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Research Article

An existence result for integro-differential degenerate sweeping process with convex sets

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Abstract

In this paper, we study the well-posedness in the sense of existence and uniqueness of a solution of integrally perturbed degenerate sweeping processes, involving convex sets in Hilbert spaces. The degenerate sweeping process is perturbed by a sum of a single-valued map satisfying a Lipschitz condition and an integral forcing term. The integral perturbation depends on two time-variables, by using a semi-discretization method. Unlike the previous works, the Cauchy's criterion of the approximate solutions is obtained without any new Gronwall's like inequality.

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1. Introduction

In the seventies, sweeping processes are introduced and deeply studied by J. J. Moreau through the series of papers [22–26] which plays an important role in elasto-plasticity, quasi-statics, dynamics, especially in mechanics [10, 27, 28].

Roughly speaking, a point is swept by a moving closed and convex set $C(t)$, which depends on time in a Hilbert space H and can be formulated in the form of first-order differential inclusion involving normal cone operators as follows

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \; a.e. \; t \in [0,T], \\ x(t) \in C(t), \quad \text{for all } t \in [0,T], \\ x(0) = x_0 \in C(0), \end{cases}$$

where $C(\cdot) : [0,T] \rightrightarrows H$ is a set-valued map defined from $[0,T]$ $(T > 0)$ to a separable Hilbert space H with nonempty, closed and convex values. Here $N_{C(t)}(\cdot)$ denotes the outward normal cone, in the sense of convex analysis, to the moving set $C(t)$ at the point $x(t)$. Why the name sweeping process? If the position $x(t)$ of a particle lies in the interior of the moving set $C(t)$, then the normal cone is reduced to the singleton $\{0\}$ and hence $\dot{x}(t) = 0$, which means that the particle remains at rest. When the boundary of $C(t)$ catches up with the particle, then this latter is pushed in an inward normal direction by the boundary of $C(t)$ to stay inside $C(t)$ and satisfies the viability constraint $x(t) \in C(t)$.

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This mechanical visualization led Moreau to call this problem the sweeping process: the particle is swept by the moving set.

Since then, many other applications have been given, such as applications in switched electrical circuits [1], nonsmooth mechanics [9,19], hysteresis in elasto-plastic models [14], among others. Over the years, many variants of the so-called Moreau's sweeping process have been developed in the literature: stochastic [11], perturbed [20], nonconvex [29], in Banach spaces framework [7].

We are interested in a particular variant of the classical sweeping process known as the degenerate sweeping process, which corresponds to the case where a linear and nonlinear operator is added "inside" the normal cone on the sweeping process. This dynamics was proposed by Kunze and Monteiro-Marques as a model for quasistatic elastoplasticity (see [16]). This problem can be formulated in the form of first-order differential inclusion involving normal cone operators as follows

$$(DSP):\begin{cases} -\dot{x}(t)\in N_{C(t)}(Ax(t)) \ a.\ e.\ t\in[0,T], \\ \\ x(0)=x_0, Ax_0\in C(0). \end{cases}$$

where $C(\cdot):[0,T]\rightrightarrows H$ is a set-valued mapping with nonempty closed values of a separable Hilbert H, $N_{C(t)}(Ax(t))$ is the normal cone to $C(t)$) at $Ax(t)\in C(t)$, and $A:H\longrightarrow H$ is a linear/nonlinear operator. Since then, the degenerate sweeping process has been studied by several authors in the framework of convex and prox-regular sets (see [3,15–18]), and when such sets vary in a Lipschitz or absolutely continuous way with respect to the Hausdorff distance, limiting the spectrum of possible applications to only bounded moving sets.

Specifically, we focused on the perturbated degenerate sweeping processes (which is called integro-differential sweeping process of Volterra type), i.e., differential inclusions of the form

$$(P_{A,f,g}):\begin{cases} -\dot{x}(t)\in N_{C(t)}(Ax(t))+f(t,x(t))+\int_0^t g(t,s,x(s))ds \quad a.\ e.\ t\in[0,T], \\ \\ x(0)=x_0, Ax_0\in C(0). \end{cases}$$

Our approach to prove the existence of a solution for the perturbed sweeping process $(P_{A,f,g})$ will use subdivisions of I and estimations depending on the initial point of each subinterval. In all the paper a set-valued map $C(\cdot)$ from $[0,T]$ to H will be involved. This is required to satisfy the following assumptions, for every $t\in[0,T]$, $C(t)$ is a closed convex and nonempty subset of H such that $t\longmapsto C(t)$ is absolutely continuous, in the sense that there is some absolutely continuous function $v(\cdot):[0,T]\longrightarrow\mathbb{R}$ such that, for any $x\in H$ and $s,t\in[0,T]$,

$$|d(x,C(t)-d(x,C(s)|\leq|v(t)-v(s)|.$$

It is worth mentioning that in the particular case where $f\equiv 0$, $g\equiv 0$, and the sets $(C(t))_t$ are convex, M. Kunze and M.D.P. Monteiro Marques [17] proved the existence and uniqueness of solution for the system $(P_{A,0,0})$ above in the case when the set-valued map $C(\cdot)$ varies in a Lipschitz continuous way with respect to the Hausdorff distance. Moreover, this solution is Lipschitz continuous. In the nonconvex case, exactly when the sets$(C(t))_t$ are prox-regular, the authors in [3] proved the well-posedness of $(P_{A,0,0})$ by using the reduction of the constrained differential inclusion $(P_{A,0,0})$ to the unconstrained differential inclusion governed by the subdifferential of distance function in the finite dimensional space. When $g\equiv 0$, problem $(PD)_{f,0}$ has been studied in [15] when the moving sets are assumed to be nonempty, closed and convex with absolutely continuous variation in time. More recently, the well-posedness of the sweeping process involving integral perturbation, i.e., $A\equiv Id$ has been studied in [5,6].

The paper is organized as follows. In Section 2, we recall some basic notations, definitions and useful results which are used throughout the paper. Next, in Section 3, we collect the hypotheses used throughout the paper. Finally, Section 4, which is the most important, is devoted to the existence result for the perturbed degenerate sweeping process. The perturbation term is the sum of a single-valued map satisfying a Lipschitz condition and an integral forcing term depends on two time-variables. The paper ends with conclusions and final remarks.

2. Notation and preliminaries

The material presented in this section is standard and, for this reason, we present it without proofs. For more details, one is invited to see [2, 4, 8, 13, 21, 30, 32], for instance. Throughout the paper, $I := [0, T]$ is an interval of $\mathbb{R}$ and H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. The closed (resp. open) ball of H centered at $x \in H$ of radius $r \in \,]0, +\infty[$ is denoted by $B\,[0, r]$ (resp. $B(x, r)$), and we will use the notation $\mathbb{B}$ for the closed unit ball centered at zero, that is, $\mathbb{B} = B\,[0, 1]$. We denote by $\mathcal{C}(I; H)$ the space of continuous functions defined on I with values in H. It is well known that $\mathcal{C}(I; H)$ is a Banach space equipped with the norm of the uniform convergence denoted by $\|\cdot\|_{\mathcal{C}(I;H)}$ or $\|\cdot\|_\infty$ and defined as follow

$$\|\varphi(t)\|_{\mathcal{C}(I;H)} := \max_{t \in I} \|\varphi(t)\|, \text{ for all } \varphi \in \mathcal{C}(I; H).$$

Given an extended real-valued function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of φ at a point $x \in \operatorname{dom} \varphi$ (in the sense of convex analysis) is the set defined by

$$\partial\varphi(x) := \{v \in H : \langle v, y - x\rangle \le \varphi(y) - \varphi(x), \text{ for all } y \in H\}, \tag{2.1}$$

where $\operatorname{dom} \varphi := \{y \in H : \varphi(y) < +\infty\}$ is the effective domain of φ. When $\varphi(x) = +\infty$, by convention $\partial\varphi(x) = \emptyset$, that is $x \notin \operatorname{Dom} \partial\varphi$, where $\operatorname{Dom} F := \{x \in H : F(x) \ne \emptyset\}$ is the domain of a set-valued map $F : H \rightrightarrows H$ and

$$\operatorname{gph} F := \{(x, y) \in H \times H : y \in F(x)\}$$

it the graph of F. The set $\partial\varphi(x)$ can expressed in terms of the directional derivative $\varphi'(x; \cdot)$ as follow

$$\partial\varphi(x) := \{v \in H : \langle v, h\rangle \le \varphi'(x; h), \text{ for all } h \in H\},$$

where $\varphi'(x; h) := \lim_{\tau \downarrow 0} \tau^{-1} (\psi(x + th) - \psi(x))$.

Let S be a nonempty closed convex subset of H. Three important functions play a central role in modern convex analysis, both in theory and algorithmically. Those particular functions correspond to the indicator function $\psi_S(\cdot)$ and support functions $\sigma(S, \cdot)$ of S respectively, and to the distance function $d_S(\cdot)$ from the set S, defined by

$$\psi_S(\cdot) : H \to \mathbb{R} \cup \{+\infty\} \text{ with } \psi_S(x) := \begin{cases} 0 & \text{if } \quad x \in S \\ +\infty & \text{if } \quad x \notin S. \end{cases}$$

$$\sigma(S, \cdot) : H \to \mathbb{R} \cup \{+\infty\} \text{ with } \sigma(S, \zeta) := \sup_{x \in S} \langle x, \zeta \rangle\,.$$

$$d_S(\cdot) : H \to \mathbb{R} \text{ with } d_S(x) := \inf_{y \in S} \|x - y\|.$$

The notion of support functions is often used to translate geometric Hahn-Banach separation theorems and in particular, it characterizes the closed convex set S through the following equivalence property: $x \in S$ if and only if $\langle \zeta, x\rangle \le \sigma(S, \zeta)$ for all $\zeta \in H$.

