MANAS Journal of Engineering
ISSN 1694-7398 | e-ISSN 1694-7398

Volume 12, Issue 1, (2024) Pages 129-134
https://doi.org/10.51354/mjen.1449637

The Analitical Solution of Linear and Non-Linear Differential-Algebraic Equations (DAEs) with Laplace-Padé Series Method

Nooriza Mirzabekova , Ercan Çelik*

Department of Applied Mathematics and Informatics, Kyrgyz-Turkish Manas University, Bishkek-Kyrgyzstan

noorizamirzabekova@gmail.com, ORCID: 0009-0004-9921-0696
ercan.celik@manas.edu.kg, ORCID: 0000-0001-5971-7653

ABSTRACT

In this paper, we apply Laplace-Padé Series method to solve linear and non-linear differential-algebraic equations (DAEs). Firstly, The basic properties of the Laplace-Padé Series method are given. Secondly, we calculate the arbitrary order power series of differential-algebraic equations (DAEs), then convert it to the series form Laplace-Padé. Then, the three differential-algebraic equations (DAEs) are solved by Laplace-Padé Series method. It was seen that the method gave effective and fast results. Therefore, the method can be easily applied to linear and non-linear differential-algebraic equations (DAEs) problems in different fields.

ARTICLE INFO

Research article

Received: 09.03.2024
Accepted: 29.04.2024

Keywords:
Differential-Algebraic Equations (DAEs), Power Series, Padé Series, Laplace-Padé Series Method
*Corresponding author

1. Introduction

Today, there are many mathematical models that can be expressed with differential-algebraic equations (DAEs). It is very difficult to solve such models analytically. Recently, researchers have begun to work on numerical solutions of systems of differential-algebraic equations (DAEs). Circuit analysis, computer aided design and real-time simulation of mechanical systems, power systems, chemical process simulation and optimum control are some of these systems in many fields. Researchers have developed some numerical methods using Backward Differentiation Formula (BDF) [1-5] and implicit Runge–Kutta methods [6] methods to solve these systems. This type of method is suitable for problems with small indexes and have a special structure. Many problems are solved by these methods. However, more general methods are needed. Many methods have been proposed by researchers[9-14], and so on[16-21]. In this article, we presented a new method for solving differential-algebraic equations (DAEs). We have applied the method to three examples. the first of the examples is linear, the second is non-linear, and the third is a physically modeled differential algebraic equations(DAEs). The difference of our method from other methods presented in the references in solving problems modeled as differential-algebraic equations (DAEs) in science and engineering is that the method we present gives fast results.

2. The Method

A system of general non-linear differential algebraic equations with initial conditions is denoted

$$G(t,y,y')=0,\ \ y(t)=y_0 \qquad (1)$$

where $G\in R^n, y\in R^n$ and $t\in R$.
Let us assume that the solutions of equation (1) are of the form

$$y=y_0+at, \qquad (2)$$

where a is a vector function. Equation (2) and its derivatives are substituted in equation (1), and if higher order terms are neglected, equation

$$\mathcal{M}a=\mathcal{N}, \qquad (3)$$

is obtained. Where $\mathcal{M}$ and $\mathcal{N}$ are constant matrixes. The a values are found by solving equation (3). Then, by applying the above process to higher order terms, arbitrary order power series of equation (1) are obtained[7,8].

3. General Power Series Solution for Differential-Algebraic Equations(DAEs)

Now let's define another kind of power series in the form following:

$$f(t)=f_0+f_1t+f_2t^2+\ldots+(f_n+q_1a_1+\ldots+q_ma_m)t^n \qquad (4)$$

Where $q_1, q_2, \cdots, q_m$ are constants. $a_1, a_2, \cdots, a_m$ are bases of vector a. Let's represent each element with the Power series in (4). Then

$$y_i = y_{i,0} + y_{i,1}t + y_{i,2}t^2 + \ldots + a_i t^n \tag{5}$$

is also generally of the form. Substituting equation (5) into equation (1), we get the following:

$$f_i = \left(f_{i,n} + q_{i,1}a_1 + \ldots + q_{i,m}a_m\right)t^{n-j} + O(t^{n-j+1}), \tag{6}$$

If (1) have y', j is 0. If (1) not have y', j is 1. From equation (6) and (3), we get the linear equation following:

$$\mathcal{M}_{i,j} = P_{i,j} \tag{7}$$

$$\mathcal{N}_i = -f_{i,n}, \tag{8}$$

When the linear equation (7) and (8) is solved, a_i $(i = 1, \ldots, m)$ is obtained. Substituting a_i in equation (5), we get y_i $(i = 1, \ldots, m)$ polynomials of degree n. If these process are repeated from equation (5) to equation (8), arbitrary order power series are obtained for equation (1).

