cap Y$. Also, $b(x_0, x_{n_0}) < r$ as (x_n) is in $\bar{D}_X(x_0; r)$. Therefore

$$b(x_0, y) \leq b(x_0, x_{n_0}) + b(x_{n_0}, x_{n_0}) + b(x_{n_0}, y) < r + \varepsilon.$$

Since ε is arbitrary, $b(x_0, y) \leq r$, and this implies $y \in \bar{D}_X(x_0; r)$. ◻

Now, we provide an example in which a left-centric closed ball is not left closed.

Example 3.2. *Let $X = [-1, 1]$, $Y = (1, \infty)$ and $b(x, y) = |x^2 - y^2|$. Consider the left-centric closed ball $\bar{D}_X(0; 3)$, which is equal to $(1, 3]$. $(y_n) = (\frac{n+1}{n})$ is a right sequence on $\bar{D}_X(0; 3)$ such that $(y_n) \rightharpoonup 1$, and also $(y_n) \rightharpoonup -1$ at the same time. However $1 \notin \bar{D}_X(0; 3)$ so that $\bar{D}_X(0; 3)$ is not left closed. Note here that, there is no convergent left sequence in $\bar{D}_X(0; 3)$. So, the fact that $\bar{D}_X(0; 3)$ is right closed is a vacuous truth in this case.*

Remark. *As a dual result of Theorem 3.4, every right-centric closed ball is left closed.*

Theorem 3.5. *A is left open, if and only if, A^c is right closed, where the complements are taken over the set $X \cup Y$.*

Proof. Let A be left open. Consider a left sequence (x_n) on A^{c}, and suppose that $(x_n) \rightharpoondown y$. We must show that $y \in A^{\mathsf{c}}$. Assume the contrary that $y \in A$. Since A is left open, there exists $\varepsilon > 0$ such that $D_Y(y; \varepsilon) \subseteq A$, and since $(x_n) \rightharpoondown y$, there is an $n_0 \in \mathbb{N}$, such that $b(x_n, y) < \varepsilon$ for $n \geq n_0$. In particular, $x_{n_0} \in D_Y(y; \varepsilon) \subseteq A$, which contradicts by $(x_n) \in A^{\mathsf{c}}$. Thus, $y \in A$ and A is a right closed set.

Conversely, suppose that A^{c} is a right closed set. To show that A is left open, consider a right point $y \in A$. Assume that there exists no $\varepsilon > 0$ such that $D_Y(y; \varepsilon) \subseteq A$. Then for each $\varepsilon > 0, D_Y(y; \varepsilon) \nsubseteq A$, or equivalently, $D_Y(y; \varepsilon) \cap A^{\mathsf{c}} \neq \emptyset$. For each $n \in \mathbb{N}$, pick an $x_n \in D_Y(y; \frac{1}{n}) \cap A^{\mathsf{c}}$. In this case, (x_n) is a left sequence since $D_Y(y; \frac{1}{n}) \subseteq X$, and $(x_n) \subseteq A^{\mathsf{c}}$. However, $(x_n) \rightharpoondown y$, since $b(x_n, y) < \frac{1}{n} \to 0$. This contradicts by the right closedness of A^{c}. Therefore $y \in A$, so that A is left open. ◻

We now present the following result on generating topologies from a given BMS.

Theorem 3.6. *Let τ_L be the family of left open subsets. Then*

(i) *$\emptyset, X \cup Y \in \tau_L$,*

(ii) *For every $i \in I$, $A_i \in \tau_L$ implies $\bigcup_{i \in I} A_i \in \tau_L$,*

(iii) *$A, B \in \tau_L$ implies $A \cap B \in \tau_L$.*

Proof. **(i)** The empty set satisfies the conditions in the definition of left open sets vacuously since it has no points, thus $\varnothing \in \tau_L$. $X \cup Y$ is a left open set, since one always have $D_Y(y;r) \subseteq X \subset X \cup Y$, for every right point $y \in X \cup Y$ and $r > 0$.

(ii) Let $\{A_i \subseteq X : i \in I\}$ be a collection of left open subsets. Take $y \in \bigcup_{i\in I} A_i$. Then, $y \in A_i$ for some $i \in I$. Since $A_i \in \tau_L$, there is an $r > 0$ such that $D_Y(y;r) \subseteq A_i$. Therefore, $D_Y(y;r) \subseteq \bigcup_{i\in I} A_i$, and $\bigcup_{i\in I} A_i \in \tau_L$.

(iii) Let A and B be left open subsets. If $y \in A \cap B$, there are $r_A, r_B > 0$ such that $D_Y(y;r_A) \subseteq A$ and $D_Y(y;r_B) \subseteq B$. Set $r = \min\{r_A, r_B\}$. Then, $D_Y(y;r) \subseteq A \cap B$. In other words, $A \cap B$ is left open. □

Remark. *In addition to the topological space (X, τ_L), there is an accompanying topology τ_R on $X \cup Y$, consisting of all right open subsets.*

For any BMS (X, Y, b), we have an associated metric space $\mathcal{Z}(X, Y, b)$, as described in Definition 2.8. A question arise then: does the topology generated by the center metric space equal to the relative topology on $X \cap Y$, corresponding to $\tau_L \cap \tau_R$? The answer is, in general, no, as will be illustrated below.