According to (2.1) and for $x \in S$, it is straightforward to see that an element $\xi \in \partial\psi_S(x)$ if and only if $\langle \xi, v - x\rangle \le 0$ for all $v \in S$, so $\partial\psi_S(x)$ is the set $N_S(x)$ of outward normals of S at the point S defined by

$$N_S(x) = \{\xi \in H : \langle \xi, v - x\rangle \le 0 \text{ for all } v \in S\}.$$

We derive from the last inequality, involving $\partial\psi_S(x)$, the following equivalence holds

$$\xi \in N_S(x) \Leftrightarrow \sigma(S,\xi) = \langle \xi, x\rangle \text{ and } x \in S.$$

Moreover

$$d_S(x) = \sup\{\langle x,y\rangle - \sigma(S,y) : y \in \mathbb{B}\},$$

and

$$N_C(-x) = -N_{-C}(x) \text{ and } N_C(y+z) = -N_{C-z}(y),$$

for any $x \in -C$ and y,z such that $y+z \in C$. It is worth emphasizing that establishing precise formulas for computing the set $\partial d_S(x)$ at a given point $x \in H$ is strongly involved in many problems arising in differential inclusions. The following equality gives a representation of $\partial d_S(x)$ at in-set points

$$\partial d_S(x) = N_S(x) \cap \mathbb{B}, \quad \text{for all } x \in S.$$

In the following, we summarize some known definitions and results concerning maximally monotone operators.

The operator $A : H \rightrightarrows H$ is called monotone if for all $x_1, x_2 \in H, y_1 \in A(x_1), y_2 \in A(x_2)$, we have $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$. In addition, A is maximal monotone if and only if it is monotone and its graph is maximal in the sense of inclusion. Another deep important property of the subdifferential in Convex Analysis concerns maximal monotonicity is the following: for a lower semicontinuous proper convex function $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$, the subdifferential $\partial f(\cdot)$ is a maximal monotone operator. As a consequence of this result, we obtain that, for any nonempty closed convex set $C \subset H$, the normal cone $N_C(\cdot)$ is a maximal monotone operator.

Before closing this section, let us remind a version of Gronwall's inequality (see, e.g. Lemma 4.1 in [31]). This auxiliary result will play a fundamental role in proving the Cauchy's criterion of the approximate solutions. Prior to this, we recall that a function $x(\cdot) : [0,T] \to H$ is absolutely continuous if there exists $v \in L^1([0,T]; H)$ such that

$$x(t) = x(0) + \int_0^t v(s)ds, \quad \text{for all } t \in [0,T].$$

In this case, $x(\cdot)$ is derivable almost everywhere (a.e., for short) on $[0,T]$ with $\dot{x}(\cdot) = v$ a.e. on $[0,T]$.

Lemma 2.1. *(Gronwall's lemma)*
Let $b(\cdot), c(\cdot), \zeta(\cdot) : [t_0, t_1] \longrightarrow \mathbb{R}^+$ be three real valued Lebesgue integrable functions. If the function $\zeta(\cdot)$ is absolutely continuous on the interval $[t_0, t_1]$ and if for almost all $t \in [t_0, t_1]$

$$\dot{\zeta}(t) \leq b(t) + c(t)\zeta(t),$$

then for all $t \in [t_0, t_1]$,

$$\zeta(t) \leq \zeta(t_0) \exp\left(\int_{t_0}^t c(s)ds\right) + \int_{t_0}^t b(r) \exp\left(\int_r^t c(s)ds\right) dr.$$

In view of an existence result of $(P_{A,f,g})$, we will make use of the following particular result which gives the well-posedness of a degenerate sweeping process without the integral perturbation. This last problem was considered in[15].

Proposition 2.2. *Let H be a real Hilbert space and $C(\cdot) : [T_0, T] \rightrightarrows H$ a multi-valued mapping. Suppose that the following hypothesis are satisfied:*
$\mathcal{H}_1$) For each $t \in [T_0, T]$, $C(t)$ is a nonempty closed and convex subset of H.
$\mathcal{H}_2$) For each $t \in [T_0, T]$, the set $C(t)$ varies in an absolutely continuous way; that is there exists an absolutely continuous function $v(\cdot) : [T_0, T] \longrightarrow \mathbb{R}$ such that

$$\forall x \in H, \forall s,t \in [T_0, T] : |d(x, C(t) - d(x, C(s)| \leq |v(t) - v(s)|.$$

$\mathcal{H}_3$) *$A : H \longrightarrow H$ is a bounded linear operator which is symmetric and ρ-coercive, that is there exists $\rho > 0$ such that*

$$\langle Ax, x\rangle = \langle x, Ax\rangle \geq \rho \|x\|^2, \forall x \in H.$$

Let $h : [T_0, T] \longrightarrow H$ be a single-valued mapping in $L^1([T_0, T], H)$. Then there exists one and unique absolutely continuous solution $x(\cdot)$ for the following differential inclusion

$$(PDPT) : \begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + h(t) \ a.e \ \ t \in [T_0, T], \\ x(T_0) = x_0, \ Ax_0 \in C(T_0). \end{cases} \tag{2.2}$$

Moreover $x(\cdot)$ satisfies the following inequality

$$\|\dot{x}(t) + h(t)\| \leq \frac{\|A\| \|h(t)\| + |\dot{v}(t)|}{\rho} \ a.e \ \ t \in [T_0, T].$$

3. Technical assumptions -list of hypotheses-

For the sake of readability, in this section we collect the hypotheses used along the paper. Before going on, let Ω_T be the triangle defined by

$$\Omega_T := \{(t, s) \in [0, T] \times [0, T] : s \leq t\}.$$

Let us start this section with listing the standing assumptions imposed throughout the paper unless otherwise stated.

Hypotheses on the set-valued map $C(\cdot) : [0, T] \rightrightarrows H$.

($\mathcal{H}_C$) For each $t \in [0, T]$, $C(t)$ is a nonempty closed convex subset of H and has an absolutely continuous variation, in the sense that there is some absolutely continuous function $v(\cdot) : [0, T] \to \mathbb{R}$ such that

$$|d(x, C(t)) - d(x, C(s))| \leq |v(t) - v(s)| \text{ for any } x \in H \text{ and } s, t \in [0, T].$$

Hypotheses on the operator $A : H \to H$.

($\mathcal{H}_A$) $A : H \to H$ is a linear, bounded and symmetric operator and ρ-coercive for some real number $\rho > 0$, that is $\langle Ax, x\rangle \geq \rho \|x\|^2, \forall x \in H$.

Hypotheses on the map $f(\cdot, \cdot) : [0, T] \times H \to H$.

($\mathcal{H}_f$) f is Bochner measurable in time such that

($\mathcal{H}_{f,1}$) f verifies the following growth condition: there exists $\pi(\cdot) \in L^1([0, T], \mathbb{R}_+)$ with

$$\|f(t, x)\| \leq \pi(t)(1 + \|x\|), \text{ for any } (t, x) \in [0, T] \times H, \text{ with } Ax \in \bigcup_{s \in I} C(s).$$

($\mathcal{H}_{f,2}$) For each real $\eta > 0$ there exists a non-negative function $k_\eta(\cdot) \in L^1(I, \mathbb{R}_+)$ such that for any $t \in [0, T]$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$

$$\|f(t, x) - f(t, y)\| \leq k_\eta(t) \|x - y\|.$$

Hypotheses on the map $g(\cdot, \cdot, \cdot) : I \times I \times H \to H$.

($\mathcal{H}_g$) $(t, s) \in \Omega_T \mapsto g(t, s, x)$ is Bochner measurable for every $x \in H$ such that

($\mathcal{H}_{g,1}$) There exists a non-negative function $\gamma(\cdot, \cdot) \in L^1(\Omega_T, \mathbb{R}_+)$ such that

$$\|g(t, s, x)\| \leq \gamma(t, s)(1 + \|x\|),$$

for all $(t, s) \in \Omega_T$ and $x \in H$ with $Ax \in \bigcup_{t \in [0, T]} C(t)$.

($\mathcal{H}_{g,2}$) For each real $\eta > 0$ there exists a non-negative function $L_\eta(\cdot) \in L^1(I, \mathbb{R}_+)$ such that for any $(t, s) \in \Omega_T$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$

$$\|g(t, s, x) - g(t, s, y)\| \leq L_\eta(t) \|x - y\|.$$

4. Main results

After establishing all the auxiliary properties above, we come now to our main result in this work which gives the existence and uniqueness result of $(P_{A,f,g})$. The proof that will be given combines ideas and techniques from [6] and [15].

Theorem 4.1. *Let H be a real Hilbert. Assume that the hypothesis $(\mathcal{H}_C), (\mathcal{H}_A), (\mathcal{H}_f)$ and $(\mathcal{H}_g)$ above hold. Then, for any initial value $x_0 \in H$ such that $Ax_0 \in C(0)$, there exists a unique absolutely continuous solution $x(\cdot) : [0,T] \to H$ of the Volterra integro-differential inclusion $(P_{A,f,g})$. Moreover, we have the following estimates*
(a) For almost all $t \in [0,T]$

$$\left\| \dot{x}(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds \right\| \leq \frac{1}{\rho}\left[\|A\|(1+l)\left(\pi(t) + \int_0^t \gamma(t,s)ds\right) + |\dot{v}(t)| \right],$$

and

$$\|f(t,x(t))\| \leq \pi(t)(1+l) \ a.\ e.\ t \in [0,T],$$
$$\|g(t,s,x(s))\| \leq \gamma(t,s)(1+l) \ a.\ e.\ (t,s) \in \Omega_T.$$

(b) For almost all $t \in [0,T]$

$$\|\dot{x}(t)\| \leq \frac{1}{\rho}\left[\|A\|(1+l)\left(\pi(t) + \int_0^t \gamma(t,s)ds\right) + |\dot{v}(t)| \right] + (1+l)\left(\pi(t) + \int_0^t \gamma(t,s)ds\right),$$

where

$$l := \|x_0\| + \exp\left[\left(\frac{\|A\|}{\rho}+1\right)\left(\int_0^T \pi(\theta)d\theta + \int_0^T\int_0^\theta \gamma(\theta,s)dsd\theta\right)\right]$$
$$\left(\frac{\|A\|}{\rho}+1\right)(1+\|x_0\|)\int_0^T\left(\pi(r) + \int_0^r \gamma(r,s)ds + \frac{1}{\rho}|\dot{v}(r)|\right)dr.$$

We are going to construct a sequence of maps $(x_n(\cdot))$ in $\mathcal{C}(I,H)$ which converges uniformly to a solution $x(\cdot)$ of $(P_{A,f,g})$.