4. Padé Series

The Padé series, in the form of a rational function, is defined as following:

$$c_0 + c_1 x + c_2 x^2 + \cdots = \frac{a_0 + a_1 x + \cdots + a_M x^M}{1 + b_1 x + \cdots + b_L x^L} \tag{9}$$

In equation (9), if the cross-multiplication multiplication is done, we have

$$(c_0 + c_1 x + c_2 x^2 + \cdots)(1 + b_1 x + \cdots + b_L x^L) = a_0 + a_1 x + \cdots + a_M x^M \tag{10}$$

From the coefficient equality of both sides in equation (10), We have

$$c_l + \sum_{k=1}^{M} c_{l-k} b_k = a_l \quad (l = 0, \cdots, M) \tag{11}$$

$$c_l + \sum_{k=1}^{L} c_{l-k} b_k = 0 \quad (l = M+1, \cdots, M+L) \tag{12}$$

By solving the linear equation (12), we get $b_k (k = 1, \cdots, L)$. And if we substitute b_k in equation (9), we get $a_k (l = 1, \cdots, M)$ [7,8].

5. Laplace-Padé Series Method (LPSM)

The following steps are done for the Laplace-Padé series method [15].

1. The Laplace transform is applied to the obtained power series.
2. In the resulting equation, s is replaced by $1/x$.
3. We convert it from order $[M/L]$ to Padé series. M and L are arbitrarily chosen, but they should be of smaller value than the order of the power series.
4. $1/s$ is written instead of x.
5. Finally, the inverse laplace transform is taken.

Thus, the exact or numerical solution of the given equation is obtained.

6. Applications

In this section, we have solved the three differential-algebraic equations (DAEs) with the the Laplace-Padé series method (LPSM).

1. We consider the following linear differential-algebraic equation [15]:

$$\begin{aligned} &y_1' + y_1 - xy_3 + y_4 = 0, \\ &y_2' - y_1 + y_2 - x^2 y_3 + xy_4 = 0, \\ &y_3' - x^3 y_1 + x^2 y_2 - y_3 = 0, \\ &xy_1 - y_2 + xy_3 - y_4 = 0, \end{aligned} \tag{13}$$

and initial values

$$y_1(0) = y_3(0) = 1,$$
$$y_2(0) = y_4(0) = 0,$$

The exact solution is

$$\begin{aligned} &y_1(x) = \exp(-x), \\ &y_2(x) = x\exp(-x), \\ &y_3(x) = \exp(x), \\ &y_4(x) = x\exp(x). \end{aligned}$$

From initial values, the solutions of (13) can be supposed as

$$\begin{aligned} &y_1(x) = y_{0,1} + a_1 x \Rightarrow y_1(x) = 1 + a_1 x \\ &y_2(x) = y_{0,2} + a_2 x \Rightarrow y_2(x) = a_2 x \\ &y_3(x) = y_{0,3} + a_3 x \Rightarrow y_3(x) = 1 + a_3 x \\ &y_4(x) = y_{0,4} + a_4 x \Rightarrow y_4(x) = a_4 x \end{aligned} \tag{14}$$

Substitute (14) into (13) and neglect higher order terms, we have

$$\begin{aligned} &a_1 + 1 + O(x) = 0 \\ &a_2 - 1 + O(x) = 0 \\ &a_3 - 1 + O(x) = 0 \\ &(2 - a_2 - a_4)x + O(x^2) = 0 \end{aligned} \tag{15}$$

These formulas correspond to (6). The linear equation corresponding to (7) and (8) can be given as:

$$\mathcal{M}a = \mathcal{N} \tag{16}$$

Where

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \mathcal{N} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$a=\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix} \quad (17)$$

From equation (16), we have linear equation

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-1&0&-1\end{pmatrix}\cdot\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}-1\\1\\1\\-2\end{pmatrix} \quad (18)$$

Solving equation (18), we have

$$a=\begin{pmatrix}-1\\1\\1\\1\end{pmatrix} \quad (19)$$

and

$$\begin{aligned}y_1(x)&=1-x,\\y_2(x)&=x,\\y_3(x)&=1+x,\\y_4(x)&=x.\end{aligned} \quad (20)$$