Example 3.3. *Consider the bipolar subspace $((-\infty, 1], [-1, \infty), b)$ of the standard metric space on $\mathbb{R}$. The center $[-1, 1]$ of this BMS is not left open as for the right point $-1 \in [-1, 1]$, there is no $r > 0$ such that $D_Y(-1;r) \subseteq [-1, 1]$, since $D_Y(-1;r) = (-r-1, r-1) \cap (-\infty, 1]$. Hence, $[-1, 1] \notin \tau_L$ and in particular $\tau_L \cap \tau_R$ cannot be a topology on $[-1, 1]$.*

As a result of Theorems 3.5, and 3.6, we have the following corollary for the family $\mathscr{K}_L$ of all left closed subsets.

Corollary 3.7. *Let $\mathscr{K}_L$ be the family of left closed subsets of $X \cup Y$.*

(i) *The empty set $\varnothing$ and $X \cup Y$ are left closed, i.e. $\varnothing, X \cup Y \in \mathscr{K}_L$.*

(ii) *Arbitrary intersections of left closed sets are left closed, i.e. for all $i \in I$, $A_i \in \mathscr{K}_L$ implies $\bigcap_{i\in I} A_i \in \mathscr{K}_L$.*

(iii) *Union of two left closed sets is left closed, i.e. $A, B \in \mathscr{K}_L$ implies $A \cup B \in \mathscr{K}_L$.*

Definition 3.4. *A left point x is called a left accumulation point of A, if $A \cap (D_X(x;r) - \{x\}) \neq \varnothing$ for every $r > 0$. The set of all left accumulation points of A is denoted by $\mathrm{acc}_X(A)$. Similarly, an $y \in Y$ is called right accumulation point if, for every $r > 0$, one has $A \cap (D_Y(y;r) - \{y\}) \neq \varnothing$ and the set of all such points is denoted by $\mathrm{acc}_Y(A)$.*

Definition 3.5. *A left point x is called a left contact point of A, if for every $r > 0$, $A \cap D_X(x;r) \neq \varnothing$. Similarly, an $y \in Y$ is a right contact point, if for every $r > 0$, $A \cap D_Y(y;r) \neq \varnothing$. The set of left contact points of A is called the left closure of A, and is denoted by $\overleftarrow{A}$. The set of right contact points of A is called the right closure of A, and it is denoted by $\overrightarrow{A}$.*

Although every left accumulation point is a left contact point, the converse is shown not to be true in the following example.

Example 3.4. *Let $\mathbb{R}_0^- = (-\infty, 0]$, $\mathbb{R}_0^+ = [0, \infty)$, and $b : \mathbb{R}_0^- \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined by $b(x, y) = \lceil y \rceil - \lfloor x \rfloor$, where $\lceil\ \rceil$ and $\lfloor\ \rfloor$ stand for ceiling and floor functions, respectively. Consider the set $\mathbb{R} = \mathbb{R}_0^+ \cup \mathbb{R}_0^-$. If $x \in \mathbb{R}_0^-$ and $x \neq 0$, then*

$$b(x, y) = \lceil y \rceil - \lfloor x \rfloor \geq \lceil y \rceil - (-1) \geq 1.$$

Therefore, $D_X(x;1) = \varnothing$, which means that $x \neq 0$ is not a left accumulation point, nor a left contact point of $\mathbb{R}$. For the only remaining left point $x = 0$, we have $D_X(0;r) = \{0\}$, if $r \leq 1$.

In this case, $\mathbb{R} \cap (D_X(0;r) - \{x\}) = \emptyset$*, but* $\mathbb{R} \cap D_X(0;r) = \{x\} \neq \emptyset$*. Thus,* $\mathrm{acc}_X(\mathbb{R}) = \emptyset$*, while* $\overleftarrow{\mathbb{R}} = \{0\}$.

Theorem 3.8. *A is left closed iff* $\mathrm{acc}_X(A) \subseteq A$.

Proof. Suppose that A is left closed, $x \in \mathrm{acc}_X(A)$, and assume that $x \notin A$. Then $x \in A^c$, and by Theorem 3.13, A^{c} is right open. In this case, since $x \in A^{\mathrm{c}}$, there exists $r > 0$ such that $D_X(x;r) \subseteq A^{\mathrm{c}}$. Therefore, $A \cap D_X(x;r) = \emptyset$. Thus, x is not a left contact point of A, so it is not a left accumulation point. However, this contradicts by $x \in \mathrm{acc}_X(A)$. Consequently $x \in A$, and we have $\mathrm{acc}_X(A) \subseteq A$.