Proof. The proof of existence of solution is divided in several steps. First of all, since the functions $\pi(\cdot)$ and $\gamma(\cdot,\cdot)$ are integrable, we can assume that

$$\int_0^T\left(\pi(t) + \int_0^r \gamma(t,s)ds\right)dt < \frac{\rho}{\rho+\|A\|}. \tag{4.1}$$

Then, we first treat the case where the condition (4.1) is assumed to be true. Then the case without it will be examined later.

Step 1. Discretization of the interval $I = [0,T]$.
For each integer $n \geq 1$, we consider the partition of the interval $[0,T]$ with the points

$$\begin{cases} t_i^n := ih \text{ with } h = \frac{T}{n} \\ I_i^n := [t_i^n, t_{i+1}^n], \end{cases}, \text{ for all } i \in \{0,\cdots,n-1\}.$$

So that

$$t_{i+1}^n = t_i^n + h \text{ and } 0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

Step 2. Construction of the approximate solutions $x_n(\cdot)$.
The approach that we use consists in considering in each sub-interval $I_k^n := \left[t_k^n, t_{k+1}^n\right], 0 \leq k \leq n-1$ a degenerate sweeping process with a single-valued perturbation of type (2.2). This technique together with the Proposition 2.2 allow us to construct a sequence of

discrete solutions $x_k^n(\cdot) : I_k^n \to H, k = 0, \ldots, n-1$.
We start by considering the following degenerate sweeping process

$$(P_0) \quad \begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + f(t, x_0) + \int\limits_0^t g(t, s, x_0) ds \quad a.\ e.\ t \in [0, t_1^n], \\ \\ x(0) = x_0, Ax_0 \in C(0) = C(t_0^n). \end{cases}$$

According to the Proposition 2.2, the inclusion (P_0) has one and only one absolutely continuous solution that we denote by $x_0^n(\cdot) : [0, t_1^n] \to H$ satisfying the following estimate

$$\|\dot{x}_0^n(t) + h_0^n(t)\| \leq \frac{1}{\rho} \left(\|A\| \, \|h_0^n(t)\| + |\dot{v}(t)|\right) \quad a.\ e.\ t \in [0, t_1^n],$$

where $[0, t_1^n] \ni t \mapsto h_0^n(t) := f(t, x_0) + \int\limits_0^t g(t, s, x_0) ds$. Further, $A(x_0^n(t)) \in C(t)$ for all $t \in [0, t_1^n]$. Indeed, h_0^n is measurable thanks to that of f and g. Further, trough the assumptions $(\mathcal{H}_{f,1})$ and $(\mathcal{H}_{g,1})$ one has

$$\begin{aligned} \|h_0^n(t)\| &\leq \|f(t, x_0)\| + \int\limits_0^t \|g(t, s, x_0)\| ds \\ &\leq (1 + \|x_0\|)\pi(t) + (1 + \|x_0\|) \int\limits_0^t \gamma(t, s) ds \\ &\leq (1 + \|x_0\|) \left[\pi(t) + \int\limits_0^t \gamma(t, s) ds\right], \end{aligned}$$

which means that h_0^n is integrable. Our efforts are now paid to establish the existence result a.e. on $[t_1^n, t_2^n]$ for the following problem

$$(P_1) \begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + f(t, x_0^n(t_1^n)) + \int\limits_0^{t_1^n} g(t, s, x_0) ds + \int\limits_{t_1^n}^t g(t, s, x_0^n(t_1^n)) ds, \\ \\ x(t_1^n) = x_0^n(t_1^n), A(x_0^n(t_1^n)) \in C(t_1^n). \end{cases}$$

To this end, let us define $h_1^n(\cdot) : [t_1^n, t_2^n] \to H$ by

$$h_1^n(t) := f(t, x_0^n(t_1^n)) + \int\limits_0^{t_1^n} g(t, s, x_0) ds + \int\limits_{t_1^n}^t g(t, s, x_0^n(t_1^n)) ds, t \in [t_1^n, t_2^n].$$

Then

$$\begin{aligned} \|h_1^n(t)\| &\leq (1 + \|x_0^n(t_1^n)\|)\pi(t) + (1 + \|x_0\|) \int\limits_0^{t_1^n} \gamma(t, s) ds + (1 + \|x_0^n(t_1^n)\|) \int\limits_{t_1^n}^t \gamma(t, s) ds \\ &\leq (1 + \max\{\|x_0^n(t_0^n)\|, \|x_0^n(t_1^n)\|\}) \left(\pi(t) + \int\limits_0^t \gamma(t, s) ds\right), \end{aligned}$$

which ensures the claimed existence and uniqueness according to Proposition 2.2.
That is, the inclusion (P_1) has a unique absolutely continuous solution $x_1^n(\cdot) : [t_1^n, t_2^n] \to H$ satisfying the following inequality

$$\|\dot{x}_1^n(t) + h_1^n(t)\| \leq \frac{1}{\rho} \left(\|A\| \, \|h_1^n(t)\| + |\dot{v}(t)|\right) \quad a.\ e.\ t \in [t_1^n, t_2^n],$$

further

$$x_0^n(t_1^n) = x_1^n(t_1^n) \text{ and } A(x_1^n(t)) \in C(t), \text{ for any } t \in [t_1^n, t_2^n].$$

Consequently, for each integer $n \geq 1$, by repeating the process just given above, we construct successively a finite sequence of absolutely continuous maps $x_k^n(\cdot) : [t_k^n, t_{k+1}^n] \to H, 0 \leq k \leq n-1$ such that

$$(P_k) \begin{cases} -\dot{x}_k^n(t) \in N_{C(t)}(Ax_k^n(t)) + f(t, x_{k-1}^n(t_k^n)) + \sum\limits_{j=0}^{k-1} \int\limits_{t_j^n}^{t_{j+1}^n} g(t, s, x_{j-1}^n(t_j^n))ds + \\ \int\limits_{t_k^n}^{t} g(t, s, x_{k-1}^n(t_k^n))ds \text{ a.e. } t \in [t_k^n, t_{k+1}^n], \\ x_k^n(t_k^n) = x_{k-1}^n(t_k^n), A\left(x_k^n(t_k^n)\right) \in C(t_k^n). \end{cases} \tag{4.2}$$

With the convention $x_{-1}^n(0) := x_0$. Moreover, for almost every $t \in [t_k^n, t_{k+1}^n]$, similar considerations bring us to the estimate

$$\|\dot{x}_k^n(t) + h_k^n(t)\| \leq \frac{1}{\rho}(\|A\|\|h_k^n(t)\| + |\dot{v}(t)|), \tag{4.3}$$

where $h_k^n(\cdot) : [t_k^n, t_{k+1}^n] \to H$ is the mapping defined by

$$h_k^n(t) := f(t, x_{k-1}^n(t_k^n)) + \sum_{j=0}^{k-1} \int_{t_j^n}^{t_{j+1}^n} g(t, s, x_{j-1}^n(t_j^n))ds + \int_{t_k^n}^{t} g(t, s, x_{k-1}^n(t_k^n))ds.$$

It is worth observing that the map $h_k^n(\cdot)$ is integrable on $[t_k^n, t_{k+1}^n]$. Indeed, taking into account the growth conditions $(\mathcal{H}_{f,1})$ and $(\mathcal{H}_{g,1})$, we obtain that for every $t \in [t_k^n, t_{k+1}^n]$

$$\|h_k^n(t)\| \leq (1 + \|x_{k-1}^n(t_k^n)\|)\pi(t) + \sum_{j=0}^{k-1}(1 + \|x_{j-1}^n(t_j^n)\|) \int_{t_j^n}^{t_{j+1}^n} \gamma(t, s)ds$$
$$+ (1 + \|x_{k-1}^n(t_k^n)\|) \int_{t_k^n}^{t} \gamma(t, s)ds,$$

the equality $x_k^n(t_k^n) = x_{k-1}^n(t_k^n)$ ensures that

$$\|h_k^n(t)\| \leq (1 + \max_{0 \leq j \leq k} \|x_j^n(t_j^n)\|) \left(\pi(t) + \int_0^t \gamma(t, s)ds\right), \tag{4.4}$$

which gives by integrating

$$\int_0^T \|h_k^n(t)\|dt \leq (1 + \max_{0 \leq j \leq k} \|x_j^n(t_j^n)\|) \left(\int_0^T \pi(t)dt + \int_0^T \int_0^t \gamma(t, s)ds\, dt\right).$$

It is clear that every mapping $x_{k-1}^n(\cdot), k = 0, \ldots, n-1$ is bounded on the interval $[t_k^n, t_{k+1}^n]$ thanks to its absolute continuity property. Combining this boundness with the fact that both mappings $\pi(\cdot)$ and $\gamma(\cdot, \cdot)$ are integrable, we deduce the claimed property on $h_k^n(\cdot)$.