From (17) the solutions of (13) can be supposed as

$$\begin{aligned}y_1(x)&=1-x+a_1x^2\\y_2(x)&=x+a_2x^2\\y_3(x)&=1+x+a_3x^2\\y_4(x)&=t+a_4x^2.\end{aligned} \quad (21)$$

In like manner, substitute (18) into (13) and neglect higher order terms, then we have

$$\begin{aligned}&(2a_1-1)x+O(x^2)=0\\&(2a_2+2)x+O(x^2)=0\\&(2a_3-1)x+O(x^2)=0\\&(-a_2-a_4)x^2+O(x^3)=0.\end{aligned} \quad (22)$$

Where

$$\mathcal{M}=\begin{pmatrix}2&0&0&0\\0&1&0&0\\0&0&2&0\\0&-1&0&-1\end{pmatrix},\ \mathcal{N}=\begin{pmatrix}1\\-1\\1\\0\end{pmatrix},$$

$$a=\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}$$

from (19) we have linear equation

$$\begin{pmatrix}2&0&0&0\\0&1&0&0\\0&0&2&0\\0&-1&0&-1\end{pmatrix}\cdot\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} \quad (23)$$

Solving equation (23), we have

$$a=\begin{pmatrix}0.5\\-1\\0.5\\1\end{pmatrix}. \quad (24)$$

Therefore

$$\begin{aligned}y_1(x)&=1-x\\y_2(x)&=x\\y_3(x)&=1+x+0.5x^2\\y_4(x)&=x+x^2\end{aligned} \quad (25)$$

Repeating above procedure we have

$$\begin{aligned}y_1(x)&=1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24},\\y_2(x)&=x-x^2+\frac{x^3}{2}-\frac{x^4}{6},\\y_3(x)&=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24},\\y_4(x)&=x+x^2+\frac{x^3}{2}+\frac{x^4}{6}.\end{aligned} \quad (26)$$

Then, Laplace transformation is applied to (26)

$$y_{L1}(s)=\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s^4}+\frac{1}{s^5},$$
$$y_{L2}(s)=\frac{1}{s^2}-\frac{2}{s^3}+\frac{3}{s^4}-\frac{4}{s^5},$$
$$y_{L3}(s)=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\frac{1}{s^5},$$
$$y_{L4}(s)=\frac{1}{s^2}+\frac{2}{s^3}+\frac{3}{s^4}+\frac{4}{s^5}.$$

and then $\frac{1}{x}$ is written in place of s.

$$y_{L1}\left(\frac{1}{x}\right)=x-x^2+x^3-x^4+x^5,$$
$$y_{L2}\left(\frac{1}{x}\right)=x^2-2x^3+3x^4-4x^5,$$
$$y_{L3}\left(\frac{1}{x}\right)=x+x^2+x^3+x^4+x^5$$
$$y_{L4}\left(\frac{1}{x}\right)=x^2+2x^3+3x^4+4x^5.$$

Afterwards, Padé approximant of order $[2/2]$ is applied

$$y_{1,[2,2]}(x)=\frac{x}{1+x},$$
$$y_{2,[2,2]}(x)=\frac{x^2}{x^2+2x+1},$$
$$y_{3,[2,2]}(x)=\frac{x}{1-x},$$
$$y_{4,[2,2]}(x)=\frac{x^2}{x^2-2x+1},$$

and $\frac{1}{s}$ is written in place of x for each variable

$$y_{1,[2/2]}\left(\frac{1}{s}\right)=\frac{1}{1+s},$$
$$y_{2,[2/2]}\left(\frac{1}{s}\right)=\frac{1}{(s+1)^2},$$
$$y_{3,[2/2]}\left(\frac{1}{s}\right)=\frac{1}{s-1},$$
$$y_{4,[2/2]}\left(\frac{1}{s}\right)=\frac{1}{(s-1)^2}.$$

Finally, by using the inverse Laplace transformation,

$$y_1(x)=\exp(-x),$$
$$y_2(x)=x\cdot\exp(-x),$$
$$y_3(x)=\exp(x),$$
$$y_4(x)=x\cdot\exp(x).$$

We obtain the exact solution for (13).