Conversely, suppose that $\mathrm{acc}_X(A) \subseteq A$ and take an $x \in X$. Consider a right sequence (y_n) on A such that $(y_n) \rightharpoonup x$. We need to show that $x \in A$. If (y_n) is an ultimately constant sequence, that is, if there is some $n_0 \in \mathbb{N}$, such that $y_n = x$ for $n \geq n_0$, then $x \in A$, since (y_n) is a right sequence on A. Otherwise, if (y_n) is not ultimately constant, then for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$, such that $b(x, y_n) < \varepsilon$ for all $n \geq n_\varepsilon$. Now consider the left-centric open ball $D_X(x;r)$. Then, $y_n \in D_X(x;r)$ for $n \geq n_r$, and since (y_n) is not ultimately constant, there is an $n^* \geq n_r$ such that $y_{n^*} \neq x$. Hence, $y_{n^*} \in A \cap (D_X(x;r) - \{x\}) \neq \emptyset$, and therefore $x \in \mathrm{acc}_X(A) \subseteq A$. □

Corollary 3.9. *A is left closed, if and only if,* $\overleftarrow{A} \subseteq A$.

Remark. *In contrast with the case of metric spaces, where a set A is closed iff* $\overline{A} = A$*, a left closed set, in general does not have the property* $\overleftarrow{A} = A$ *in a BMS. For instance, in Examlple 3.4,* $\mathbb{R}$ *is left closed, since for any convergent right sequence* (y_n) *in* $\mathbb{R}$*,* $(y_n) \rightharpoonup x$ *is possible, only if* (y_n) *is ultimately zero, and* $x = 0$*. However* $\overleftarrow{\mathbb{R}} = \{0\} \neq \mathbb{R}$.

The following propositions are direct consequences of definitions.

Proposition 3.10. $\mathrm{acc}_X(A \cup B) = \mathrm{acc}_X(A) \cup \mathrm{acc}_X(B)$.

Proposition 3.11. *If* $A \subseteq B$*, then* $\mathrm{acc}_X(A) \subseteq \mathrm{acc}_X(B)$.

In classical metric spaces, and more generally in topological spaces, the closure operator satisfies four conditions known as the Kuratowski closure axioms; namely it preserves the empty set ($\overline{\emptyset} = \emptyset$), is extensive ($A \subseteq \overline{A}$), idempotent ($\overline{\overline{A}} = \overline{A}$), and distributes over unions of two sets ($\overline{A \cup B} = \overline{A} \cup \overline{B}$). When idempotency is removed from Kuratowski axioms, the remaining three axioms are called Čech closure axioms. Although the Kuratowski and Čech closure axioms are especially prominent because they provide necessary and sufficient conditions to provide equivalent definitions for pretopological and topological spaces, respectively, there are many more properties satisfied by the closure operator of a metric space. We now investigate the extent to which the left closure operator provides similar properties.

Proposition 3.12. *The following hold.*

1. $\overleftarrow{A} \subseteq X$.
2. $\overleftarrow{A}$ *is left closed.*
3. *If* $A \subseteq X \cap Y$*, then* $A \subseteq \overleftarrow{A}$.
4. *If* $A \subseteq K$ *and* K *is left closed, then* $\overleftarrow{A} \subseteq K$.
5. *A is left closed, if and only if* $\overleftarrow{A} \subseteq A$.
6. *If* $A \subseteq B$*, then* $\overleftarrow{A} \subseteq \overleftarrow{B}$.
7. $\overleftarrow{\overleftarrow{A}} \subseteq \overleftarrow{A}$.

8. $\overleftarrow{\varnothing} = \varnothing$.
9. $\overleftarrow{A \cup B} = \overleftarrow{A} \cup \overleftarrow{B}$.

Proof. 1. It follows from the left closure definition.

2. Let (u_n) be a right sequence on $\overleftarrow{A}$ such that $(u_n) \rightharpoonup x$. Since $\overleftarrow{A} \subseteq X$ by definition, (u_n) is in fact a central sequence on $\overleftarrow{A}$. We also have $A \cap D_X(u_n; \frac{1}{n}) \neq \varnothing$ for all $n \in \mathbb{N}^+$. In particular, since $D_X(u_n; \frac{1}{n}) \subseteq Y$, the sets $A \cap D_X(u_n; \frac{1}{n})$ consist of right points. Form a right sequence (y_n) such that $y_n \in A \cap D_X(u_n; \frac{1}{n})$. In this case (y_n) is a right sequence on A and

$$b(x, y_n) \leq b(x, u_n) + b(u_n, u_n) + b(u_n, y_n) \leq b(x, u_n) + \frac{1}{n}.$$

Taking limits on $\mathbb{R}$ as $n \to \infty$ on both sides, we get $(y_n) \rightharpoonup x$ by Proposition 2.4. Then for any given $\varepsilon > 0$, there exists such an $n_0 \in \mathbb{N}$ that $n \geq n_0$ implies $b(x, y_n) < \varepsilon$ for $n \in \mathbb{N}$. Particularly, $b(x, y_{n_0}) < \varepsilon$, or in other terms, $y_{n_0} \in A \cap D_X(x; \varepsilon) \neq \varnothing$. Hence $x \in \overleftarrow{A}$ and the set $\overleftarrow{A}$ is left closed.