Based on the discrete sequences $(x_k^n(\cdot))$ that we have constructed above, we are now in a position to define the sequence of approximate solutions $(x_n(\cdot))_n$ on the whole interval $[0, T]$. For each integer $n \geq 1$, let $x_n(\cdot) : [T_0, T] \to H$ be such that

$$x_n(t) := x_k^n(t) \quad \text{whenever } t \in \left[t_k^n, t_{k+1}^n\right], k \in \{0, 1, \ldots, n-1\}. \tag{4.5}$$

It follows that (x_n) is absolutely continuous and represents the solution of (P_k) on $[t_k^n, t_{k+1}^n]$, further

$$x_n(t_k^n) = x_k^n(t_k^n) = x_{k-1}^n(t_k^n). \tag{4.6}$$

So, in order to present the differential inclusions (P_k) in a more convenient form involving only the sequence (x_n), we introduce the function $\theta_n(\cdot) : [0,T] \to [0,T]$ defined by

$$\begin{cases} \theta_n(0) := 0, \\ \theta_n(t) := t_k^n, \text{ if } t \in \left]t_k^n, t_{k+1}^n\right], k \in \{0,1,\ldots,n-1\}. \end{cases} \tag{4.7}$$

Therefore, we deduce from (4.2), (4.6) and (4.7) that

$$\begin{cases} -\dot{x}_n(t) \in N_{C(t)}(Ax_n(t)) + f(t, x_n(\theta_n(t))) + \int\limits_0^t g(t,s,x_n(\theta_n(s)))ds \quad a.\ e.\ t \in [0,T], \\ x_n(0) = x_0, Ax_0 \in C(0), \end{cases} \tag{4.8}$$

and from (4.3) that

$$\left\| \dot{x}_n(t) + f(t, x_n(\theta_n(t))) + \int\limits_0^t g(t,s,x_n(\theta_n(s)))ds \right\| \leq$$

$$\frac{1}{\rho}\left(\|A\| \left\| f(t, x_n(\theta_n(t))) + \int\limits_0^t g(t,s,x_n(\theta_n(s)))ds \right\| + |\dot{v}(t)| \right) \quad a.\ e.\ t \in [0,T]. \tag{4.9}$$

Step 3. We show that the sequence $(x_n(\cdot))$ converges uniformly to some absolutely continuous mapping $x(\cdot) : [0,T] \to H$.
First, we are going to prove that the sequence $(\dot{x}_n(\cdot))$ is uniformly dominated by an integrable function. According to (4.3), (4.4) and (4.5) we have for almost every $t \in [t_k^n, t_{k+1}^n]$

$$\|\dot{x}_n(t)\| \leq \left(1 + \frac{\|A\|}{\rho}\right)\|h_k^n(t)\| + \frac{1}{\rho}|\dot{v}(t)|$$

$$\leq \left(1 + \frac{\|A\|}{\rho}\right)\left(1 + \max_{1\leq j\leq n}\|x_n(t_j^n)\|\right)E(t) + \frac{1}{\rho}|\dot{v}(t)|,$$

where $E(t) := \pi(t) + \int\limits_0^t \gamma(t,s)ds$. Thus, for each $k \in \{0,1,\ldots,n-1\}$

$$\|x_n(t_{k+1}^n)\| \leq \|x_0\| + \int\limits_0^{t_{k+1}^n} \|\dot{x}_n(t)\|dt$$

$$\leq \|x_0\| + \left(1 + \frac{\|A\|}{\rho}\right)\left(1 + \max_{1\leq j\leq n}\|x_n(t_j^n)\|\right)\int\limits_0^{t_{k+1}^n} E(t)dt + \frac{1}{\rho}\int\limits_0^{t_{k+1}^n} |\dot{v}(t)|dt$$

$$\leq \|x_0\| + \left(1 + \frac{\|A\|}{\rho}\right)\int\limits_0^T E(t)\,dt + \left(1 + \frac{\|A\|}{\rho}\right)\max_{1\leq j\leq n}\|x_n(t_j^n)\|\int\limits_0^T E(t)dt + \frac{1}{\rho}\int\limits_0^T |\dot{v}(t)|dt.$$

Since k is selected arbitrary in $\{0,1,\ldots,n-1\}$, we deduce that

$$\left(\left(1 - \frac{\rho + \|A\|}{\rho}\right)\int\limits_0^T E(t)dt\right)\max_{0\leq j\leq n}\|x_n(t_j^n)\| \leq \|x_0\| + \left(1 + \frac{\|A\|}{\rho}\right)\int\limits_0^T E(t)\,dt + \frac{1}{\rho}\int\limits_0^T |\dot{v}(t)|\,dt.$$

Which gives by (4.1)

$$\max_{0\leq j\leq n}\left\|x_n(t_j^n)\right\| \leq M, \tag{4.10}$$

where

$$M := \frac{1}{\left(1-\frac{\rho+\|A\|}{\rho}\right)\|E(\cdot)\|_{L^1([0,T],\mathbb{R}_+)}}\left(\|x_0\|+\left(1+\frac{\|A\|}{\rho}\right)\|E(\cdot)\|_{L^1([0,T],\mathbb{R}_+)}+\frac{1}{\rho}\int_0^T|\dot{v}(t)|\,dt\right).$$

On one hand, from assumptions $(\mathcal{H}_{f,1})$, $(\mathcal{H}_{g,1})$ and (4.7) and (4.10) we have for all n

$$\|f(t,x_n(\theta_n(t)))\|\le\pi(t)(1+\|x_n(\theta_n(t))\|)\le(1+M)\pi(t) \text{ for all } t\in[0,T], \tag{4.11}$$

and

$$\|g(t,s,x_n(\theta_n(s)))\|\le\gamma(t,s)(1+\|x_n(\theta_n(s))\|)\le(1+M)\gamma(t,s) \text{ for all } (t,s)\in\Omega_T. \tag{4.12}$$

On the other hand, the inequalities (4.9) , (4.11) and (4.12) imply for almost all t and for all n

$$\left\|\dot{x}_n(t)+f(t,x_n(\theta_n(t)))+\int_0^t g(t,s,x_n(\theta_n(s))ds\right\|\le\alpha(t), \tag{4.13}$$

where

$$\alpha(t) := \frac{\|A\|}{\rho}(1+M)\left(\pi(t)+\int_0^t\gamma(t,s)ds\right)+\frac{1}{\rho}|\dot{v}(t)|.$$

Which gives for almost all t and for all n

$$\|\dot{x}_n(t)\|\le\psi(t), \tag{4.14}$$

where

$$\psi(t) := (1+M)\left(1+\frac{\|A\|}{\rho}\right)\left(\pi(t)+\int_0^t\gamma(t,s)ds\right)+\frac{1}{\rho}|\dot{v}(t)|.$$

We can now prove that $(x_n(\cdot))$ is a Cauchy sequence in the Banach space $(\mathcal{C}(I,H),\|.\|_\infty)$. Let $m,n\in\mathbb{N}$, for almost all $t\in[0,T]$, we have

$$\begin{cases} -\dot{x}_n(t)-f(t,x_n(\theta_n(t))-\int_0^t g(t,s,x_n(\theta_n(s))ds\in N_{C(t)}(Ax_n(t)),\\ -\dot{x}_m(t)-f(t,x_m(\theta_m(t))-\int_0^t g(t,s,x_m(\theta_m(s))ds\in N_{C(t)}(Ax_m(t)).\end{cases}$$

Using the fact that the normal cone is monotone, we get the following

$$\frac{1}{2}\frac{d}{dt}\langle x_n(t)-x_m(t),A\left(x_n(t)-x_m(t)\right)\rangle\le$$

$$\left\langle f(t,x_n(\theta_n(t))-f(t,x_m(\theta_m(t))+\int_0^t\left(g(t,s,x_n(\theta_n(s))-g(t,s,x_m(\theta_m(s))\right)ds, A\left(x_m(t)-x_n(t)\right)\right\rangle.$$

We obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle x_n(t)-x_m(t),A\left(x_n(t)-x_m(t)\right)\rangle\le\\ &\|A\|\,\|x_n(t)-x_m(t)\|\,\|f(t,x_n(\theta_n(t))-f(t,x_m(\theta_m(t))\|+\\ &\|A\|\,\|x_n(t)-x_m(t)\|\int_0^t\|(g(t,s,x_n(\theta_n(s))-g(t,s,x_m(\theta_m(s)))\|\,ds\end{aligned}$$

On the other hand, the absolute continuity of $x_n(\cdot)$ gives by (4.14)

$$\|x_n(t)\|\le\|x_0\|+\int_0^T\psi(r)dr \text{ for all } t\in[0,T].$$

It follows that, for some $\eta > 0$, for all $t \in [0, T]$ and for all $n \in \mathbb{N}$,

$$x_n(t) \in B\left[0, \eta\right], \tag{4.15}$$

with

$$\eta := \|x_0\| + \int_0^T \psi(r)dr.$$

Which gives by the assumptions $(\mathcal{H}_{f,2})$ and $(\mathcal{H}_{g,2})$ with $J_n(t) := \|x_n(\theta_n(t)) - x_n(t)\|$ that

$$\frac{1}{2}\frac{d}{dt}\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle \leq$$
$$k_\eta(t)\|A\|\|x_n(t) - x_m(t)\|\|x_n(\theta_n(t)) - x_m(\theta_m(t))\| +$$
$$L_\eta(t)\|A\|\|x_n(t) - x_m(t)\| \int_0^t \|x_n(\theta_n(s)) - x_m(\theta_m(s))\| ds \leq$$
$$k_\eta(t)\|A\|\|x_n(t) - x_m(t)\| \left(\|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x_m(t)\| + \|x_m(t) - x_m(\theta_m(t))\|\right) +$$
$$L_\eta(t)\|A\|\|x_n(t) - x_m(t)\| \left(\int_0^t J_n(s)ds + \int_0^t \|x_n(s) - x_m(s)\| ds + \int_0^t J_m(s)ds\right)$$