2. We consider the following linear differential-algebraic equation[15]:

$$\begin{aligned} y_1' &= 2y_1y_2z_1z_2,\\ y_2' &= -y_1y_2z_2^2,\\ z_1' &= (y_1y_2+z_1z_2)u,\\ z_2' &= -y_1y_2^2z_2^2u,\\ y_1y_2^2 &= 1. \end{aligned} \qquad (27)$$

and initial values

$$y_1(0)=y_2(0)=1,$$
$$z_1(0)=z_2(0)=1,$$
$$u(0)=1.$$

where prime denotes derivative with respect to x.
Applying the above method to Equation (27)

$$\begin{aligned} y_1(x)&=1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4,\\ y_2(x)&=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4,\\ z_1(x)&=1+2x+2x^2+\frac{4}{3}x^3+\frac{101}{174}x^4,\\ z_2(x)&=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{59}{696}x^4,\\ u(x)&=1+x+\frac{1}{2}x^2-\frac{1}{174}x^3-\frac{25}{58}x^4. \end{aligned} \qquad (28)$$

Then, Laplace transformation is applied to (28) and then $\frac{1}{x}$ is written in place of s . Afterwards, Padé approximant of order $[2/2]$ is applied and $\frac{1}{s}$ is written in place of x for each variable. Finally, by using the inverse Laplace transformation, we obtain the exact solution for (27).

$$
\begin{aligned}
&y_1(x)=\exp(2x),\\
&y_2(x)=\exp(-x),\\
&z_1(x)=\exp(2x),\\
&z_2(x)=\exp(-x),\\
&u(x)=\exp(x).
\end{aligned}
\qquad (29)
$$

3. Physical Problem

Consider the linear circuit given in Figure 1. The modified nodal analysis leads directly to the system

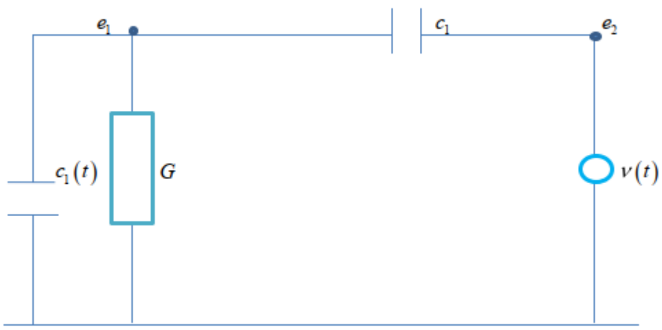

Fig. 1. Linear circuit with a time-dependent capacitance.

$$
\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1(t)k_1(t) \\ \\ B_2(t)k_1(t)+B_2(t)k_2(t) \end{pmatrix}' +
\begin{pmatrix} H & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} k_1(t) \\ k_2(t) \\ j(t) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ W(t) \end{pmatrix} \qquad (30)
$$

Choosing

$$B_1(t)=1+0.25(\sin(t)+\cos(t)),\quad \mathrm{B}_2=1,\quad \mathrm{H}=2$$

(31)

And input voltage

$$W(t)=4\sin(t)+0.25\sin(2t). \qquad (32)$$

Substitute equation (31) and (32) into equation (30), we can get the following

$$
\begin{aligned}
&\left[2+0.25\left(\cos(t)-\sin(t)\right)\right]k_1(t)+\left[2+0.25\left(\cos(t)+\sin(t)\right)\right]k_1'(t)-k_2'(t)=0,\\
&-k_1'(t)+k_2'(t)-j(t)=0,\\
&k_2(t)=4\sin(t)+0.25\sin(2t)
\end{aligned}
$$

(33)

Exact solution of equation (33) is

$$
\begin{aligned}
&k_1(t)=\sin(t)+\cos(t),\\
&k_2(t)=4\sin(t)+0.25\sin(2t),\\
&j(t)=3\cos(t)+0.5\cos(2t)+\sin(t)
\end{aligned}
$$

For the consistent initial values

$$k_1(0)=1, k_2(0)=0, j(0)=3.5.$$

Equation (33) is form of equation (1).
Applying the above method to equation (33), we have

$$
\begin{aligned}
&k_1(t)=1+t-0.5t^2-0.1666666667t^3+0.04166666667t^4,\\
&k_2(t)=4.5t-t^3,\\
&j(t)=3.5+t-2.5t^2-0.1666666667t^3+0.4583333333t^4.
\end{aligned}
$$

(34)

Then, Laplace transformation is applied to (34) and then $\frac{1}{t}$ is written in place of s. Afterwards, Padé approximant of order $[2/2]$ is applied and $\frac{1}{s}$ is written in place of t for each variable. Finally, by using the inverse Laplace transformation, we obtain the exact solution for (30).

$$
\begin{aligned}
&k_1(t)=\sin(t)+\cos(t),\\
&k_2(t)=4\sin(t)+0.25\sin(2t),\\
&j(t)=3\cos(t)+0.5\cos(2t)+\sin(t).
\end{aligned}
$$

Conclusion

This work presented Laplace-Padé series method as a combination of the classic series method and method based on the Laplace and Padé series. By solving three problems, we presented the Laplace-Padé series method as a useful tool with high potential to solve linear/non-linear differential-algebraic equations. The Laplace-Padé Series Method(LPMS) is used for rapid convergence of solutions or to find exact solutions. The proposed method possesses a straightforward procedure, suitable for science and engineers. The method presented in our next work will be applied to different problems modeled as linear and non-linear differential-algebraic equations (DAEs).