3. Suppose $A \subseteq X \cap Y$. Let $u \in A \cap X = A$. Then $u \in D_X(u; r)$, and $A \cap D_X(u; r) \neq \varnothing$ for all $r > 0$. Therefore $u \in \overleftarrow{A}$.

4. Let $A \subseteq K$ and K be a left closed set. Given $x \in \overleftarrow{A}$. Then $A \cap D_X(x, \frac{1}{n}) \neq \varnothing$ for all $n \in \mathbb{N}^+$. Form a right sequence (y_n) such that $y_n \in A \cap D_X(x; \frac{1}{n}) \subseteq K \cap D_X(x; \frac{1}{n})$. (y_n) is a right sequence on K, and $(y_n) \rightharpoonup x$, as $(b(x, y_n)) \to 0$ on $\mathbb{R}$. Since K is left closed, $x \in K$, and $\overleftarrow{A} \subseteq K$.

5. This is Corollary 3.9.

6. Follows immediately from the definitions.

7. A direct consequence of (2) and (5).

8. It is clear, as $\varnothing \cap D_X(x; r)$ will always be empty, for any $x \in X$.

9. By (6), $A, B \subseteq A \cup B$ implies $\overleftarrow{A} \subseteq \overleftarrow{A \cup B}$ and $\overleftarrow{B} \subseteq \overleftarrow{A \cup B}$, so that

$$\overleftarrow{A} \cup \overleftarrow{B} \subseteq \overleftarrow{A \cup B}.$$

On the other hand, if $x \in \overleftarrow{A \cup B}$, then $(A \cup B) \cap D_X(x; r) \neq \varnothing$, so that either $A \cap D_X(x; r) \neq \varnothing$ or $B \cap D_X(x; r) \neq \varnothing$, that is $x \in \overleftarrow{A} \cup \overleftarrow{B}$. ◻

As can be understood from the proposition above, the left closure operator of a BMS, in general, does not satisfy two of the Kuratowski closure axioms, namely $A \subseteq \bar{A}$ and $\bar{\bar{A}} = \bar{A}$. The following example illustrates that, both $A \subseteq \overleftarrow{A}$ and $\overleftarrow{\overleftarrow{A}} = \overleftarrow{A}$ are in fact not required in BMSs.

Example 3.5. *Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq x\}$ and $Y = \{(x, y) \in \mathbb{R}^2 : y < x\}$. Let $b : X \times Y \to \mathbb{R}_0^+$ be the restriction of Euclidean metric on $\mathbb{R}^2$. Consider the unit disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Observe that $\overleftarrow{D} = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1,\ y = x\}$, and $\overleftarrow{\overleftarrow{D}} = \varnothing$. Thus, neither $D \subseteq \overleftarrow{D}$, nor $\overleftarrow{\overleftarrow{D}} = \overleftarrow{D}$.*

By Proposition 3.12 (2), $\overleftarrow{A}$ is always a left closed set. The following example illustrates that $\overleftarrow{A}$ does not have to be right closed.

Example 3.6. *Let $X = (-\infty, 5)$, $Y = (-5, \infty)$, $A = (-10, 10)$, and $b(x, y) = |x - y|$. Then $\overleftarrow{A} = [-5, 5)$. (x_n) is a left sequence on $\overleftarrow{A}$, where $x_n = \frac{5n}{n+1}$, but $(x_n) \rightharpoondown 5 \notin \overleftarrow{A}$. So, $\overleftarrow{A}$ is not right closed.*

Theorem 3.13. *In a BMS, every singleton is closed.*

Proof. Let $x_0 \in X$ and $C = \{x_0\}$. We must show that C is both left and right closed.

Suppose that (y_n) is a right sequence from C, and $(y_n) \rightharpoonup x$. Since $C = \{x_0\}$, $(y_n) = (x_0, x_0, x_0, \ldots)$ is a constant sequence. In this case, x_0 is both a left and a right point and (y_n) is a central sequence. Then clearly $(y_n) \rightharpoonup x_0$ as $b(x_0, x_0)$ is defined, and x_0 is the only limit by Theorem 2.6. So $x = x_0 \in C$, and C is left closed.

On the other hand, if (x_n) is a left sequence from C, then $(x_n) = (x_0, x_0, x_0, \ldots)$, and only limit of (x_n) is x_0 (if x_0 is a central point), or (x_n) is not convergent (if x_0 is noncentral left point). In both cases, C is right closed. □

Definition 3.6. *The left interior of A is given by*

$$A^{\triangleleft} = \{x \in X : D_X(x; r) \subseteq A, \textit{ for some } r > 0\},$$

and the right interior of A is the set

$$A^{\triangleright} = \{y \in Y : D_Y(y; r) \subseteq A, \textit{ for some } r > 0\}.$$

The points of $A^{\triangleleft}$ are called left interior points, and the points of $A^{\triangleright}$ are called right interior points.

Proposition 3.14. *A set $A \subseteq X \cup Y$ is left open in a BMS (X, Y, b), if and only if, all of its right points are right interior points, that is $A \cap Y \subseteq A^{\triangleright}$.*

Proof. It is a direct consequence of Definitions 3.2 and 3.6. □

Now we give an analog of Proposition 3.12 for left interiors.