By (4.14), we have for all $t \in [0, T]$ and for all $n \in \mathbb{N}$,

$$\|x_n(t) - x_n(\theta_n(t))\| \leq \int_{\theta_n(t)}^t \psi(r)dr.$$

Moreover, by (4.15), we have

$$\|x_n(t) - x_m(t)\| \leq 2\eta.$$

This implies that

$$\frac{d}{dt}\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle \leq 2k_\eta(t)\|A\|\|x_n(t) - x_m(t)\|^2 +$$
$$4\eta k_\eta(t)\|A\| \left[\int_{\theta_n(t)}^t \psi(r)dr + \int_{\theta_m(t)}^t \psi(r)dr\right] + 4\eta L_\eta(t)\|A\| \int_0^t \left(\int_{\theta_n(s)}^s \psi(r)dr + \int_{\theta_m(s)}^s \psi(r)dr\right) ds +$$
$$2L_\eta(t)\|A\|\|x_n(t) - x_m(t)\| \int_0^t \|x_n(s) - x_m(s)\| ds.$$

On the other hand, using the fact that A is $\rho-$coercive, we have

$$\frac{d}{dt}\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle \leq \frac{2}{\rho}k_\eta(t)\|A\|\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle +$$
$$4\eta k_\eta(t)\|A\| \left[\int_{\theta_n(t)}^t \psi(r)dr + \int_{\theta_m(t)}^t \psi(r)dr\right] + 4\eta L_\eta(t)\|A\| \int_0^t \left(\int_{\theta_n(s)}^s \psi(r)dr + \int_{\theta_m(s)}^s \psi(r)dr\right) ds +$$
$$2L_\eta(t)\|A\| \frac{1}{\rho}\sqrt{\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle} \int_0^t \sqrt{\left\langle x_n(s) - x_m(s), A\left(x_n(s) - x_m(s)\right)\right\rangle} ds.$$

Putting for all $t \in [0, T]$

$$a_{n,m}(t) := 4\eta k_\eta(t)\|A\| \left[\int_{\theta_n(t)}^t \psi(r)dr + \int_{\theta_m(t)}^t \psi(r)dr\right], \tag{4.16}$$

and

$$b_{n,m}(s) := \int_{\theta_n(s)}^{s} \psi(r)dr + \int_{\theta_m(s)}^{s} \psi(r)dr, \tag{4.17}$$

we arrive to

$$\frac{d}{dt}\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle \leq \frac{2}{\rho}k_\eta(t)\left\|A\right\|\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle +$$

$$2L_\eta(t)\left\|A\right\|\frac{1}{\rho}\sqrt{\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle}\int_0^t \sqrt{\left\langle x_n(s) - x_m(s), A\left(x_n(s) - x_m(s)\right)\right\rangle}ds +$$

$$a_{n,m}(t) + 4\eta L_\eta(t)\left\|A\right\|\int_0^t b_{n,m}(s)ds.$$

Since $\psi(\cdot) \in L^1(I, \mathbb{R}_+)$ and for each $t \in I$, we have $\theta_n(t), \theta_m(t) \longrightarrow t$, then

$$\lim_{n,m\longrightarrow\infty} a_{n,m}(t) = 0 \text{ and } \lim_{n,m\longrightarrow\infty} b_{n,m}(t) = 0 \quad a.\ e.\ t \in [0,T]. \tag{4.18}$$

On the other hand, for each $n \in \mathbb{N}$ writing

$$\int_{\theta_n(t)}^{t} \psi(s)ds \leq \int_0^T \psi(s)ds. \tag{4.19}$$

The relations (4.16), (4.17) and (4.19) imply that

$$\left|a_{n,m}(t)\right| \leq 8\eta k_\eta(t)\left\|A\right\|\int_0^T \psi(s)ds,$$

and

$$\left|b_{n,m}(s)\right| \leq 2\int_0^T \psi(s)ds.$$

It follows from the dominated convergence theorem that for all $t \in [0,T]$

$$\lim_{n,m\longrightarrow\infty}\int_0^T a_{n,m}(t)dt = 0 \text{ and } \lim_{n,m\longrightarrow\infty}\int_0^T b_{n,m}(s)ds = 0. \tag{4.20}$$

Note also by (4.18) that

$$\frac{d}{dt}\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle \leq \frac{2}{\rho}k_\eta(t)\left\|A\right\|\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle +$$

$$a_{n,m}(t) + 4\eta L_\eta(t)\left\|A\right\|\int_0^T b_{n,m}(s)ds +$$

$$2L_\eta(t)\left\|A\right\|\frac{1}{\rho}\sqrt{\left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle}\int_0^t \sqrt{\left\langle x_n(s) - x_m(s), A\left(x_n(s) - x_m(s)\right)\right\rangle}ds. \tag{4.21}$$

For each $t \in [0,T]$, let us set

$$\sigma_{n,m}(t) := \left\langle x_n(t) - x_m(t), A\left(x_n(t) - x_m(t)\right)\right\rangle, \chi_{n,m}(t) := a_{n,m}(t) + 4\eta L_\eta(t)\left\|A\right\|\int_0^T b_{n,m}(s)ds.$$

These notations together with the integration of (4.21) give

$$\sigma_{n,m}(t) \leq$$

$$\sigma_{n,m}(0) + \int_0^t \left(\frac{2\|A\|}{\rho} k_\eta(s)\sigma_{n,m}(s) + \chi_{n,m}(s) + \frac{2\|A\|}{\rho} L_\eta(s)\sqrt{\sigma_{n,m}(s)} \int_0^s \sqrt{\sigma_{n,m}(r)}dr \right) ds,$$

On the other hand, it is clear that for any $n, m \in \mathbb{N}$, the function

$$\Gamma_{n,m}(t) :=$$

$$\sigma_{n,m}(0) + \int_0^t \left(\frac{2\|A\|}{\rho} k_\eta(s)\sigma_{n,m}(s) + \chi_{n,m}(s) + \frac{2\|A\|}{\rho} L_\eta(s)\sqrt{\sigma_{n,m}(s)} \int_0^s \sqrt{\sigma_{n,m}(r)}dr \right) ds,$$

is nondecreasing on I. Further, for each real $t \in [0, T]$

$$\dot{\Gamma}_{n,m}(t) = \frac{2\|A\|}{\rho} k_\eta(t)\sigma_{n,m}(t) + \chi_{n,m}(t) + \frac{2\|A\|}{\rho} L_\eta(t)\sqrt{\sigma_{n,m}(t)} \int_0^t \sqrt{\sigma_{n,m}(s)}ds \leq$$

$$\frac{2\|A\|}{\rho} k_\eta(t)\Gamma_{n,m}(t) + \chi_{n,m}(t) + \frac{2\|A\|}{\rho} L_\eta(t)\sqrt{\Gamma_{n,m}(t)} \int_0^t \sqrt{\Gamma_{n,m}(s)}ds$$

Since $\Gamma_{n,m}(\cdot)$ is nondecreasing, it results that

$$\dot{\Gamma}_{n,m}(t) \leq \frac{2\|A\|}{\rho} k_\eta(t)\Gamma_{n,m}(t) + \chi_{n,m}(t) + \frac{2\|A\|}{\rho} tL_\eta(t)\Gamma_{n,m}(t).$$

Finally, for almost every $t \in [0, T]$, one has

$$\dot{\Gamma}_{n,m}(t) \leq \frac{2\|A\|}{\rho} \left(k_\eta(t) + tL_\eta(t)\right)\Gamma_{n,m}(t) + \chi_{n,m}(t).$$

Applying the classical Gronwall's inequality brings us to the following estimate

$$\Gamma_{n,m}(t) \leq \Gamma_{n,m}(0) \exp\left(\frac{2\|A\|}{\rho} \left(\int_0^t k_\eta(s)ds + T\int_0^t L_\eta(s)ds \right) \right) +$$

$$\int_0^t \exp\left(\frac{2\|A\|}{\rho} \left(\int_s^t k_\eta(r)dr + T\int_s^t L_\eta(r)dr \right) \right) \chi_{n,m}(s)ds.$$

Consequently

$$\sigma_{n,m}(t) \leq \sigma_{n,m}(0) \exp\left(\frac{2\|A\|}{\rho} \left(\|k_\eta\|_{L^1([0,T];\mathbb{R}_+)} + T\|L_\eta\|_{L^1([0,T];\mathbb{R}_+)} \right) \right) +$$

$$\exp\left(\frac{2\|A\|}{\rho} \left(\|k_\eta\|_{L^1([0,T];\mathbb{R}_+)} + T\|L_\eta\|_{L^1([0,T];\mathbb{R}_+)} \right) \right) \int_0^T \chi_{n,m}(s)ds.$$

Using the $\rho-$coercivity of A and the equality $\sigma_{n,m}(0) = 0$ bring us to

$$\|x_n(t) - x_m(t)\|^2 \leq \frac{1}{\rho} \exp\left(\frac{2\|A\|}{\rho} \left(\|k_\eta\|_{L^1([0,T];\mathbb{R}_+)} + T\|L_\eta\|_{L^1([0,T];\mathbb{R}_+)} \right) \right) \int_0^T \chi_{n,m}(s)ds.$$

On the other hand, it follows from the dominated convergence theorem that for all $t \in [0, T]$,

$$\lim_{n,m \longrightarrow \infty} \int_0^T \chi_{n,m}(s)ds = 0.$$

As a consequence, we obtain

$$\lim_{n,m\longrightarrow\infty} \|x_n(t) - x_m(t)\| = 0.$$

The above equality being true for all $t \in [0,T]$, it follows that the sequence $(x_n(\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{C}([0,T],H), \|\cdot\|_\infty)$ and hence converges uniformly to some map $x(\cdot) : [0,T] \longrightarrow H$.