Acknowledgments

The authors have been supported by the Kyrgyz-Turkish Manas University BAP Coordination Unit with Project number KTMU-BAP-2023.FB.02.

REFERENCES

[1]. K. E. Brenan, S.L. Campbell and L.R. Petzold, Numerical Solution of Initial Value Problems in Differential Algebraic Equations, Elsevier, New York, 1989.

[2]. E. Hairer, S.P. Norsett and G. Wanner, Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems, Springer, New York, 1992.

[3]. L.R. Petzold, Numerical solution of differential-algebraic equations, Adv. Numer. Anal. 4 (1995).

[4]. U.M. Ascher, On symmetric schemes and differential-algebraic equations, SIAM J. Sci. Stat. Comput. 10, 937–949, 1989.

[5]. C.W. Gear and L.R. Petzold, ODE systems for the solution of differential-algebraic systems, SIAM J. Numer. Anal. 21, 716–728, 1984.

[6]. U.M. Ascher and L.R. Petzold, Projected implicit Runge–Kutta methods for differential-algebraic equations, SIAM J. Numer. Anal. 28, 1097–1120, 1991.

[7]. E. Çelik and M. Bayram, On the numerical solution of differential-algebraic equation by Pade´ series, Appl. Math. Comput. 137, 151–160, 2003.

[8]. E. Çelik and M. Bayram, The numerical solution of physical problems modeled as a systems of differential-algebraic equations(DAEs), Journal of Franklin Institute, 342, 1-6, 2005.

[9]. U.M. Ascher and P. Lin, Sequential regularization methods for higher index differential-algebraic equations with constant singularities: the linear index-2 case, SIAM J. Numer. Anal. 33, 1921–1940, 1996.

[10]. U.M. Ascher and P. Lin, Sequential regularization methods for non-linear higher index differential-algebraic equations, SIAM J. Sci. Comput. 18, 160–181, 1997.

[11]. S.L. Campbell, A computational method for general higher index singular systems of differential equations, IMACS Trans. Sci. Comput. 89, 555–560, 1989.

[12]. S.L. Campbell, E. Moore and Y. Zhong, Utilization of automatic differentiation in control algorithms, IEEE Transactions on Automatic Control, 39 (5), 1047 – 1052,1994.

[13]. S. K. Vanani and A. Aminataei, "Numerical solution of differential algebraic equations using a multiquadric approximation scheme," Mathematical and Computer Modelling, 53 (5-6), 659 – 666, 2011.

[14]. Y. Khan and N. Faraz, Application of modified laplace decomposition method for solving boundary layer equation, Journal of King Saud University - Science, 23 (1), 115 – 119, 2011.

[15]. H.Vazquez-Leal, Exact solutions for Differential-Algebraic Equations, Miskolc Mathematical Notes, 15 (1), 227-238, 2014.

[16]. H. Singh and A. Wazwaz, Computational Method for Reaction Diffusion-Model Arising in a Spherical Catalyst, International Journal of Applied and Computational Mathematics, 7 (3), 65, 2021.

[17]. H. Singh, Analysis for fractional dynamics of Ebola virus model, Chaos solitons & fractals, 138, 109992, 2020.

[18]. H. Singh, Analysis of drug treatment of the fractional HIV infection model of CD4+ T-cells, Chaos Solitons & Fractals, 146, 110868, 2021.

[19]. H. Singh, Jacobi collocation method for the fractional advection-dispersion equation arising in porous media, Numerical methods for partial differential equations, 2020.

[20]. H. Singh, H.M. Srivastava and D. Kumar, A reliable numerical algorithm for the fractional vibration equation, Chaos, Solitons and Fractals, vol. 103, 131-138, 2017.

[21]. H. Singh, Chebyshev spectral method for solving a class of local and nonlocal elliptic boundary value problems, International Journal of Nonlinear Sciences and Numerical Simulation, 24(3), 899-915, 2021