Proposition 3.15. *Then the following holds.*

1. $A^{\triangleleft} \subseteq X$.
2. $A^{\triangleleft}$ *is left open.*
3. *If $A^{\mathsf{c}} \subseteq Y$, then $A^{\triangleleft} \subseteq A$, where the complement is taken in $X \cup Y$.*
4. *If $B \subseteq A$ and B is right open, then $B \cap X \subseteq A^{\triangleleft}$.*
5. *A is right open if and only if $A \cap X \subseteq A^{\triangleleft}$.*
6. *If $A \subseteq B$, then $A^{\triangleleft} \subseteq B^{\triangleleft}$.*
7. $A^{\triangleleft\triangleleft} \subseteq A^{\triangleleft}$.
8. $Y^{\triangleleft} = X$.
9. $(A \cap B)^{\triangleleft} = A^{\triangleleft} \cap B^{\triangleleft}$.

Proof. 1. $A^{\triangleleft} \subseteq X$ by the definition.

2. Let $u \in A^{\triangleleft}$ be a right point. Since $A^{\triangleleft} \subseteq X$, u is a central point. By $u \in A^{\triangleleft}$, there is some $r > 0$, such that $D_X(u; r) \subseteq A$. To show that $A^{\triangleleft}$ is left open, we must find a right-centric open ball with center u contained in $A^{\triangleleft}$.

Consider the right-centric open ball $D_Y(u; \frac{r}{2}) \subseteq X$. To see that $D_Y(u; \frac{r}{2}) \subseteq A^{\triangleleft}$, we must verify that for each $x \in D_Y(u; \frac{r}{2})$, there exists $\varepsilon > 0$ such that $D_X(x; \varepsilon) \subseteq A$. We set $\varepsilon = \frac{r}{2}$. In this case, if $y \in D_X(x; \frac{r}{2})$, then $b(x, y) < \frac{r}{2}$. On the other hand by $x \in D_Y(u; \frac{r}{2})$, we have $b(x, u) < \frac{r}{2}$. Combining these yields,

$$b(u, y) \leq b(u, u) + b(x, u) + b(x, y) < 0 + \frac{r}{2} + \frac{r}{2} = r.$$

Hence $y \in D_X(u; r) \subseteq A$, that is $D_X(x; \frac{r}{2}) \subseteq A$, and $x \in A^{\triangleleft}$. This means that $D_Y(u; \frac{r}{2}) \subseteq A^{\triangleleft}$, and $A^{\triangleleft}$ is left open.

3. Suppose that $A^{\mathsf{c}} \subseteq Y$. We show that $A^{\mathsf{c}} \subseteq (A^{\triangleleft})^{\mathsf{c}}$. Let $y \in A^{\mathsf{c}}$. Assume that $y \in A^{\triangleleft}$. Then y is a central point, and $D_X(y; r) \subseteq A$. But by centrality of y, $b(y, y) = 0$ and $y \in D_X(y; r) \subseteq A$, which contradicts by $y \in A^{\mathsf{c}}$.

4. Given $x \in B \cap X$. Since B is right open, there is some $r > 0$ such that $D_X(x;r) \subseteq B$. By $B \subseteq A$, we also have $D_X(x;r) \subseteq A$, which means that $x \in A^{\triangleleft}$.

5. Suppose that $A \cap X \subseteq A^{\triangleleft}$. If x is a left open in A, then $x \in A \cap X \subseteq A^{\triangleleft}$, and thus there exist an $r > 0$, such that $D_X(x;r) \subseteq A$. Hence A is right open. Conversely, if A is right open and $x \in A \cap X$, then there exists some $r > 0$, such that $D_X(x;r) \subseteq A$, and this gives $x \in A^{\triangleleft}$.

6. It is clear from the definition.

7. We know that $A^{\triangleleft}, A^{\triangleleft\triangleleft} \subseteq X$. Suppose for a left point x that $x \notin A^{\triangleleft}$. Then for each $r > 0$, one have $D_X(x;r) \nsubseteq A$, so that there exists at least a $y_r \in D_X(x;r)$, such that $y_r \notin A$. Now we claim that also $y_r \notin A^{\triangleleft}$. Assume the contrary that $y_r \in A^{\triangleleft}$. Since $A^{\triangleleft} \subseteq X$, y_r is a central point. By $y_r \in A^{\triangleleft}$, there is an $\varepsilon > 0$ such that $D_X(y_r;\varepsilon) \subseteq A$. However, $y_r \in D_X(y_r;\varepsilon)$, by centrality of y_r. Thus $y_r \in A$, and this is a contradiction. Consequently, our assumption $y_r \in A^{\triangleleft}$ is false, and $y_r \notin A^{\triangleleft}$. Therefore $D_X(x;r) \nsubseteq A^{\triangleleft}$. Since $r > 0$ is arbitrary chosen, x is not an interior point of $A^{\triangleleft}$, that is $x \notin A^{\triangleleft\triangleleft}$.