Step 4: We show that $x(\cdot)$ is absolutely continuous.
By virtue of (4.14), extracting a subsequence if necessary, we assume without loss of generality that $(\dot{x}_n(\cdot))$ converges weakly in $L^1(I,H)$ to some mapping $g(\cdot) \in L^1(I,H)$. This means that,

$$\int_0^T \langle \dot{x}_n(s), h(s)\rangle\, ds \longrightarrow \int_0^T \langle g(s), h(s)\rangle\, ds, \forall h \in L^\infty(I,H).$$

For any $z \in H$, and any $n \in \mathbb{N}$, we can write

$$\int_0^T \left\langle \dot{x}_n(s), z \cdot 1_{[0,t]}(s)\right\rangle ds = \int_0^t \langle \dot{x}_n(s), z\rangle\, ds,$$

and

$$\int_0^T \left\langle g(s), z \cdot 1_{[0,t]}(s)\right\rangle ds = \int_0^t \langle g(s), z\rangle\, ds.$$

So, from the weak convergence we deduce that

$$\int_0^t \dot{x}_n(s)ds \longrightarrow \int_0^t g(s)ds \text{ weakly in } H.$$

This and the absolute continuity of $x_n(\cdot)$ imply that

$$x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s)ds \longrightarrow x(0) + \int_0^t g(s)ds \text{ weakly in } H.$$

For each $t \in [0,T]$, the strong convergence of $(x_n(t))_{n\in\mathbb{N}}$ to $x(t)$ in H and the equality $x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s)ds$ valid for all $n \in \mathbb{N}$ entail

$$x(t) = x(0) + \int_0^t g(s)ds.$$

We deduce that $x(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{x}(t) = g(t)$ for almost everywhere on I and hence

$$\dot{x}_n(\cdot) \longrightarrow \dot{x}(\cdot) \text{ weakly in } L^1(I,H).$$

Step 5: We show that $x(\cdot)$ is a solution of $(P_{A,f,g})$.
First, it is obvious that $x(0) = x_0$ and $Ax_0 \in C(0)$. Now, it remains to prove that

$$\dot{x}(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds \in -N_{C(t)}(Ax(t)) \quad a.\ e.\ t \in [0,T].$$

Since $\theta_n(t) \longrightarrow t$ for any $t \in I$ and $x_n(\cdot)$ converges uniformly to $x(\cdot)$, on has $x_n(\theta_n(t)) \longrightarrow x(t)$ for each $t \in I$. On the other hand, the continuity of $f(t,\cdot)$ ensures that, for all $t \in I$,

$$f(t, x_n(\theta_n(t))) \longrightarrow f(t,x(t)) \text{ in } H.$$

According to (4.11), we also have

$$\int_0^T \|f(t, x_n(\theta_n(t)))\| \, dt \leq (1+M) \int_0^T \pi(t) dt = (1+M) \|\pi\|_{L^1(I,\mathbb{R}_+)}.$$

Hence $f(\cdot, x_n(\theta_n(\cdot)))$ is a sequence in $L^1(I, H)$, it follows from the dominated convergence theorem that

$$f(\cdot, x_n(\theta_n(\cdot))) \longrightarrow f(\cdot, x(\cdot)) \text{ strongly in } L^1(I, H),$$

which implies that

$$f(\cdot, x_n(\theta_n(\cdot))) \longrightarrow f(\cdot, x(\cdot)) \text{ weakly in } L^1(I, H).$$

On the other hand, we have shown in the above step that $\dot{x}_n(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^1(I, H)$.

Now, let us put

$$z_n(\cdot) := z_n^{(1)}(\cdot) + z_n^{(2)}(\cdot) + z_n^{(3)}(\cdot),$$

where

$$\begin{cases} z_n^{(1)}(\cdot) := \dot{x}_n(\cdot), \\ z_n^{(2)}(\cdot) := f(\cdot, x_n(\theta_n(\cdot))), \end{cases}$$

and for all $t \in [0, T]$,

$$z_n^{(3)}(t) = \int_0^t g(t, s, x_n(\theta_n(s))) ds.$$

Let us show that $z_n^{(3)}(\cdot)$ converges weakly in $L^1(I, H)$ to $z^{(3)}(\cdot)$ such that for all $t \in [0, T]$

$$z^{(3)}(t) := \int_0^t g(t, s, x(s) ds.$$

From the Lipschitz property of g with respect to x, we have

$$\begin{aligned} \int_0^T \left\| z_n^{(3)}(t) - z^{(3)}(t) \right\| dt &\leq \int_0^T \int_0^t \|g(t, s, x_n(\theta_n(s))) - g(t, s, x(s))\| \, dsdt \\ &\leq \int_0^T L_\eta(t) \int_0^t \|x_n(\theta_n(s)) - x(s)\| \, dsdt \\ &\leq \int_0^T L_\eta(t) \int_0^T \|x_n(\theta_n(s)) - x(s)\| \, dsdt \\ &\leq \|L_\eta(\cdot)\|_{L^1(I,\mathbb{R}_+)} \|x_n(\theta_n(\cdot)) - x(\cdot)\|_{L^1(I,\mathbb{R}_+)}. \end{aligned}$$

Therefore, using the fact that $x_n(\cdot)$ converges uniformly to $x(\cdot)$, it follows that

$$\lim_{n \longrightarrow \infty} \int_0^T \left\| z_n^{(3)}(t) - z^{(3)}(t) \right\| dt = 0.$$

This means that $z_n^{(3)}(\cdot)$ converges strongly in $L^1(I, H)$ to $z^{(3)}(\cdot)$. Consequently $z_n^{(3)}(\cdot)$ converges weakly in $L^1(I, H)$ to $z^{(3)}(\cdot)$. This implies that

$$z_n(\cdot) \longrightarrow z(\cdot) := \dot{x}(\cdot) + f(\cdot, x(\cdot)) + z^{(3)}(\cdot) \text{ weakly in } L^1(I, H). \tag{4.22}$$

Now, we apply a classical technique due to C. Castaing (see ([12])). Thanks to (4.22), by Mazur's lemma, there exists a sequence $(v_n(\cdot))_n$ which converges strongly in $L^1(I,H)$ to $z(\cdot)$ with for each n and for all $t \in I$

$$v_n(t) \in co\{z_k(t), k \geq n\}.$$

Extracting a subsequence, we may suppose that

$$v_n(t) \longrightarrow z(t) := \dot{x}(t) + f(t, x(t)) + \int_0^t g(t,s,x(s))ds \quad a.\ e.\ t \in I,$$

which allows us to write, for almost all $t \in I$

$$z(t) \in \bigcap_n \overline{co}\{z_k(t), k \geq n\}.$$

Here $\overline{co}$ denotes the closed convex hull. The last relation above yields, for almost all $t \in I$, for any $\xi \in H$,

$$\left\langle \xi, \dot{x}(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds \right\rangle \leq$$

$$\inf_n \sup_{k\geq n} \left\langle \xi, \dot{x}_k(t) + f(t, x_k(\theta_k(t))) + \int_0^t g(t,s,x_k(\theta_k(s)))ds \right\rangle.$$

On the other hand, coming back to (4.8) and (4.13), we arrive to the inclusion

$$\dot{x}_n(t) + f(t, x_n(\theta_n(t))) + \int_0^t g(t,s,x_n(\theta_n(s))ds \in a(t)\mathbb{B}_H \quad a.\ e.\ t \in I.$$

The latter inclusion and relation (4.8) entail for almost all $t \in I$

$$-\frac{1}{\alpha(t)}\left(\dot{x}_n(t) + f(t, x_n(\theta_n(t))) + \int_0^t g(t,s,x_n(\theta_n(s))ds\right) \in N_{C(t)}(Ax_n(t)) \cap \mathbb{B}_H.$$

It follows that for almost all $t \in I$,

$$\frac{1}{\alpha(t)}\left(\dot{x}_n(t) + f(t, x_n(\theta_n(t))) + \int_0^t g(t,s,x_n(\theta_n(s))ds\right) \in \Big(-\partial d_{C(t)}(Ax_n(t))\Big).$$

Which implies that, for almost all $t \in I$, for all $\xi \in H$

$$\left\langle \xi, \dot{x}_n(t) + f(t, x_n(\theta_n(t))) + \int_0^t g(t,s,x_n(\theta_n(s))ds \right\rangle \leq \alpha(t).\sigma(-\partial d_{C(t)}(x_n(t)), \xi).$$

So, for almost all $t \in I$, for all $\xi \in H$

$$\left\langle \xi, \dot{x}(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds \right\rangle \leq$$

$$\alpha(t) \limsup_{n\longrightarrow\infty} \sigma(-\partial d_{C(t)}(Ax_n(t)), \xi) \leq \alpha(t)\sigma(-\partial d_{C(t)}(Ax(t)), \xi),$$

where the second inequality follows from the upper continuity of $\sigma(-\partial d_{C(t)}(\cdot), \xi)$. This implies that

$$\left\langle \xi, \dot{x}(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds \right\rangle - \sigma(-\alpha(t)\partial d_{C(t)}(x(t)), \xi) \leq 0.$$

Since ξ is arbitrary, we have

$$\sup_{\xi\in H}\left[\left\langle \xi, \dot{x}(t)+f(t,x(t))+\int_0^t g(t,s,x(s))ds\right\rangle - \sigma(-\alpha(t)\partial d_{C(t)}(Ax(t)),\xi)\right]\leq 0. \quad (4.23)$$

This implies by the closedness and convexity of $\partial d_{C(t)}(Ax(t))$ and by properties of support function that for almost all $t\in I$

$$d\left(\dot{x}(t)+f(t,x(t))+\int_0^t g(t,s,x(s))ds, -\alpha(t)\partial d_{C(t)}(Ax(t))\right)=$$

$$\sup_{\xi\in\mathbb{B}_H}\left\{\left\langle \xi, \dot{x}(t)+f(t,x(t))+\int_0^t g(t,s,x(s))ds\right\rangle - \sigma(-\alpha(t)\partial d_{C(t)}(Ax(t)),\xi)\right\}$$

$$\leq \sup_{\xi\in H}\left\{\left\langle \xi, \dot{x}(t)+f(t,x(t))+\int_0^t g(t,s,x(s))ds\right\rangle - \sigma(-\alpha(t)\partial d_{C(t)}(Ax(t)),\xi)\right\}.$$

This and inequality (4.23) give

$$d\left(\dot{x}(t)+f(t,x(t))+\int_0^t g(t,s,x(s))ds, -\alpha(t)\partial d_{C(t)}(Ax(t))\right)=0.$$

Therefore, for almost all $t\in I$

$$-\dot{x}(t)-f(t,x(t))-\int_0^t g(t,s,x(s))ds\in\alpha(t)\partial d_{C(t)}(Ax(t))\subset N_{C(t)}(Ax(t)).$$

Consequently, as desired it follows that

$$\dot{x}(t)+f(t,x(t))+\int_0^t g(t,s,x(s))ds\in -N_{C(t)}(Ax(t)) \quad a.e.\ t\in I.$$

Therefore, the function $x(\cdot)$ is a solution of $(P_{A,f,g})$.

Step 6: Uniqueness of solution.

Now, we turn to the uniqueness. If $x_1(\cdot)$ and $x_2(\cdot)$ are two solutions of $(P_{A,f,g})$, the monotonicity property of the normal cone yields, for almost all $t\in[0,T]$,

$$\langle \dot{x}_1(t)-\dot{x}_2(t), A\left(x_1(t)-x_2(t)\right)\rangle\leq$$

$$\left\langle f(t,x_1(t))-f(t,x_2(t))+\int_0^t g(t,s,x_1(s))ds-\int_0^t g(t,s,x_2(s))ds, A\left(x_2(t)-x_1(t)\right)\right\rangle$$

$$\leq \|A\left(x_2(t)-x_1(t)\right)\|\left(\|f(t,x_1(t))-f(t,x_2(t))\|+\int_0^t\|g(t,s,x_1(s))-g(t,s,x_2(s))\|\,ds\right).$$

Since A is a bounded linear map and by the assumptions $(\mathcal{H}_{f,2}),(\mathcal{H}_{g,2})$ we have for almost all $t\in[0,T]$

$$\langle \dot{x}_1(t)-\dot{x}_2(t), A\left(x_1(t)-x_2(t)\right)\rangle\leq$$

$$\leq k_\eta(t)\,\|A\|\,\|x_1(t)-x_2(t)\|^2+L_\eta(t)\,\|A\|\,\|x_1(t)-x_2(t)\|\int_0^t\|x_1(s)-x_2(s)\|\,ds.$$

Using the fact that A is ρ-coercive, we have

$$\frac{d}{dt}\langle x_1(t)-x_2(t), A\left(x_1(t)-x_2(t)\right)\rangle\leq\frac{2}{\rho}k_\eta(t)\,\|A\|\,\langle x_1(t)-x_2(t), A\left(x_1(t)-x_2(t)\right)\rangle+$$

$$2L_\eta(t)\,\|A\|\,\frac{1}{\rho}\sqrt{\langle x_1(t)-x_2(t), A\left(x_1(t)-x_2(t)\right)\rangle}\int\limits_0^t\sqrt{\langle x_1(s)-x_2(s), A\left(x_1(s)-x_2(s)\right)\rangle}ds$$

Let us set for each $t\in[0,T]$,

$$\phi(t):=\langle x_1(t)-x_2(t), A\left(x_1(t)-x_2(t)\right)\rangle,$$

applying the classical Gronwall's inequality brings us to the following estimate

$$\phi(t)\le\phi(0)\exp\left(\frac{2\,\|A\|}{\rho}\left(\|k_\eta\|_{L^1([0,T],\mathbb{R}_+)}+T\|L_\eta\|_{L^1([0,T],\mathbb{R}_+)}\right)\right).$$

Using the $\rho-$coercivity of A and the equality $\phi(0)=0$ bring us to

$$\|x_1(t)-x_2(t)\|=0.$$

The above equality being true for all $t\in[0,T]$, it follows that

$$x_1(\cdot)=x_2(\cdot).$$

Case 2: Now assume that

$$\int\limits_0^T\left(\pi(r)+\int\limits_0^r\gamma(r,s)ds\right)dr\ge\frac{\rho}{\rho+\|A\|}.$$

Consider a subdivision of $[0,T]$ given

$$T_0=0,T_1,...,T_k=T,$$

such that, for any $i\in\{0,1,...,k-1\}$

$$\int\limits_{T_i}^{T_{i+1}}\left(\pi(r)+\int\limits_0^r\gamma(r,s)ds\right)dr<\frac{\rho}{\rho+\|A\|}.$$

Then, by what precedes, there exists an absolutely continuous map $x_0:[0,T_1]\to H$ such that $x_0(0)=x_0, Ax_0\in C(0)$ for all $t\in[0,T_1]$, and

$$-\dot{x}_0(t)\in N_{C(t)}(Ax_0(t))+f(t,x_0(t))+\int\limits_0^t g(t,s,x_0(s))ds\quad a.\ e.\ t\in[0,T_1].$$

In the same vein, by what precedes again, there exists an absolutely continuous map $x_1(\cdot):[T_1,T_2]\longrightarrow H$ such that $x_1(T_1)=x_0(T_1), Ax_1(t)\in C(t)$ for all $t\in[T_1,T_2]$, and

$$-\dot{x}_1(t)\in N_{C(t)}(Ax_1(t))+f(t,x_1(t))+\int\limits_0^t g(t,s,x_1(s))ds\quad a.\ e.\ t\in[T_1,T_2].$$

Inductively, there exists a finite sequence of absolutely continuous maps $x_i(\cdot):[T_i,T_{i+1}]\longrightarrow H$ $(0\le i\le k-1)$ such that, for each $i\in\{0,1,...,k-1\}$ (we set $x_{-1}(0)=x_0$), $x_i(T_i)=x_{i-1}(T_i), Ax_i(t)\in C(t)$ for all $t\in[T_i,T_{i+1}]$, and

$$-\dot{x}_i(t)\in N_{C(t)}(Ax_i(t))+f(t,x_i(t))+\int\limits_0^t g(t,s,x_i(s))ds\quad a.\ e.\ t\in[T_i,T_{i+1}].$$

Now, let $x(\cdot):[0,T]\longrightarrow H$ be the map defined by

$$x(t)=x_i(t),\ \text{if } t\in[T_i,T_{i+1}],(0\le i\le k-1).$$

Obviously, $x(\cdot)$ is an absolutely continuous map satisfying $x(0) = x_0, Ax(t) \in C(t)$ for all $t \in [0,T]$ and

$$-\dot{x}(t) \in N_{C(t)}(Ax(t)) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds \quad a.\ e.\ t \in [0,T]\,.$$

Step 7. We prove the estimations.
It remains to prove the predicted estimations. Let $x(\cdot)$ be the unique solution of $(P_{A,f,g})$. According to Proposition 2.2, one has

$$\left\| \dot{x}(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds \right\| \leq$$

$$\frac{1}{\rho}\left[\|A\| \, \|f(t, x(t))\| + \|A\| \int_0^t \|g(t, s, x(s)\| \, ds + |\dot{v}(t)| \right] \quad a.\ e.\ (t,s) \in \Omega_T. \tag{4.24}$$

On the other hand

$$\|\dot{x}(t)\| \leq \left\| \dot{x}(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds \right\| + \|f(t, x(t))\| + \int_0^t \|g(t, s, x(s))\| \, ds.$$

Therefore, by (4.24), we obtain

$$\|\dot{x}(t)\| \leq \frac{1}{\rho}\left[\|A\| \, \|f(t, x(t))\| + \|A\| \int_0^t \|g(t, s, x(s)\| \, ds + |\dot{v}(t)| \right] +$$

$$\|f(t, x(t))\| + \int_0^t \|g(t, s, x(s))\| \, ds \quad a.\ e.\ t \in [0,T].$$

From the growth conditions of f and g, we have for almost all $(t,s) \in \Omega_T$

$$\|\dot{x}(t)\| \leq \left(\frac{\|A\|}{\rho} + 1 \right) (1 + \|x(t)\|)\, E(t) + \frac{1}{\rho} |\dot{v}(t)|\,,$$

where

$$E(t) := \pi(t) + \int_0^t \gamma(t,s)ds.$$

On the other hand the fact that $x(\cdot)$ is absolutely continuous implies

$$\|\dot{x}(t)\| \leq$$

$$\left(\frac{\|A\|}{\rho} + 1 \right) (1 + \|x_0\|)\, E(t) + \frac{1}{\rho} |\dot{v}(t)| + \left(\frac{\|A\|}{\rho} + 1 \right) E(t) \int_0^t \|\dot{x}(s)\| \, ds \quad a.\ e.\ t \in [0,T].$$

By Gronwall's lemma we obtain, for all $t \in [0,T]$,

$$\int_0^t \|\dot{x}(s)\| \, ds \leq \int_0^t \left[\left(\frac{\|A\|}{\rho} + 1 \right) (1 + \|x_0\|)\, E(r) + \frac{1}{\rho} |\dot{v}(r)| \right] \exp\left(\int_r^t \left(\frac{\|A\|}{\rho} + 1 \right) E(\theta)\, d\theta \right) dr.$$