8. By definition $Y^{\triangleleft} \subseteq X$, whereas for any $x \in X$, $D_X(x;r) \subseteq Y$ for all $r > 0$, by the definition of a left-centric open ball. Hence $x \in Y^{\triangleleft}$.

9. By (6), $A \cap B \subseteq A$ and $A \cap B \subseteq B$ imply $(A \cap B)^{\triangleleft} \subseteq A^{\triangleleft}$, $(A \cap B)^{\triangleleft} \subseteq B^{\triangleleft}$, and $(A \cap B)^{\triangleleft} \subseteq A^{\triangleleft} \cap B^{\triangleleft}$. On the other side, if $x \in A^{\triangleleft} \cap B^{\triangleleft}$, then there are $r, s > 0$ such that $D_X(x;r) \subseteq A$, and $D_X(x;s) \subseteq B$. If we set $\varepsilon = \min\{r,s\}$, then $D_X(x;\varepsilon) \subseteq A \cap B$, so that $x \in (A \cap B)^{\triangleleft}$. □

There is a well-known duality between inteiror and closure operators on metric, and more generally topological spaces, namely $A^{\circ} = (\overline{A^{\mathsf{c}}})^{\mathsf{c}}$. A weaker analog of this for left interior and left closure operations on a BMS, is stated below.

Theorem 3.16. $A^{\triangleleft} \subseteq \left(\overleftarrow{A^{\mathsf{c}}}\right)^{\mathsf{c}}$.

Proof. Let $x \in A^{\triangleleft}$. Then $A^{\triangleleft} \subseteq X$, and there exists $r > 0$ with $D_X(x;r) \subseteq A$, or equivalently $D_X(x;r) \cap A^{\mathsf{c}} = \varnothing$. This means that $x \notin \overleftarrow{A^{\mathsf{c}}}$, hence $x \in (\overleftarrow{A^{\mathsf{c}}})^{\mathsf{c}}$. □

Applying the dual result to Theorem 3.16 for the set A^{c}, we immediately have the following result.

Corollary 3.17. $\overleftarrow{A} \subseteq ((A^{\mathsf{c}})^{\triangleleft})^{\mathsf{c}}$.

Now we provide a counterexample on falsity of some expectable properties of left interiors.

Example 3.7. *Let* $X = [0,3]$*,* $Y = [1,4]$*,* $A = (1,4)$*, and* $b(x,y) = |x-y|$*. Observe that* $X^{\triangleleft} = [0,3)$*,* $A^{\triangleleft} = [0,1) \cup (1,3]$*,* $A^{\triangleleft\triangleleft} = [0,1) \cup (1,3)$*,* $A^{\mathsf{c}} = [0,1] \cup \{4\}$*,* $\overleftarrow{A^{\mathsf{c}}} = \{1\}$*, and* $(\overleftarrow{A^{\mathsf{c}}})^{\mathsf{c}} = [0,1) \cup (1,4]$*. Then, one have* $X^{\triangleleft} \subsetneq X$*,* $A^{\triangleleft\triangleleft} \subsetneq A^{\triangleleft}$*, and* $A^{\triangleleft} \subsetneq (\overleftarrow{A^{\mathsf{c}}})^{\mathsf{c}}$.

In Theorem 3.16 and Corollary 3.17, if one takes some complements in X, instead of $X \cup Y$, then also the equalities are satisfied. The key here is to prevent noncentral right points from falling into the right hand sets by restricting only the final complements.

Theorem 3.18. $A^{\triangleleft} = X \setminus \overleftarrow{A^{\mathsf{c}}}$ *and* $\overleftarrow{A} = X \setminus (A^{\mathsf{c}})^{\triangleleft}$.

Proof. $A^{\triangleleft} \subseteq X \setminus \overleftarrow{A^{\mathsf{c}}}$ follows from Proposition 3.12 (1) and Theorem 3.17. On the other side if $x \in X \setminus \overleftarrow{A^{\mathsf{c}}}$, then $x \in X$, but $x \notin \overleftarrow{A^{\mathsf{c}}}$. Therefore $D_X(x;r) \cap A^{\mathsf{c}} = \varnothing$ for some $r > 0$, which impilies $D_X(x;r) \subseteq A$, thence $x \in A^{\triangleleft}$. The other equality is similarly shown. □

Remark. *It worths noting that the left and right closure operators coincide if $X = Y$, and in this case they are equal to the closure operator of metric space $(X, b) = (Y, b)$. Thus, properties that are not satisfied for the closure operator, are also not available for left closures. For example, in general $\overleftarrow{A \cap B} \neq \overleftarrow{A} \cap \overleftarrow{B}$, and similarly $(A \cup B)^{\triangleleft} \neq A^{\triangleleft} \cup B^{\triangleleft}$.*

Maybe the most conspicuous topological defect of BMSs is the presence of null interior points and missing contanct points. More specifically, it is possible that $\varnothing^{\triangleleft} \neq \varnothing$, $\varnothing^{\triangleright} \neq \varnothing$, $\overleftarrow{X} \subsetneq X$, and $\overrightarrow{Y} \subsetneq Y$. The following theorem provides more information on these two cases.