Using the inequality

$$\|x(t)\| \leq \|x_0\| + \int_0^t \|\dot{x}(s)\| \, ds,$$

we get for all $t \in [0,T]$

$$\|x(t)\| \leq$$
$$\|x_0\| + \int_0^t \left[\left(\frac{\|A\|}{\rho}+1\right)(1+\|x_0\|)\,E(r) + \frac{1}{\rho}\,|\dot v(r)|\right] \exp\left(\int_r^t \left(\frac{\|A\|}{\rho}+1\right) E\,(\theta)\,d\theta\right) dr.$$

As a result, for

$$l := \|x_0\| + \exp\left[\left(\frac{\|A\|}{\rho}+1\right)\int_0^T E(\theta)d\theta\right]\left(\frac{\|A\|}{\rho}+1\right)(1+\|x_0\|)\int_0^T \left(E(r) + \frac{1}{\rho}\,|\dot v(r)|\right) dr,$$

one has

$$\|x\,(\cdot)\|_\infty \leq l.$$

Consequently

$$\|f(t,x(t))\| \leq \pi(t)(1+l) \quad a.\ e.\ t \in [0,T]\,, \tag{4.25}$$

and

$$\|g(t,s,x(s))\| \leq \gamma(t,s)(1+l) \quad a.\ e.\ (t,s) \in \Omega_T. \tag{4.26}$$

And $(4.24)\,,(4.25)$ and (4.26) together imply, for almost all $t \in [0,T]$

$$\left\|\dot x(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds\right\| \leq \frac{1}{\rho}\left[\|A\|\,(1+l)E(t) + |\dot v(t)|\right]. \tag{4.27}$$

Further, we have

$$\|\dot x(t)\| \leq \left\|\dot x(t) + f(t,x(t)) + \int_0^t g(t,s,x(s))ds\right\| + \|f(t,x(t))\| + \int_0^t \|g(t,s,x(s))\|\,ds.$$

Coming back to (4.27), it follows that, for almost all $t \in [0,T]$,

$$\|\dot x(t)\| \leq \frac{1}{\rho}\left(\|A\|\,(1+l)E(t) + |\dot v(t)|\right) + (1+l)E(t).$$

Then, the proof of the theorem is complete. □

Let $\mathcal{S}_A$ be the set defined by

$$\mathcal{S}_A := \{e \in H : Ae \in C(0)\}\,.$$

For each $e \in \mathcal{S}_A$, denote by $x_e(\cdot)$ the unique solution of $(P_{A,f,g})$ with the initial data $x_e(0) = e, Ae \in C(0)$.

The following proposition gives a topological result concerning the map $e \longmapsto x_e(\cdot)$ which associates with each $e \in \mathcal{S}_A$ the unique solution $x_e(\cdot)$ of $(P_{A,f,g})$ with the initial data $x_e(0) = e, Ae \in C(0)$. For completeness of the paper, we sketch the proof.

Proposition 4.2. *Assume that the assumptions of Theorem 4.1 hold. For each $e \in \mathcal{S}_A$, the map*

$$\begin{array}{rccl} \psi: & \mathcal{S}_A & \longrightarrow & \mathcal{C}([0,T]\,,H) \\ & e & \longmapsto & \psi(e) = x_e(\cdot), \end{array}$$

endowed with the uniform convergence norm is Lipschitz on any bounded subset of $\mathcal{S}_A$.

Proof. Let $R > 0$ be any fixed positive real number. We are going to prove that ψ is Lipschitz on $\mathcal{S}_A \cap R.\mathbb{B}$. According to Theorem 4.1 (case (a)) and since the constant l depends on the initial condition, one can find a real number R_1 depending only on R such that, for all $z \in \mathcal{S}_A \cap R.\mathbb{B}$ and for almost all $t \in [0,T]$,

$$\left\|\dot x_z(t) + f(t,x_z(t)) + \int_0^t g(t,s,x_z(s))ds\right\| \leq G(t),$$

where

$$G(t) := \frac{1}{\rho}\left[\|A\|\,(1+R_1)\left(\pi(t)+\int_0^t \gamma(t,s)ds\right)+|\dot{v}(t)|\right].$$

Which entails that (by (a) and (b)) there exists some real number $\eta > 0$ depending only on R for which

$$\|x_z(t)\| \leq \eta, \tag{4.28}$$

for all $z \in \mathcal{S}_A \cap R.\mathbb{B}$ and for all $t \in [0,T]$.
Fix any $e, d \in \mathcal{S}_A \cap R.\mathbb{B}$. By the monotonicity property of the normal cone, we have for almost all $(t,s) \in \Omega_T$

$$\left\langle \dot{x}_e(t) + f(t,x_e(t)) + \int_0^t g(t,s,x_e(s))ds - \dot{x}_d(t) - f(t,x_d(t)) - \int_0^t g(t,s,x_d(s))ds, Ax_e(t) - Ax_d(t)\right\rangle$$
$$\leq 0,$$

from which we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\langle x_e(t)-x_d(t), A\left(x_e(t)-x_d(t)\right)\right\rangle \leq\\ &\|A\|\,\|f(t,x_e(t))-f(t,x_d(t))\|\,\|x_e(t)-x_d(t)\| +\\ &\|A\|\,\|x_e(t)-x_d(t)\|\int_0^t \|g(t,s,x_e(s))-g(t,s,x_d(s))\|\,ds.\end{aligned}$$

Since, by the assumptions $(\mathcal{H}_{f,2})$ and $(\mathcal{H}_{g,2})$, the above inequality along with (4.28) entails that for almost all $t \in [0,T]$

$$\frac{d}{dt}\left\langle x_e(t)-x_d(t), A\left(x_e(t)-x_d(t)\right)\right\rangle \leq$$
$$2k_\eta(t)\,\|A\|\,\|x_e(t)-x_d(t)\|^2 + 2L_\eta(t)\,\|A\|\,\|x_e(t)-x_d(t)\|\int_0^t \|x_e(s)-x_d(s)\|\,ds.$$

On the other hand, using the fact that A is ρ-coercive, we have

$$\frac{d}{dt}\left\langle x_e(t)-x_d(t), A\left(x_e(t)-x_d(t)\right)\right\rangle \leq$$
$$\frac{2k_\eta(t)\,\|A\|}{\rho}\left\langle A\left(x_e(t)-x_d(t)\right), x_e(t)-x_d(t)\right\rangle + 2L_\eta(t)\,\|A\|\,\|x_e(t)-x_d(t)\|\int_0^t \|x_e(s)-x_d(s)\|\,ds.$$

Finally, one has

$$\frac{d}{dt}\left\langle x_e(t)-x_d(t), A\left(x_e(t)-x_d(t)\right)\right\rangle \leq$$
$$\frac{2k_\eta(t)\,\|A\|}{\rho}\left\langle A\left(x_e(t)-x_d(t)\right), x_e(t)-x_d(t)\right\rangle +$$
$$\frac{2L_\eta(t)\,\|A\|}{\rho}\sqrt{\left\langle A\left(x_e(t)-x_d(t)\right), x_e(t)-x_d(t)\right\rangle}\int_0^t \sqrt{\left\langle A\left(x_e(s)-x_d(s)\right), x_e(s)-x_d(s)\right\rangle}ds.$$

Based on this last inequality and proceeding in the same way as for the Cauchy criterion stated above, we arrive at the following inequality

$$\xi(t) \leq \xi(0)\exp 2\left(\frac{\|A\|}{\rho}\left(\|k_\eta\|_{L^1([0,T],\mathbb{R}_+)} + T\,\|L_\eta\|_{L^1([0,T],\mathbb{R}_+)}\right)\right),$$

where

$$\xi(t) := \left\langle x_e(t)-x_d(t), A\left(x_e(t)-x_d(t)\right)\right\rangle.$$

Using the ρ-coercivity of A and the equalities $x_e(0) = e, x_d(0) = d$ bring us to

$$\|x_e(t) - x_d(t)\|^2 \leq \frac{1}{\rho}\xi(0) \exp 2\left(\frac{\|A\|}{\rho}\left(\|k_\eta\|_{L^1([0,T],\mathbb{R}_+)} + T\|L_\eta\|_{L^1([0,T],\mathbb{R}_+)}\right)\right).$$

Therefore

$$\|x_e(t) - x_d(t)\|^2 \leq \frac{1}{\rho}\|e-d\|^2 \|A\| \exp 2\left(\frac{\|A\|}{\rho}\left(\|k_\eta\|_{L^1([0,T],\mathbb{R}_+)} + T\|L_\eta\|_{L^1([0,T],\mathbb{R}_+)}\right)\right).$$

This implies that

$$\sup_{t\in I}\|x_e(t) - x_d(t)\| \leq Lip\,\|e-d\|,$$

where

$$Lip := \sqrt{\left(\frac{\|A\|}{\rho}\right)} \exp\left(\frac{\|A\|}{\rho}\left(\|k_\eta\|_{L^1([0,T],\mathbb{R}_+)} + T\|L_\eta\|_{L^1([0,T],\mathbb{R}_+)}\right)\right).$$

The proof is then complete. □

5. Conclusion

In this paper, using tools from convex analysis, we have introduced and studied the well-posedness of integrally perturbed degenerate sweeping processes under the absolute continuity in time t of the closed sets $C(t)$ and their convexity, by using a semi-discretization method. The existence and uniqueness of solutions for this class of sweeping processes are obtained under the coercivity assumption of the involved operator. Regarding the integrally perturbed degenerate sweeping processes, many questions remain that require further investigation. For example, it would be interesting to study the case of degenerate state-dependent sweeping processes. Another unexplored research topic is when the sets $C(t)$ are prox-regular, which would also be of great interest.

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