Theorem 3.19. $\varnothing^{\triangleleft} = X \setminus \overleftarrow{X} = \left\{x \in X : \inf\limits_{y \in Y} b(x, y) \neq 0\right\}$ *and* $\varnothing^{\triangleright} = Y \setminus \overrightarrow{Y} = \left\{y \in Y : \inf\limits_{x \in X} b(x, y) \neq 0\right\}$.

Proof. We only show the equality for the left interior and closure. Other results follows from the duality. For $x \in X$,

$$\begin{aligned} x \in \varnothing^{\triangleleft} &\iff \exists r > 0,\ D_X(x; r) \subseteq \varnothing \\ &\iff \exists r > 0,\ \forall y \in Y,\ b(x, y) \geq r \\ &\iff \inf_{y \in Y} b(x, y) = 0, \end{aligned}$$

and as we have both $\varnothing^{\triangleleft}, \overleftarrow{X} \subseteq X$

$$\begin{aligned} x \notin \overleftarrow{X} &\iff \exists r > 0,\ D_X(x, r) \cap X = \varnothing \\ &\iff \exists r > 0,\ D_X(x, r) = \varnothing \\ &\iff \exists r > 0,\ \forall y \in Y,\ b(x, y) \geq r. \end{aligned}$$

Hence, $x \in \varnothing^{\triangleleft}$ iff $x \notin \overleftarrow{X}$ □

In this context, we now define a better-behaved subclass of BMSs.

Definition 3.7. *A BMS is called nondegenerate, if $\varnothing^{\triangleleft} = \varnothing = \varnothing^{\triangleright}$. Otherwise it is called degenerate.*

Example 3.8. *The BMSs in Example 3.1 and Example 3.3 are degenerate with $\varnothing^{\triangleleft} = (-\infty, -1)$ and $\varnothing^{\triangleright} = (1, \infty)$, while Example 3.6 has a degeneracy with $\varnothing^{\triangleleft} = (-\infty, -5)$ and $\varnothing^{\triangleright} = (5, \infty)$. The space in Example 3.2 is degenerate with $\varnothing^{\triangleleft} = (-1, 1)$ and $\varnothing^{\triangleright} = (1, \infty)$. The space in Example 3.4 is degenerate with $\varnothing^{\triangleleft} = (-\infty, 0)$ and $\varnothing^{\triangleright} = (0, \infty)$. The space in Example 3.5 is degenerate with $\varnothing^{\triangleleft} = \{(x, y) \in \mathbb{R}^2 : y > x\}$ and $\varnothing^{\triangleright} = \{(x, y) \in \mathbb{R}^2 : y < x\}$. And similarly, the space in Example 3.7 is degenerate with $\varnothing^{\triangleleft} = [0, 1)$ and $\varnothing^{\triangleright} = (3, 4]$. On the other hand every metric space is a nondegenerate BMS. However, the class of nondegenerate BMSs is properly larger than the class of metric spaces. An example of a nondegenerate BMS is (X, Y, b), where $X = \mathbb{Q}^2$, $Y = S^1$, the unit circle, and b is the restriction of the Euclidean metric. Another example is the BMS $(\mathbb{Q}, \mathbb{Q}^{\mathsf{c}}, b)$, where $b(x, y) = |x^2 - y^2|$.*

Having both left and right closure and interior operators, it is also possible to talk about left and right boundaries and exteriors on a BMS.

Definition 3.8. *The boundary of A is defined to be the set $\partial_L(A) = \overleftarrow{A} \setminus A^{\triangleleft}$, and the right boundary of A is $\partial_R(A) = \overrightarrow{A} \setminus A^{\triangleright}$, the left exterior of A is $\mathrm{ext}_L(A) = (A^{\mathsf{c}})^{\triangleleft}$, and $\mathrm{ext}_R(A) = (A^{\mathsf{c}})^{\triangleright}$.*

Proposition 3.20. *$\partial_L(A)$ is left closed.*

Proof. Let (u_n) be a right sequence on $\partial_L(A) = \overleftarrow{A} \setminus A^{\lhd}$ and $(u_n) \rightharpoonup x \in X$. As $\overleftarrow{A} \subseteq X$, (u_n) is a central sequence, and $x \in \overleftarrow{A}$ by Proposition 3.12 (2). It remains to show that $x \notin A^{\lhd}$. Since $u_n \notin A^{\lhd}$, there is no $r > 0$ such that $D_X(u_n; r) \subseteq A$, so that $D_X(u_n; r) \cap A^{\mathsf{c}} \neq \varnothing$, and $u_n \in \overleftarrow{A^{\mathsf{c}}}$. Then (u_n) is a right sequence on the left closed set $\overleftarrow{A^{\mathsf{c}}}$, and $(u_n) \rightharpoonup x$ yields $x \in \overleftarrow{A^{\mathsf{c}}}$. By applying Corollary 3.17 for A^{c}, one have $\overleftarrow{A^{\mathsf{c}}} \subseteq (A^{\lhd})^{\mathsf{c}}$, and this implies $x \notin A^{\lhd}$ as desired. □

The trichotomy rule $M = L^{\circ} \cup \partial(L) \cup \mathrm{ext}(L)$, which is valid for any subset L in a metric space (M, d), is in general does not work (at least perfectly) for BMSs. By Proposition 3.15 (6), both $A^{\lhd}$ and $\mathrm{ext}_L(A)$ are subsets of $\varnothing^{\lhd}$. Since it is possible that $\varnothing^{\lhd} \neq \varnothing$ on a BMS, $A^{\lhd}$ and $\mathrm{ext}_L(A)$ does not have to be disjoint. Many instances of this case, can be found in Example 3.8. Nevertheless, by removing the requirement for the sets to be pairwise disjoint, the following weaker result can be stated.

Theorem 3.21. *$X = A^{\lhd} \cup \partial_L(A) \cup \mathrm{ext}_L(A)$ and $X = A^{\rhd} \cup \partial_R(A) \cup \mathrm{ext}_R(A)$.*

Proof. By the definitions, Proposition 3.15 (1), and Theorem 3.18,

$$\begin{aligned} A^{\lhd} \cup \partial_L(A) \cup \mathrm{ext}_L(A) &= A^{\lhd} \cup (\overleftarrow{A} \setminus A^{\lhd}) \cup (A^{\mathsf{c}})^{\lhd} \\ &= A^{\lhd} \cup \overleftarrow{A} \cup (A^{\mathsf{c}})^{\lhd} \\ &= A^{\lhd} \cup (X \setminus (A^{\mathsf{c}})^{\lhd}) \cup (A^{\mathsf{c}})^{\lhd} \\ &= A \cup X = X. \end{aligned}$$

On the other hand, $X = A^{\rhd} \cup \partial_R(A) \cup \mathrm{ext}_R(A)$ follows from the duality. □

While the class τ_L in Theorem 3.6 is a topology on $X \cup Y$, the left closure operator do not correspond to τ_L. In fact, except for some special cases, it does not correspond to the closure operator of any topology on $X \cup Y$, as it does not satisfy the Kuratowski closure axioms in general. In this context, we finally introduce a modified kind of left and right interior and closure operators on a BMS, which fit better with the topologies τ_L and τ_R.

Definition 3.9. *The set $\overset{\Longleftarrow}{A} := \overleftarrow{A} \cup A$ is called the normalized left closure of A, and the set $\overset{\Longrightarrow}{A} := \overrightarrow{A} \cup A$ is called the normalized right closure of A, the set $A^{\blacktriangleleft} := A \cap (A^{\lhd} \cup X^{\mathsf{c}})$ is called the normalized left interior of A, and the set $A^{\blacktriangleright} := A \cap (A^{\rhd} \cup Y^{\mathsf{c}})$ is called the normalized right interior of A.*

Theorem 3.22. *A is left closed iff $\overset{\Longleftarrow}{A} = A$, and A is right closed iff $\overset{\Longrightarrow}{A} = A$.*

Proof. By Theorem 3.8, A is left closed iff $\mathrm{acc}_X(A) \subseteq A$, that is $\mathrm{acc}_X(A) \cup A = A$. Comparing definitions 3.4 and 3.5, we have $\mathrm{acc}_X(A) \subseteq \overleftarrow{A}$. If $\overset{\Longleftarrow}{A} = A$, then $\overleftarrow{A} \cup A = A$, and $\overleftarrow{A} \subseteq A$. In this case also $\mathrm{acc}_X(A) \subseteq A$, and A is left closed. On the other hand, if A is left closed, then $\overleftarrow{A} \subseteq A$ by Proposition 3.12 (4), and $\overset{\Longleftarrow}{A} = \overleftarrow{A} \cup A = \overleftarrow{A}$. The result for right closed sets follows from the duality. □

Theorem 3.23. *$A^{\blacktriangleleft} = (\overset{\Longleftarrow}{A^{\mathsf{c}}})^{\mathsf{c}}$ and $A^{\blacktriangleright} = (\overset{\Longrightarrow}{A^{\mathsf{c}}})^{\mathsf{c}}$.*

Combining Theorem 3.5, Theorem 3.22, and Theorem 3.23 gives rise to the following corollary.

Corollary 3.24. *A is right open iff $A^{\blacktriangleleft} = A$, and A is left open iff $A^{\blacktriangleright} = A$.*

4. Conclusion

Open sets in metric spaces are studied through two weaker concepts, right open and left open sets, in BMSs. Of course, the same situation applies to the case of closed sets. The duality between left open and right closed sets is particularly interesting. On the other hand, in BMSs, the left closure operator does not satisfy two of the Kuratowski closure axioms, namely extensivity and idempotency. In this respect, left closure operators do not determine a topology on the left pole, nor do they determine a pretopology. Therefore, it is certain that the topology τ_L, and the left closure operator represent different structures. Hence, it is understood that it is not necessary to study topological concepts from only a single point of view in BMSs. Instead, different perspectives can be brought to the structure. Undoubtedly, this attempt, which initiated an independent review of topological concepts in BMSs, is only the beginning, and there is still a long way to advance.

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